Lovász local lemma

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Paper Presentation

04/12/2013

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- Lovász local lemma (Symmetric Case)
- Lovász local lemma (Asymmetric Case)
- Example of application of Lovász local lemma.
- Algorithmic Lovász local lemma.
- Applications and proving a theorem using the Lovász local lemma.

Lemma: Symmetric Case

- Let $A_{1,}$ $A_{2,}$ ……., A_n be events in an arbitrary probability space. Suppose each event A_i is independent of all the other events, except at most d of them, and that $Pr[A_i] \le p$ for all A_i , and if certain conditions on p and d are met, then:

$$\Pr\left[\bigwedge_{i=1}^{n} \bar{A}_i\right] > 0$$

Conditions on p and d

- **Lemma I** (Lovász and Erdős 1973; published 1975)

$$4pd \le 1$$

- **Lemma II** (Lovász 1977; published by Joel Spencer)

$$ep(d+1) \le 1$$

Lemma III (Shearer 1985)

$$\begin{cases} p < \frac{(d-1)^{d-1}}{d^d} & d > 1 \\ p = \frac{1}{2} & d = 1 \end{cases}$$

Lemma: Asymmetric (general) Case
(events can have different probability bounds)

- Let $A_{1,} A_{2,}$ ...., A_n be events in an arbitrary probability space. A directed digraph D=(V,E) on the set of vertices V = {1, 2, ..., n} is called a dependency digraph for the events $A_{1,} A_{2,}$ .., A_n if for each 1≤ i ≤n, the event A_i is mutually independent of all the events $\{A_j : (i,j) \notin E\}$. Suppose D=(V,E) is a dependency digraph for the above events, and suppose there are real numbers $x_{1,} x_{2,}$ ....., x_n such that $0 \le x_i < 1$ and

$$\Pr\,[\bigwedge_{i=1}^{n} \bar{A}_i] \le x_i \prod_{(i,j)\in E} (1 - x_j)$$

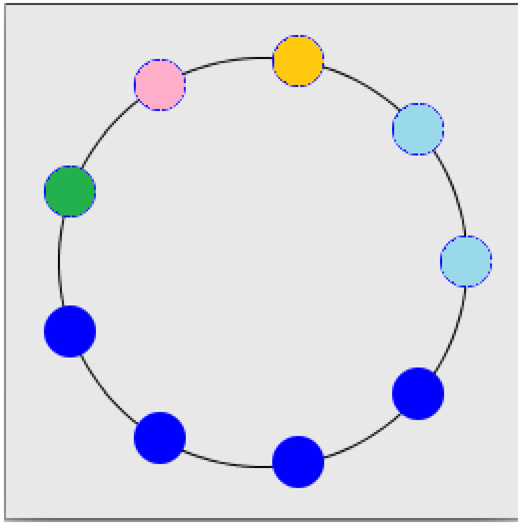
for all 1≤ i ≤n, then

$$\Pr\,[\bigwedge_{i=1}^{n} \bar{A}_i] \ge \prod_{i=1}^{n} (1 - x_i)$$

Example

- Suppose 11n points are placed on a circle and colored with n different colors.
- Each color is applied to exactly 11 points.
- In any such coloring, there must be subset of n points each colored a different color and not containing any adjacent points.
- Let's pick a point of each color randomly (p=1/11).
- Event we want to avoid corresponds to picking a pair (a,b) of adjacent points (p=1/121).
- 21 pairs include same color as a,21 pairs include same color as b, if disjoint, (a,b) is dependent on 42 pairs. d=42.

$$ep(d+1) \approx 0.966 < 1.$$



Algorithmic Lovász local lemma

Algorithm

Given a set of bad events A that we wish to avoid, and that are determined by a collection of mutually independent random variables P:

1. For every random variable P, make an assignment.
2. While there exists a satisfied event A:
    1. Pick a satisfied event A.
    2. For all P that determine A, give a new assignment to P.
3. Return the final assignment

Applications

- Probabilistic method proofs.
- Random Graphs.

Theorem:

Let H=(V,E) be a hypergraph in which every edge has at least k elements, and suppose that each edge of H intersects at most d other edges. If $e(d+1) \leq 2^{k-1}$ then H is two-colorable.

Proof

- Color each vertex v of H randomly and independently, either blue or red with equal probability. For each edge f of H, let A_f be the event that f is monochromatic.

$$\Pr\left[A_f\right] \geq 2/2^{|f|} \leq 1/2^{\mathrm{k}-1}$$

- Therefore if ep(d+1) ≤1 (Lovász local lemma) then $e(d+1) \leq 2^{k-1}$

and H is two-colorable.

References

- Alon Noga, Spencer Joel H. “5. The Local Lemma”, *The Probabilistic Method*. Hoboken, NJ: John Wiley & Sons, Inc. 2008. 3rd ed.
- http://en.wikipedia.org:Lovász local lemma
- http://en.wikipedia.org:Algorithmic Lovász local lemma