THE LOVÁSZ LOCAL LEMMA

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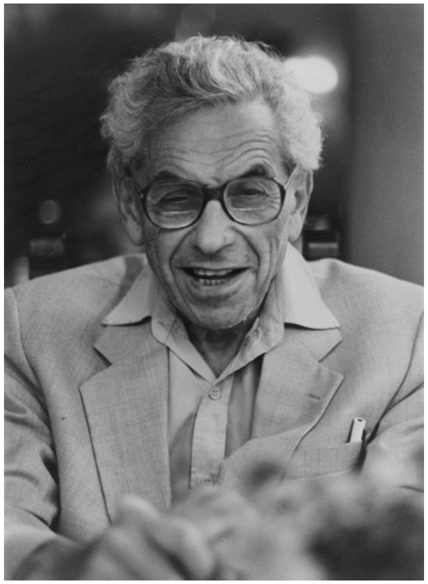
László Lovász
1948 – Present

Paul Erdős
1913 – 1996



- Two Hungarian mathematicians
- Published the Lovász local lemma in 1975 in the article *Problems and results on 3-chromatic hypergraphs and some related questions*

Introduction to the Probabilistic Method

- A method primarily used in combinatorics and pioneered by Erdős
- Method works by showing:

If one randomly chooses objects from a specified class, then the probability the result satisfies some desired properties is greater than zero.

Constructive vs. Non-constructive

- Constructive proof – method of proof that demonstrates the existence of a mathematical object by creating or proving a method for creating the object
- Non-constructive proof (existence proof) – proves the existence of a particular kind of object without providing an example

Does the probabilistic method use a constructive or non-constructive strategy?

Intuition Behind Local Lemma

- Let A_1, A_2, ... , A_n be a set of "bad" events, whose occurrences render the object undesirable
- Desired property = avoidance of all bad events

If $\sum P[A_i] < 1$, then clearly there is a positive probability that none of them occurs. However, $\sum P[A_i]$ is often much larger than $P[\bigcup A_i]$.

Intuition Behind Local Lemma (con.)

- We are interested in the probability that no bad events occur: $$P[\bigcap_{i=1}^{n} \overline{A}_i] > 0$$
- If $A_1, \ldots, A_n$ are independent events and $P[A_i] \leq p$ then: $$P[\bigcap_{i=1}^{n} \overline{A}_i] = \prod_{i=1}^{n} P[\overline{A}_i] \geq (1-p)^n,$$ which is strictly positive (provided that the trivial condition $p < 1$ holds).

Intuition Behind Local Lemma (con.)

- Local lemma extends prior result to a setting that allows "limited dependencies"
- Typically applies to situations of rare events, where the probability that no bad events occurs is very small, but exists even if $P[A_i]$ is arbitrarily close to 1

Useful Definitions

1. Mutual independence

Definition. An event A is said to be *mutually independent* of a set of events $\{B_i\}$ if for any subset β of events or their complements contained in $\{B_i\}$, we have $Pr[A|\beta] = Pr[A]$.

2. Dependency digraph

Definition. Let $A_1, A_2, ..., A_n$ be events in a probability space. A directed graph $D = (V, E)$ with $V = \{1, ..., n\}$ is a *dependency digraph* for $A_1, ..., A_n$ if each event A_i is mutually independent of all the events A_j with $(i, j) \notin E$.

Lovász Local Lemma

Lemma. (General Lovász Local Lemma) Let $A_1, A_2, ..., A_n$ be a set of "bad" events and $D = (V, E)$ their dependency digraph. If a set of real numbers $x_1, ..., x_n \in [0, 1)$ is assigned to the events such that

$$Pr[A_i] \leq x_i \prod_{(i,j) \in E} (1 - x_j),$$

then

$$Pr\left[\bigcap_{i=1}^{n} \overline{A_i}\right] \geq \prod_{i=1}^{n} (1 - x_i) > 0.$$

Lovász Local Lemma (Symmetric)

Lemma. (Symmetric Lovász Local Lemma) Let $A_1, A_2, ..., A_n$ be a set of "bad" events with $Pr[A_i] \leq p$ for all $1 \leq i \leq n$ and all outdegrees in dependency digraph $D = (V, E)$ are at most d. If

