

The introduction of a paper that presents others' results should

- 1) indicate the paper's focus
- 2) indicate why that focus is important or interesting
- 3) be worded to be understood by the target audience and remain relatively nontechnical
- 4) preview the paper's structure.
- 5) Acknowledge the source(s) of the content.

The third goal often conflicts with the first two.

These are the first 2 pages of W. G. Dwyer and J. Spalinski's "Homotopy theories and model categories," Chapter 1 of *Handbook of Algebraic Topology*, Edited by I.M. James, (c) 1996 Elsevier Science B.V.

## 1. Introduction

### 1. The paper's focus.

This paper is an introduction to the theory of “model categories”, which was developed by Quillen in [22] and [23]. By definition a model category is just an ordinary category with three specified classes of morphisms, called *fibrations*, *cofibrations* and *weak equivalences*, which satisfy a few simple axioms that are deliberately reminiscent of properties of topological spaces. Surprisingly enough, these axioms give a reasonably general context in which it is possible to set up the basic machinery of homotopy theory. The machinery can then be used immediately in a large number of different settings, as long as the axioms are checked in each case. Although many of these settings are geometric (spaces (§8), fibrewise spaces (3.11),  $G$ -spaces [11], spectra [5], diagrams of spaces [10] . . .), some of them are not (chain complexes (§7), simplicial commutative rings [24], simplicial groups [23] . . .). Certainly each setting has its own technical and computational peculiarities, but the advantage of an abstract approach is that they can all be studied with the same tools and described in the same language. What is the suspension of an augmented commutative algebra? One of incidental appeals of Quillen’s theory (to a topologist!) is that it both makes a question like this respectable and gives it an interesting answer (11.3).

2&3. This paragraph does a nice job of indicating why the focus is interesting, while remaining relatively nontechnical.

5. This underlined text acknowledges the paper's main source while indicating the contribution of the authors.

We have tried to minimize the prerequisites needed for understanding this paper; it should be enough to have some familiarity with CW-complexes, with chain complexes, and with the basic terminology associated with categories. Almost all of the material we present is due to Quillen [22], but we have replaced his treatment of suspension functors and loop functors by a general construction of homotopy pushouts and homotopy pullbacks in a model category. What we do along these lines can certainly be carried further. This paper is not in any sense a survey of everything that is known about model categories; in fact we cover only a fraction of the material in [22]. The last section has a discussion of some ways in which model categories have been used in topology and algebra.

*Organization of the paper.* Section 2 contains background material, principally a discussion of some categorical constructions (limits and colimits) which come up almost immediately in any attempt to build new objects of some abstract category out of old ones. Section 3 gives the definition of what it means for a category  $\mathbf{C}$  to be a model category, establishes some terminology, and sketches a few examples. In §4 we study the notion of “homotopy” in  $\mathbf{C}$  and in §5 carry out the construction of the homotopy category  $\text{Ho}(\mathbf{C})$ . Section §6 gives  $\text{Ho}(\mathbf{C})$  a more conceptual significance by showing that it is equivalent to the “localization” of  $\mathbf{C}$  with respect to the class of weak equivalences. For our purposes the “homotopy theory” associated to  $\mathbf{C}$  is the homotopy category  $\text{Ho}(\mathbf{C})$  together with various related constructions (§10).

Sections 7 and 8 describe in detail two basic examples of model categories, namely the category  $\mathbf{Top}$  of topological spaces and the category  $\mathbf{Ch}_R$  of nonnegative chain complexes of modules over a ring  $R$ . The homotopy theory of  $\mathbf{Top}$  is of course familiar, and it turns out that the homotopy theory of  $\mathbf{Ch}_R$  is what is usually called homological algebra. Comparing these two examples helps explain why Quillen called the study of model categories “homotopical algebra” and thought of it as a

4. The introduction concludes by outlining the structure and contents of the paper. As much as possible, this outline should be specific but should also make some sense to the target audience before they've read the paper. This balance can be challenging to achieve.

generalization of homological algebra. In §9 we give a criterion for a pair of functors between two model categories to induce equivalences between the associated homotopy categories; pinning down the meaning of “induce” here leads to the definition of derived functor. Section 10 constructs homotopy pushouts and homotopy pullbacks in an arbitrary model category in terms of derived functors. Finally, §11 contains some concluding remarks, sketches some applications of homotopical algebra, and mentions a way in which the theory has developed since Quillen.

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## 2. Categories

In this section we review some basic ideas and constructions from category theory; for more details see [17]. The reader might want to skip this section on first reading and return to it as needed.

*2.1. Categories.* We will take for granted the notions of *category*, *subcategory*, *functor* and *natural transformation* [17, I]. If  $\mathbf{C}$  is a category and  $X$  and  $Y$  are objects of  $\mathbf{C}$ , we will assume that the morphisms  $f : X \rightarrow Y$  in  $\mathbf{C}$  form a *set*  $\text{Hom}_{\mathbf{C}}(X, Y)$ , rather than a class, a collection, or something larger. These morphisms are also called *maps* or *arrows* in  $\mathbf{C}$  from  $X$  to  $Y$ . Some categories that come up in this paper are:

- (i) the category **Set** whose objects are sets and whose morphisms are functions from one set to another,
- (ii) the category **Top** whose objects are topological spaces and whose morphisms are continuous maps,
- (iii) the category  $\mathbf{Mod}_R$  whose objects are left  $R$ -modules (where  $R$  is an associative ring with unit) and whose morphisms are  $R$ -module homomorphisms.

*2.2. Natural equivalences.* Suppose that  $F, F' : \mathbf{C} \rightarrow \mathbf{D}$  are two functors, and that  $t$  is a natural transformation from  $F$  to  $F'$ . The transformation  $t$  is called a *natural equivalence* [17, p. 16] if the morphism  $t_X : F(X) \rightarrow F'(X)$  is an isomorphism in  $\mathbf{D}$  for each object  $X$  of  $\mathbf{C}$ . The functor  $F$  is said to be an *equivalence of categories* if there exists a functor  $G : \mathbf{D} \rightarrow \mathbf{C}$  such that the composites  $FG$  and  $GF$  are related to the appropriate identity functors by natural equivalences [17, p. 90].

*2.3. Full and faithful.* A functor  $F : \mathbf{C} \rightarrow \mathbf{D}$  is said to be *full* (resp. *faithful*) if for each pair  $(X, Y)$  of objects of  $\mathbf{C}$  the map

$$\text{Hom}_{\mathbf{C}}(X, Y) \rightarrow \text{Hom}_{\mathbf{D}}(F(X), F(Y))$$

It's professional to thank anyone who helped substantially with the paper.

It's very helpful to readers if you use guiding text to tell them what you're doing.

These subheads will help readers find these definitions later.

