Generating Functions Part 2

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April 26, 2013

Main Topics

- ❖Exponential Generating function
- ❖Operations on exponential Generating Functions
- ❖Enumerating Trees using Generating functions

Reminder: Catalan Sequence

- The Catalan sequence 1, 1, 2, 5, 14, 42, 132, …

has the generating function

$$\boldsymbol{Cat(x) = \frac{1-\sqrt{1-4x}}{2x}}$$

With coefficients satisfying:

$$c_{n+1} = c_0 c_n + c_1 c_{n-1} + \cdots + c_n c_0$$

$$\mathrm{c_n} = \frac{1}{\mathrm{n}+1}\binom{2\mathrm{n}}{\mathrm{n}}$$

(Derivation of coefficients in “Lectures on Generating Functions” by S. K. Lando, chapter 2)

Enumeration of Binary Trees

- Enumerate them using

$$B(x) = b_0 + b_1 x + b_2 x^2 + \ldots = \sum_{k=0}^{\infty} b_k x^k$$

- It turns out

$$b_{n+1} = b_0 b_n + b_1 b_{n-1} + \cdots + b_n b_0$$

- think of tree with n+1 nodes as root+2 sub-trees
- Sub-tree 1 has k nodes
- Sub-tree 2 has n-k nodes

Exponential Generating Functions

- Are of the form $F(x) = \sum_{n=0}^{\infty} f(n) \frac{x^n}{n!}$
- Are used to enumerate f(n); the number of structures on n-element set.
- Difference between exponential GF and ordinary GF? Exponential GF describe marked objects, while ordinary GF describe unmarked objects (example: unmarked binary tree in previous slide)

Exponential Generating Functions of Trivial Structures

- Structure of :

“being a set”: f(n) = 1 $F(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$

“1-element set”: f(n)=1 if n=1, 0 otherwise; F(x)=x

“empty set”: f(n)=1 if n=0, 0 otherwise; F(x)=1

“non-empty set”: F(x)= $e^x - 1$ (based on addition principal which is intuitive)

Operations on Exponential Generating Functions

- Addition:

  F(x) = G(x) + H(x)

- Multiplication:

Definition:

Let g and h be two structures on finite sets. A structure g * h on a set A consists of:

(i) An ordered partition of A into disjoint subsets $A = A_1 \bigcup A_2$

(ii) A g-structure on A_1

(iii) An h-structure on A_2

g and h are independent

$$F(x) = G(x)\, H(x)$$

(generalizes to product of 3 or more GF)

Multiplication Principle Proof

- f(n) is number of g*h structures on n-element set A.
- We partition A into A_1 and A_2, with g structure on A_1 and h structure on A_2.
- $|A_1|=k$, $\binom{n}{k}$ choices of the partition, g(k) choices of the g-structure
- e, h(n-k) choices of h-structure.

$$f(n) = \sum_{k=0}^{n} \binom{n}{k} g(k)h(n-k) = \sum_{k=0}^{n} n! \frac{g(k)}{k!} \frac{h(n-k)}{(n-k)!}.$$

Hence

$$f(n)/n! = \sum_{k=0}^{n} \frac{g(k)}{k!} \frac{h(n-k)}{(n-k)!}.$$

- The coefficient implies F(x) = G(x) H(x).

Multiplication Principle Examples

- Number of subsets of n-element set: $A = A_1 \bigcup A_2$
- A_1 is the subset and A_2 is its complement. Both structures g and h on A_1 and A_2 are the trivial structures of being a set.
- $F(x) = H(x)G(x) = e^x e^x = e^{2x} = F(x) = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!}$
- Hence $f(n)=2^n$

Labeled Binary Trees

- Let us count t(n) the number of labeled binary trees on n-element set of vertices.
- Empty tree counts.
- For non empty tree, we partition vertices into 3 partitions: A_1 (root), A_2 (left sub-tree), A_3 (right sub-tree). The structure on A_1 is the trivial 1-element set structure. The structures on A_2 and A_3 are the structures of a binary tree.
- Let T(x) be the exponential generating function for labeled binary trees, then $T(x)=1+xT(x)^2$.

$$T(x) = \frac{1-\sqrt{1-4x}}{2x} = \sum_n C_n x^n = \sum_n n! C_n \frac{x^n}{n!},$$

- $t(n)=n!\,C_n$ where C_n is the nth Catalan number.

The functional Composition Principle

Definition:

Let g and h be two structures on finite sets. Assume H(0)=0 (no h-structure on empty set n=0). A composite structure $g \circ h$ on a set A consists of:

(i) A partition of A into disjoint subsets

(ii) On each block we independently choose an h-structure.

(iii) g-structure on the set of blocks.

g and h must be independent

$$F(x)=G(H(x))$$

Composition Principle Proof

- On a set A we have a number of blocks, let's say k. The partition of blocks is ordered and there is an h-structure on each block, and a g-structure on the set of blocks. There are g(k) ways to choose the g-structure, and since we choose an h-structure for each block, then we need to multiply H(x) by itself k times. The GF is therefore

$$g(k)H(x)^k$$

Composition Principle Proof (continued)

$g(k)H(x)^k$

- There are k! structures of each composite $g \circ h$ structure with k blocks, so the GF for one such structure is

$$(g(k)H(x)^k)/k!$$

Therefore the generating function for composite structures is

$$F(x) = \sum_{k=0}^{\infty} g(k) \frac{H(x)^k}{k!} = G(H(x))$$

Rooted Labeled Trees

- Let t(n) be the number of unordered labeled rooted trees on n-element set, the corresponding exponential GF is therefore:

$$T(x) = \sum_{n=0}^{\infty} t(n) \frac{x^n}{n!}$$

- Note: to ensure T(0) = 0 (needed for composition principle), we don't count the empty tree.

Rooted Labeled Trees

- Let us view a tree as a product structure, with one vertex chosen as the root, and the rest are sub-trees with the roots being children of the main root. Each sub-tree is a block, and the vertices of that block have the structure of rooted tree (H(x)=T(x)), and the structure of the blocks is the trivial structure of being a set (G(x)= e^x).
- By the composition principle, the exponential GF describing the sub-trees is $e^{T(x)}$
- The product of structures describing the root and sub-trees is (one-element set)* $e^{T(x)}$
- Hence $T(x) = xe^{T(x)}$

Rooted Labeled Trees

$$T(x) = xe^{T(x)}$$

Using principles of Lagrange Inversion we get

$$t(n) = n^{n-1}$$

References

- Lando, S. K."2. Generating Functions for Well-known Sequences”, *Lectures on Generating Functions*. Providence, RI: American Mathematical Society, 2003. STML v. 23.
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- http://ulcar.uml.edu/~iag/CS/Catalan.html