Enumerative Combinatorics

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Permutations

- The number of all permutations of an n-element set is n!
- How many ways are there to arrange the letters ABCDE in a row?

$$5! = 120$$

Permutations with Repeats

- Let n, m, a_1, a_2, …, a_m be non-negative integers satisfying $a_1 + a_2 + \ldots + a_m = n$ and where there are a_i objects of type i, for $1 \leq i \leq m$. The number of ways to linearly order these objects is $\frac{n!}{a_1!a_2!\ldots a_m!}$
- For m = 2, and letting $k = a_1$, we get the binomial coefficient of

$$\frac{n!}{a_1!a_2!} = \frac{n!}{k!(n-k)!} = \binom{n}{k} = \binom{n}{n-k}$$

Permutations with Repeats

- A garden has 3 identical red flowers, 4 identical green flowers, and 3 identical yellow flowers. How many ways are there to arrange all the flowers in a row?

$$\frac{10!}{3!4!3!} = 4200$$

Strings Over Alphabets

- For $n \geq 0$ and $k \geq 1$, the number of k-digit strings that can be formed over an n-element alphabet is n^k
- How many 3 digit odd numbers are there?

$$9 \bullet 10 \bullet 5 = 450$$

Strings Over Alphabets

- For $n \geq 1$, $k \geq 1$, and $n \geq k$, the number of k-digit strings that can be formed over an n-element alphabet in which no letter is used more than once is $n(n-1)\ldots(n-k+1) = \frac{n!}{(n-k)!}$
- How many 3 digit numbers are there with distinct digits?

$$9 \bullet 9 \bullet 8 = 648$$

Bijections

- Let X and Y be two finite sets. A function f: $X \to Y$ is bijective if and only if every element of X is mapped to exactly one element of Y, and for every element of Y, there is exactly one element in X that maps to it
- If there exists a bijection f from X onto Y, then X and Y have the same number of elements
- A bijection allows us to count in two different ways, which helps us set up a ton of identities and simplifies many hard counting problems

Subsets

- The number of k-element subsets of $\{a_1, a_2, \ldots, a_n\}$ is $\binom{n}{k} = \frac{n!}{k!(n-k)!}$
- How many 3-digit numbers are there such that the digits, read from left to right, are in strictly decreasing order?

$$\binom{10}{3} = 120$$

Subsets

- A multiset is a set which members are allowed to appear more than once
- The number of k-element multi-subsets of $\{a_1, a_2, \ldots, a_n\}$ is

$$\binom{n+k-1}{k}$$

Binomial Theorem And Related Identities

- $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$
- $2^n = \sum_{k=0}^{n} \binom{n}{k}$
- $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$
- $n2^{n-1} = \sum_{k=1}^{n} k \binom{n}{k}$

Functions
(n distinct objects, k distinct boxes)

- We have 10 different presents and 5 people to give the presents to. How many different ways can the people receive the presents?

$$5^{10}$$

- The number of ways to put n distinct objects into k distinct boxes is k^n

Weak Compositions

(n identical objects, k distinct boxes)

- Chocolate Problem: We have 20 identical chocolates and 13 people in the class. How many ways are there to give out the chocolates such that each person receives a nonnegative amount?

$$\binom{20+13-1}{13-1} = \binom{32}{12} = \binom{32}{20} = 225{,}792{,}840$$

- A sequence $(a_1, a_2, \ldots, a_k)$ of integers fulfilling $a_i \geq 0$ for all i, and $a_1 + a_2 + \ldots + a_k = n$ is called a weak composition of n.
- For all positive integers n and k, the number of weak compositions of n into k distinct parts is

$$\binom{n+k-1}{k-1}$$

Weak Compositions
(n identical objects, k distinct boxes)

- When expanding $(a + b + c)^{10}$ and combining like-terms, how many terms do we get?

$$\binom{10 + 3\ - 1}{3 - 1} = \binom{12}{2} = 66$$

- How many ordered quadruples $(x_1, x_2, x_{3,} x_4)$ of odd positive integers satisfy $x_1 + x_2 + x_3 + x_4 = 98$?

$$\binom{47 + 4\ - 1}{4 - 1} = \binom{50}{3} = 19600$$

Strong Compositions

- A sequence $(b_1, b_2, \ldots, b_k)$ of integers fulfilling $b_i \geq 1$ for all i, and $b_1 + b_2 + \ldots + b_k = n$ is called a strong composition of n.
- For all positive integers n and k, the number of strong compositions of n into k parts is
$$\binom{n-1}{k-1}$$

Set Partitions

(n distinct objects, k identical boxes)

- A set partition involves partitioning the set $\{a_1, a_2, \ldots, a_n\}$ into k nonempty subsets
- There are 7 ways that we can partition the set $\{a_1, a_2, a_3, a_4\}$ into 2 nonempty subsets

Set Partitions
(n distinct objects, k identical boxes)

- There are $S(n, k)$ ways to partition a set of n elements into k nonempty subsets
  - Stirling numbers of the second kind
  - $S(0, 0) = 0$ and $S(n, k) = 0$ if $n < k$ by convention
- With empty boxes allowed, there are $\sum_{i=1}^{k} S(n, i)$ ways to put n distinct objects into k identical boxes
- $S(n, k) = S(n\text{-}1, k\text{-}1) + k \bullet S(n\text{-}1, k)$

Set Partitions
(n distinct objects, k identical boxes)

- What is a closed form formula for S(n, 2)?

$$2^{n-1} - 1$$

- What is a closed form formula for S(n, n-1)?

$$\binom{n}{2}$$

Integer Partitions
(n identical objects, k identical boxes)

- Let $a_1 \geq a_2 \geq \ldots \geq a_k \geq 1$ be integers so that $a_1 + a_2 + \ldots + a_k = n$. Then the sequence $(a_1, a_2, \ldots, a_k)$ is called a partition of integer n.
- The integer 5 has 7 partitions

Integer Partitions
(n identical objects, k identical boxes)

- The number of all partitions of integer n is $p(n)$
- The number of partitions of integer n into exactly k parts is $p_k(n)$
- With empty boxes allowed, there are $\sum_{i=1}^{k} p_i(n)$ ways to put n identical objects into k identical boxes

Integer Partitions
(n identical objects, k identical boxes)

- Ferrers Diagram: A diagram of a partition $p = (a_1, a_2, \ldots, a_k)$ that has a set of n square boxes with horizontal and vertical sides so that in the row i, we have a_i boxes and all rows start at the same vertical line
- The number of partitions of n into at most k parts is equal to that of partitions of n into parts not larger than k
- Let $q(n)$ be the number of partitions of n in which each part is at least two. Then $q(n) = p(n) - p(n-1)$ for all positive integers $n \geq 2$