Generating Functions: The Basics

A formal power series $P(x)$ is an expression of the form

$$P(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{i=0}^{\infty} a_i x^i$$

A generating function is a power series formed from a sequence $\{a_0, a_1, \ldots\}$.

Important Power Series Identity (ID1):

$$\frac{1}{(1-x)^k} = \sum_{n=0}^{\infty} \binom{n+k-1}{n} x^n$$

Methodology:

1) Define generating function $A(x)$ from sequence
2) Multiply both sides of recurrence by x^n and sum over all n
3) Express both sides of identity in terms of $A(x)$ and solve for $A(x)$
4) Expand $A(x)$ as a power series

Example 1: Fibonacci Sequence

$$F_{n+2} = F_{n+1} + F_n, \qquad F_0 = 0, F_1 = 1$$

Finding the general term:

Define $F(x) = \sum F_i x^i$. Then:

$$\sum_{n=0}^{\infty} F_{n+2} x^n = \sum_{n=0}^{\infty} F_{n+1} x^n + \sum_{n=0}^{\infty} F_n x^n$$
$$\Rightarrow \frac{1}{x^2} \sum_{n=0}^{\infty} F_{n+2} x^{n+2} = \frac{1}{x} \sum_{n=0}^{\infty} F_{n+1} x^{n+1} + \sum_{n=0}^{\infty} F_n x^n$$
$$\Rightarrow \frac{A(x) - x}{x^2} = \frac{A(x)}{x} + A(x)$$
$$\Rightarrow A(x) = \frac{x}{1 - x - x^2}$$

Define also

$$r_+ = \frac{1+\sqrt{5}}{2}, \qquad r_- = \frac{1-\sqrt{5}}{2}$$

Then:

$$\begin{aligned} A(x) &= \frac{x}{1-x-x^2} \\ &= \frac{x}{(1-r_-x)(1-r_+x)} \\ &= \frac{1}{r_+-r_-}\left[\frac{1}{1-r_+x} - \frac{1}{1-r_-x}\right] \\ &= \frac{1}{\sqrt{5}}\left(\sum_{i=1}^{\infty} r_+^i x^i - \sum_{i=1}^{\infty} r_-^i x^i\right) \end{aligned}$$

Therefore, we have that

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

Here is a more practical application:

Example 2: What is the generating function for a partition of n into a sum of even positive integers?

Solution:

$$(1+x^2+x^4+\cdots)(1+x^4+x^8+\cdots)(1+x^6+x^{12}+\cdots)$$