18.304. Talk: Integral Independence & Triangle Cutting.

Introduction:

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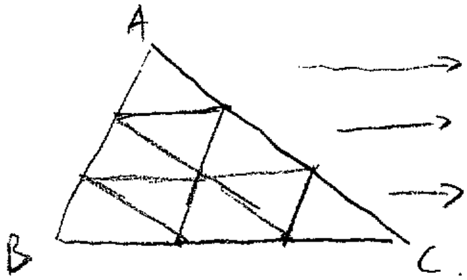


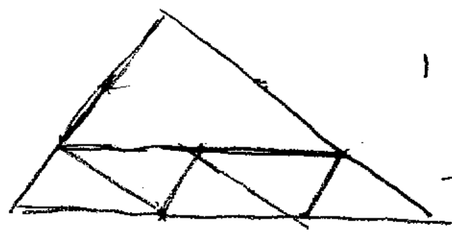
$$\left.\begin{array}{l} \longrightarrow 1 \\ \longrightarrow 3 \\ \longrightarrow 5 \end{array}\right\} = 9 \text{ congruent } \triangle.$$

$1+3+5+\cdots+(2n-1) = n^2 \rightarrow$ Achievable.

Problem: Find all $n \in \mathbb{Z}^+$, such that any $\triangle$ can be cut into n $\left\{\begin{array}{l} \text{I. congruent } \triangle \\ \quad\Downarrow \\ \text{II. similar } \triangle. \end{array}\right.$

II:

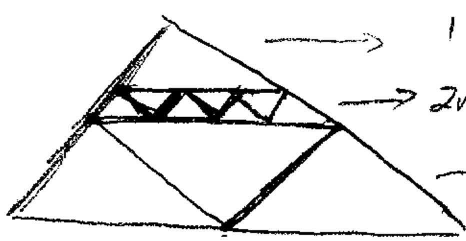
Case 1:



$$\left.\begin{array}{l} 1 \\ \rightarrow 2n-1 \end{array}\right\} 2n.$$

$n \geq 2 \Rightarrow$ every even $\# \geq 4$ is possible.

Case II:



$$\left.\begin{array}{l} \longrightarrow 1 \\ \rightarrow 2n-1 \\ \rightarrow 3 \end{array}\right\} 2n+3$$

$n \geq 2 \Rightarrow$ every odd $\# \geq 7$ is possible.

Conclude: All # is possible except $\underline{2, 3, 5}$

?

- Integral Independence:

def: for: $a_1x + a_2y = 0$ $\Rightarrow$ Generalize: $a_1x_1 + a_2x_2 + a_3 \cdot x_3 + \cdots a_nx_n = 0$

if has non-zero soln $\longrightarrow$. are int. dep.

if has only zero soln $\longrightarrow$. are int. indep.

<u>Example 1: $\sqrt{2}$ & 1 are Int. Indep. ?</u> $\Rightarrow$ Irrational & Integer are Int. Indep.

$\sqrt{2} \neq \frac{x}{y} \Rightarrow \sqrt{2}y + (-x) \neq 0$ unless $x = y = 0$

<u>Example 2</u>: $\sqrt{2}$, $\sqrt{3}$ & $180 - \sqrt{2} - \sqrt{3}$ are int. <u>indep</u>?

$$\sqrt{2}x + \sqrt{3}y + (180 - \sqrt{2} - \sqrt{3})z = 0$$

$$\sqrt{2}\underbrace{(x-z)}_{p} + \sqrt{3}\underbrace{(y-z)}_{q} + 180z = 0.$$

if $x = z \Rightarrow y = z$ & $z = 0 \Rightarrow x = 0$

if $y = z \Rightarrow x = z$ & $z = 0 \Rightarrow y = 0$

if $x \neq z$ & $y \neq z \Rightarrow p \neq 0, q \neq 0$.

$$\sqrt{2}p + \sqrt{3}q = -180z$$

$$\Downarrow$$

$$1\underbrace{(2p^2 + 3q^2 - 180z^2)}_{=0} + \sqrt{6}\underbrace{(2pq)}_{=0} = 0$$

$\Rightarrow$ only zero solution

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<u>Example 3</u>: Angles of a right $\triangle$ are int. ______ ?

$A, B, C = \frac{\pi}{2}$ $\quad A + B = C \longrightarrow A + B + C(-1) = 0$

Example 4:

let real # A, B, C, α, β be related by:

$$\begin{aligned} A &= a_{11}\alpha + a_{12}\beta \\ B &= a_{21}\alpha + a_{22}\beta \quad , \quad a_{ij} \in \mathbb{Z}. \\ C &= a_{31}\alpha + a_{32}\beta \end{aligned}$$

Then A, B, C are int. <u>dep</u> ?

$$\begin{aligned} a_{21} \cdot A &= a_{21}a_{11}\alpha + a_{21}a_{12}\beta \\ a_{11} B &= a_{11}a_{21}\alpha + a_{11}a_{22}\beta \end{aligned} \downarrow$$

$$a_{21}A - a_{11}B = (a_{21}a_{12} - a_{11}a_{22})\beta.$$

if $a_{21}a_{12} - a_{11}a_{22} = 0$

$A a_{21} - B a_{11} + C\,0 = 0. \to$ a non-zero soln.

if $a_{21}a_{12} - a_{11}a_{22} \neq 0$

$$\beta = f(A, B), \quad \alpha = g(A, B)$$

$A \_\_ + B \_\_ + C \_\_ = 0 \to$ non-zero soln

Example 5

What if $a_{ij} \in \mathbb{Q}$?