$$ep(d+1) \leq 1,$$

(where $e = 2.7128...$ is the basis of natural logarithms), then

$$Pr\left[\bigcap_{i=1}^{n} \overline{A_i}\right] > 0.$$

Proof of Local Lemma

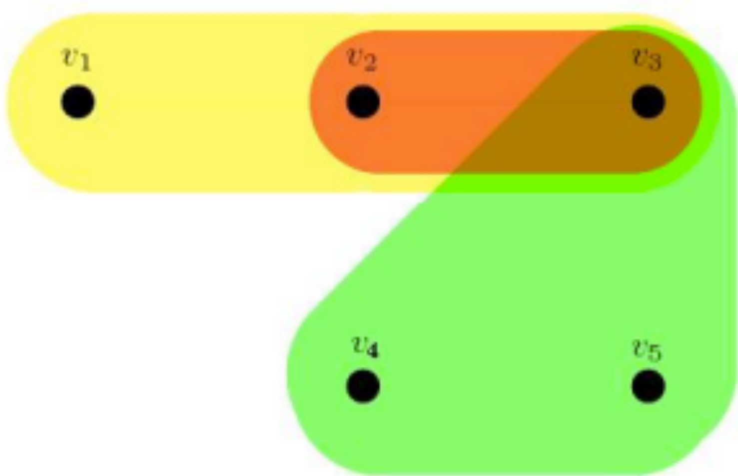
- First, prove how symmetric lemma extends from general case
  - Let $x_i = 1/(d+1)$
- Prove general lemma

Back to the Circle Coloring Problem

- $e = 2.718...$
- $p = 1/121$
- $d = 42$
- $ep(d+1) \approx 0.966 < 1$
- Therefore, by the local lemma, a set satisfying a valid circle coloring.

Application: Hypergraph Coloring

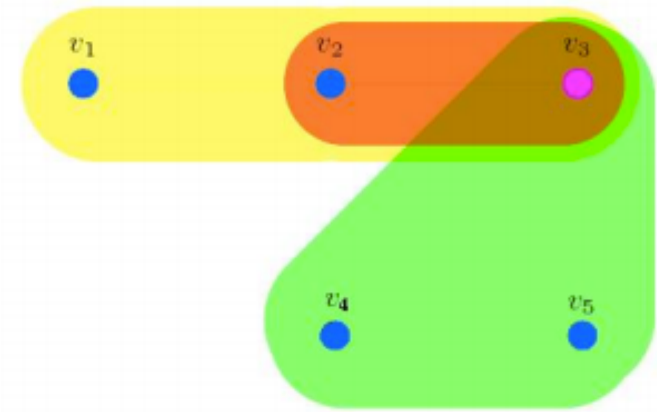
- Hypergraph H (X, E) – generalization of a graph in which an edge can connect any number of vertices
  - While graph edges are pairs of nodes, hyperedges are arbitrary sets of nodes
  - *Degree* of a node is number of hyperedges containing it

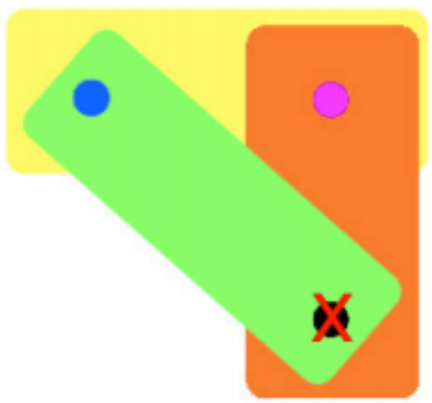


Define $E = \{e_1, e_2, e_3\}$. What is the degree of v_3?

2-Coloring Hypergraphs

- 2-coloring – assignment of two colors to vertices where no edge is monochromatic
- If there are two many intersections among edges, a hypergraph may be impossible to 2-color





How many intersections can we allow and guarantee a 2-coloring exists?

Applying Symmetric Local Lemma

- K-uniform hypergraph – hypergraph such that all hyperedges have size k
- Hypergraph coloring result: Any k-uniform hypergraph with less than 2^{k-1} edges is 2-colorable
- Obtain similar result by applying local lemma (but hypergraph need not be k-uniform):

Theorem. Let H be a hypergraph in which every edge has at least k vertices and intersects at most d other edges. If $e(d+1) \leq 2^{k-1}$, then H is 2-colorable.

Proof of hypergraph application

Proof set-up:

- Define bad events as edges that are monochromatic
- Let d = $(2^{k-1} / e) - 1$
- p = $1/2^{k-1}$ where $P[A_i] \leq p$ for all i

Lemma. (Symmetric Lovász Local Lemma) Let $A_1, A_2, ..., A_n$ be a set of "bad" events with $Pr[A_i] \leq p$ for all $1 \leq i \leq n$ and all outdegrees in dependency digraph $D = (V, E)$ are at most d. If

$$ep(d+1) \leq 1,$$

(where $e = 2.7128...$ is the basis of natural logarithms), then

$$Pr\left[\bigcap_{i=1}^{n} \overline{A_i}\right] > 0.$$

More Applications

- Satisfying Boolean formulas with limited intersection of clauses
- Packet routing where the problem is to select possibly overlapping paths and a schedule to move the packets along the paths while satisfying edge capacity constraints

Extensions

- Algorithmic approach published by József Beck in 1991 (instead of proving the existence of a 2-coloring, find a valid hypergraph 2-coloring in polynomial time)
- Constructive proof published by Robin Moser in 2008