COUNTING & SAMPLING

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MAY 3, 2013

OVERVIEW

Definitions

- Plane Tree
- Binary Tree
- Dyck Path

Counting Dyck Paths

Bijection

Sampling Dyck Paths

PLANE TREE

A rooted tree in which the order of the children matters

T_n: set of plane trees with *n* edges.

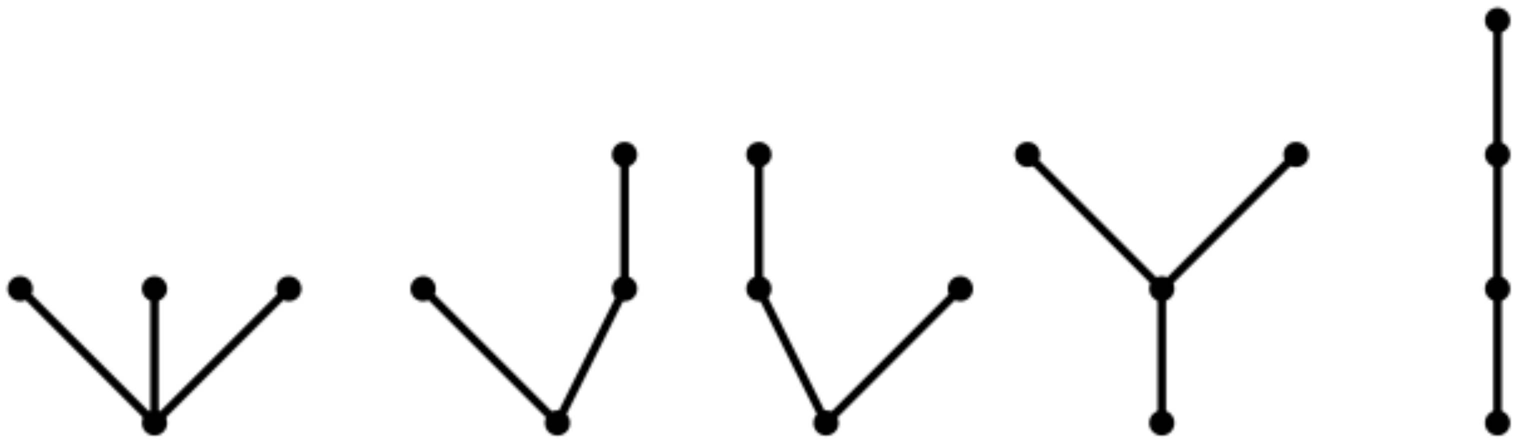


Figure 1: The set $\mathcal{T}_3$ of plane trees.

BINARY TREE

- A plane tree where each vertex is either:
  - Node: has 2 children
  - Leaf: has 0 children
- B_n: set of binary trees with n nodes

Figure 2: The set $\mathcal{B}_3$ of binary trees.

DYCK PATH

- **A lattice path made of up steps and down steps**
  - **starting and ending at level 0, and**
  - **remaining non-negative**
- **D_n: set of Dyck paths with $2n$ steps**

Figure 3: The set $\mathcal{D}_3$ of Dyck paths.

CLAIM

We will prove that

$$|\mathcal{T}_n| = |\mathcal{B}_n| = |\mathcal{D}_n| = \frac{1}{n+1}\binom{2n}{n}$$

COUNTING DYCK PATHS

Let $P_n^{(k)}$ be the set of lattice paths of length $2n$* that start at 0 and end at k.

Then $P_n^{(0)}$ = ?

$$\mathcal{P}_n^{(0)} = \binom{2n}{n}$$

* If *k* even. If *k* odd, the length is 2*n*+1

COUNTING DYCK PATHS

Let us define

$$\overline{\mathcal{D}}_n \equiv \mathcal{P}_n^{(0)} \setminus \mathcal{D}_n$$

Claim:

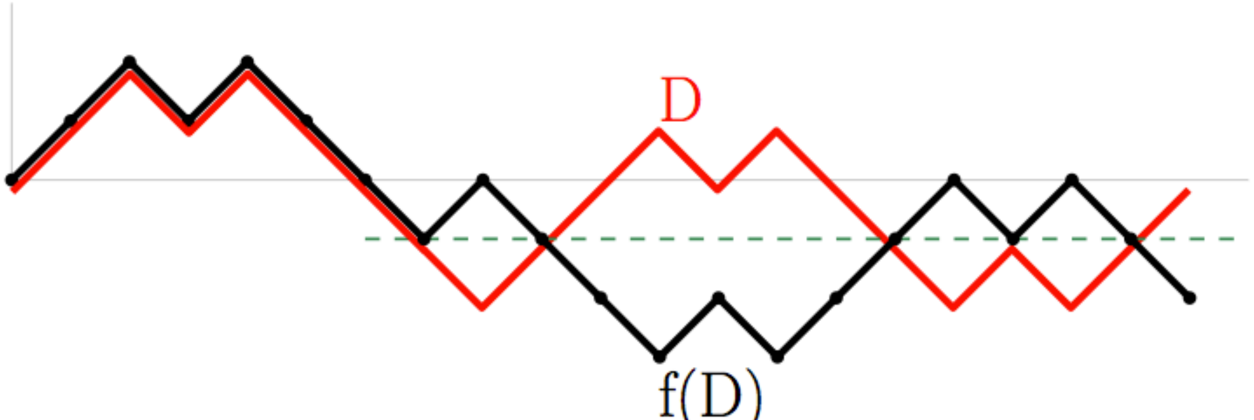
$$|\overline{\mathcal{D}}_n| = \binom{2n}{n-1}$$

COUNTING DYCK PATHS

Consider $P_n^{(-2)}$, the set of paths that start at 0 and end at -2

$$|\mathcal{P}_n^{(-2)}| = \binom{2n}{n-1}$$

Then we have a bijection f from $P_n^{(-2)}$ to $\overline{\mathcal{D}}_n$



COUNTING DYCK PATHS

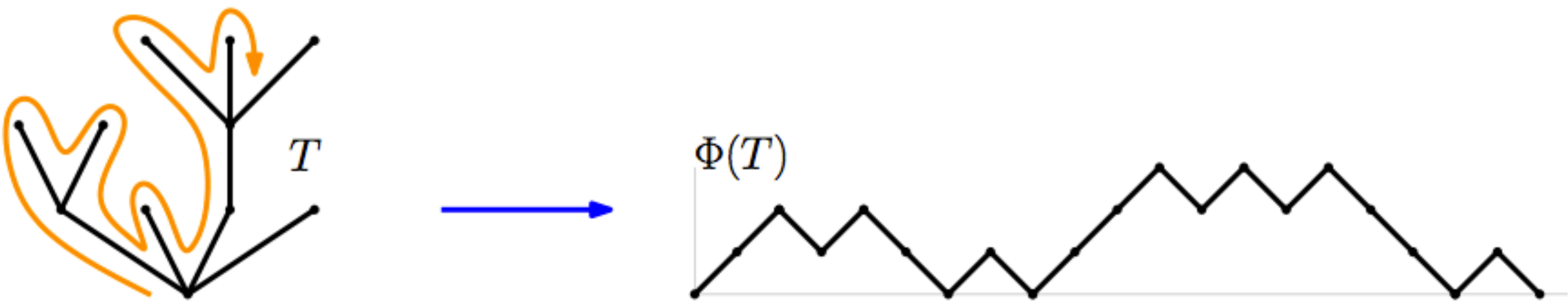
$$|\mathcal{D}_n| = |\mathcal{P}_n^{(0)}| - |\overline{\mathcal{D}}_n| = \binom{2n}{n} - \binom{2n}{n-1}$$

$$= \frac{(2n)!}{n!n!} - \frac{(2n)!}{(n+1)!(n-1)!}$$

$$= \frac{(2n)!}{n+1!n!} = \frac{1}{n+1}\binom{2n}{n}$$

BIJECTION

We can find a bijection between the Dyck paths and the plane trees:



$$|\mathcal{T}_n| = |\mathcal{D}_n| = \frac{1}{n+1}\binom{2n}{n}$$

BIJECTION

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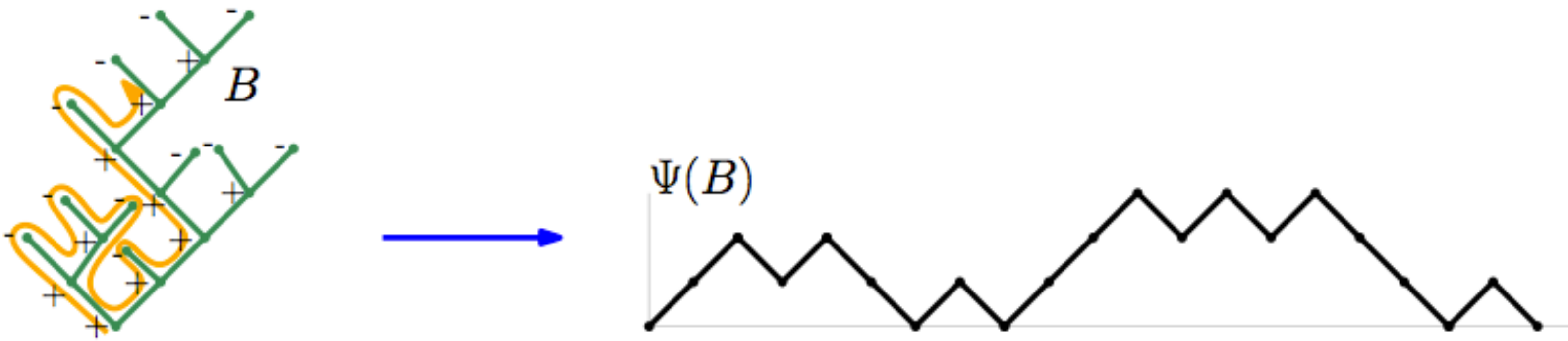


$$|\mathcal{B}_n| = |\mathcal{D}_n| = \frac{1}{n+1}\binom{2n}{n}$$

SAMPLING

Consider $P_n^{(-1)}$, the set of paths of length 2*n*+1 that start at 0 and end at -1

We define a random sampling procedure for $P_n^{(-1)}$ then show the construction of a Dyck path from the output.

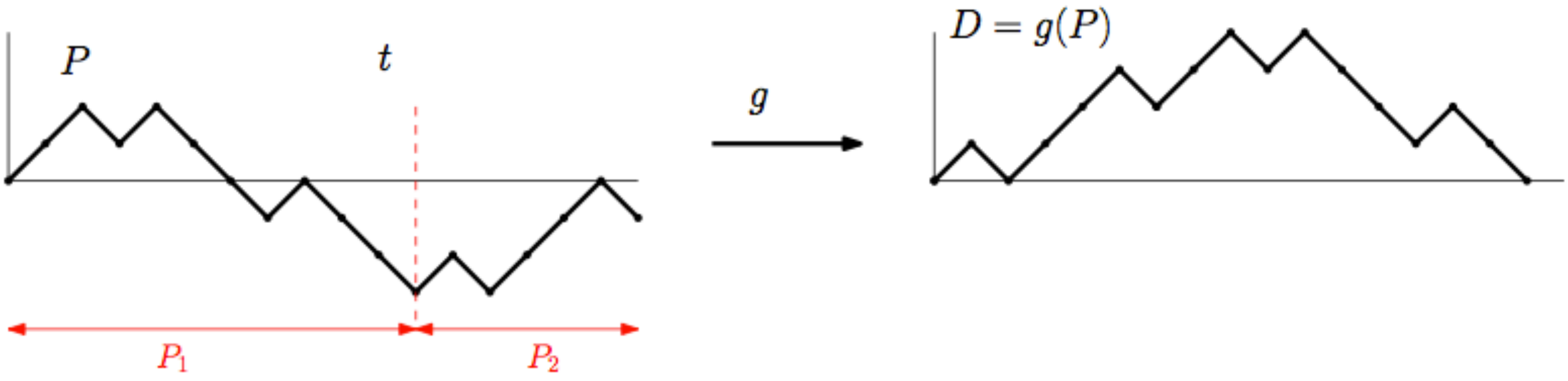
To generate an element of $P_n^{(-1)}$ at random:

Input an integer n.
- Initialize an array V of length $2n+1$ with value 1 in the first n entries and value -1 in the remaining $n+1$ entries.
- For $i = 1$ to $2n+1$ do
  - Choose an integer r uniformly at random in $\{i, i+1, \ldots, 2n+1\}$.
  - Swap the values at position i and r in V.

Output the array V.

SAMPLING

The mapping g of a Dyck path from an element of $P_n^{(-1)}$ is as follows:



Therefore we have a random sampling of D_n

SUMMARY

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Bijections

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SOURCES

18.310 *Counting, Coding, and Sampling* Notes, Spring 2012