A, B, C are <u>Int dep</u>.

Will 2, 3, 5 work?

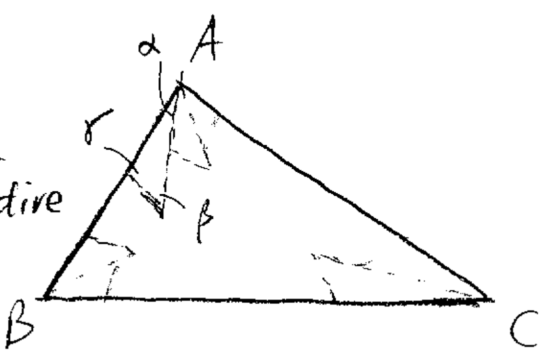
Assume A, B, C are Int. Indep.

$A = a_{11}\alpha + a_{12}\beta + a_{13}\gamma$ $\quad a_{ij}$ non-negative

$B = a_{21}\alpha + a_{22}\beta + a_{23}\gamma$

$C = a_{31}\alpha + a_{32}\beta + a_{33}\gamma$ $\quad +$

$\Sigma_1 \quad \Sigma_2 \quad \Sigma_3$



$\pi = \Sigma_1\alpha + \Sigma_2\beta + \Sigma_3\gamma$

Case 1: at least one of $\Sigma_i = 0$, say Σ_3

$a_{13} = a_{23} = a_{33} = 0$

Example 4 $\Rightarrow$ A, B, C are Int. Dep #

Case 2: Σ_1, Σ_2 & Σ_3 are all positive.

$\Sigma_1\alpha + \Sigma_2\beta + \Sigma_3\gamma \geq \alpha + \beta + \gamma = \pi$

$\Sigma_1 \to 1, \quad \Sigma_2 \to 1, \quad \Sigma_3 \to 1$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

each row & column has exactly one 1 & two 0.

$\Rightarrow A, B, C$ is a permutation of α, β, γ

$A = \alpha, B = \beta, C = \gamma$

Lemma: ① Triangles of partition $\sim$ to each other to original triangle.

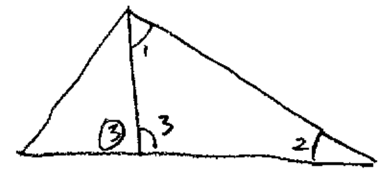
② Angles of $\triangle ABC$ are not split, they're cut off

$\Rightarrow$ 2, 3 are not possible! in general $\Rightarrow$

③ If a $\triangle$ is cut into 2 triangles T_1 & T_2, $T_1 \sim T_2$.

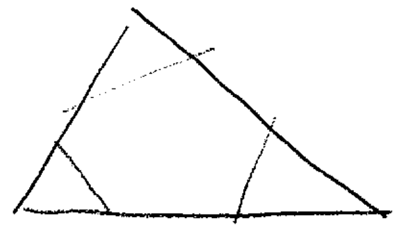
then T_1 & T_2 are right $\triangle$.

$\angle ③ = \angle 3 \Rightarrow \angle ③ = \angle 3 = 90°$.



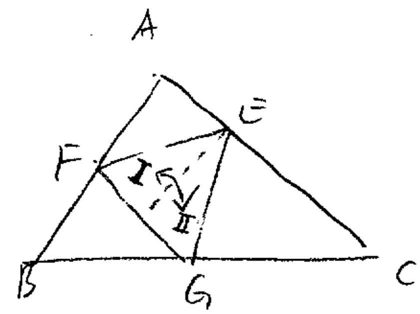
5?

Say a $\triangle$ with Int. Indep. angles is cut into 5 similar triangles. middle piece:



- i) Triangle
- ii) Quadrilateral

i) $\triangle EFG$:



Lemma ③ $\downarrow$

$\triangle$I & $\triangle$II are right $\triangle$

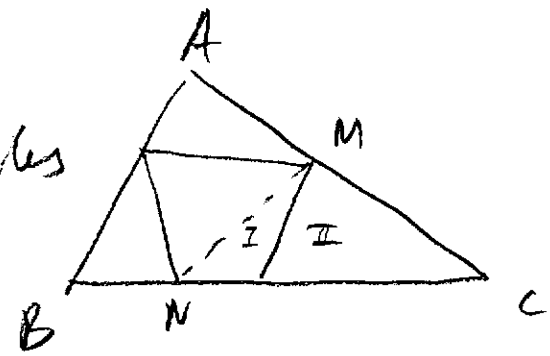
Lemma ② $\downarrow$

All 5 triangles are right $\triangle$ & $\triangle ABC$ is Rt$\triangle$

Example 4 $\downarrow$

$\angle A$, $\angle B$, $\angle C$ are Int. Rep #.

ii) Quadrilateral:
it must be cut into 2 triangles
by its diagonal



Look at $\triangle MNC$

Lemma ③ $\downarrow$

$\triangle I$ & $\triangle II$ are Right $\triangle$ $\rightarrow$ $\angle A$ & $\angle B$ & $\angle C$ are Int. Rep.

Conclusion: $\triangle T$ with Int. Indep angles cannot be cut into 2, 3 or 5 triangles similar to each other.

Does $\triangle T$ exist? $\longrightarrow$ Example 2.