Finding a Maximum Matching in Non-Bipartite Graphs

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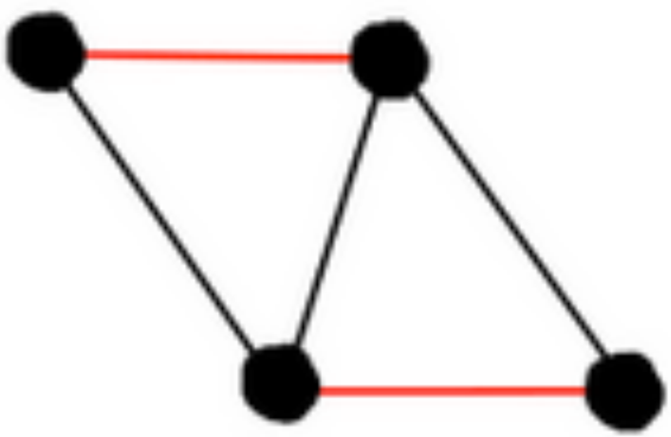
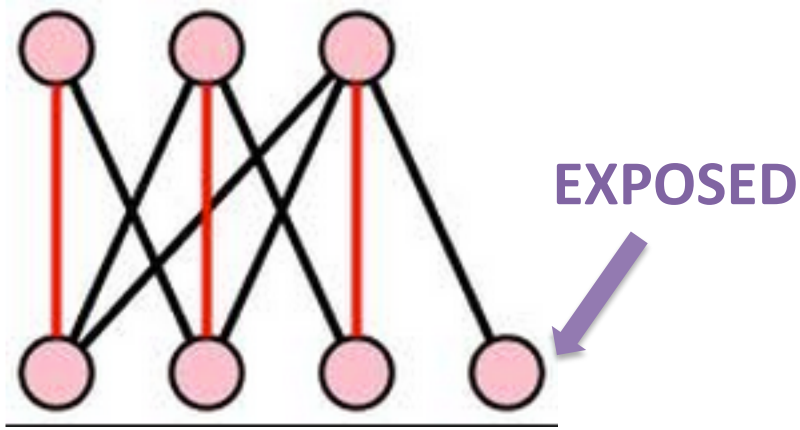
Our Goal

- Make an algorithm to find the **largest cardinality matching** (most sets of partners) in ANY graph.
- *Method: Generalize the maximum matching algorithm for bipartite graphs*

Review (1/2)

- **Matching:** a set of edges without common vertices

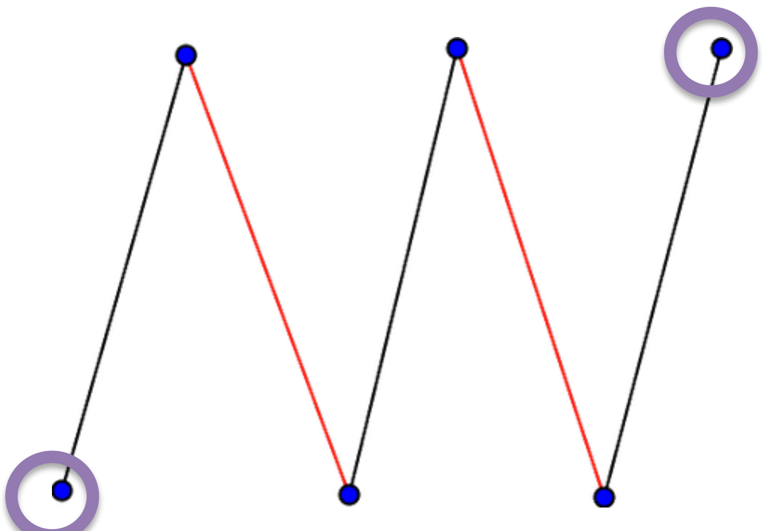




  - **Maximum Cardinality Matching:** largest # of edges

Review (2/2)

- An **alternating path** with respect to **M** alternates between edges in **M** and in **E-M**



- An **augmenting path** with respect to **M** is an alternating path with **first and last vertices exposed**

Bipartite Graph Algorithm

1 – Start with any matching **M** (let's say M = {})

2 – As long as there exists an **augmenting path** with respect to **M**:

  3 – Find augmenting path **P** with respect to **M**

  4 – Augment **M** along **P**: **M' = M Δ P**

  5 – Replace **M** with the new **M'**

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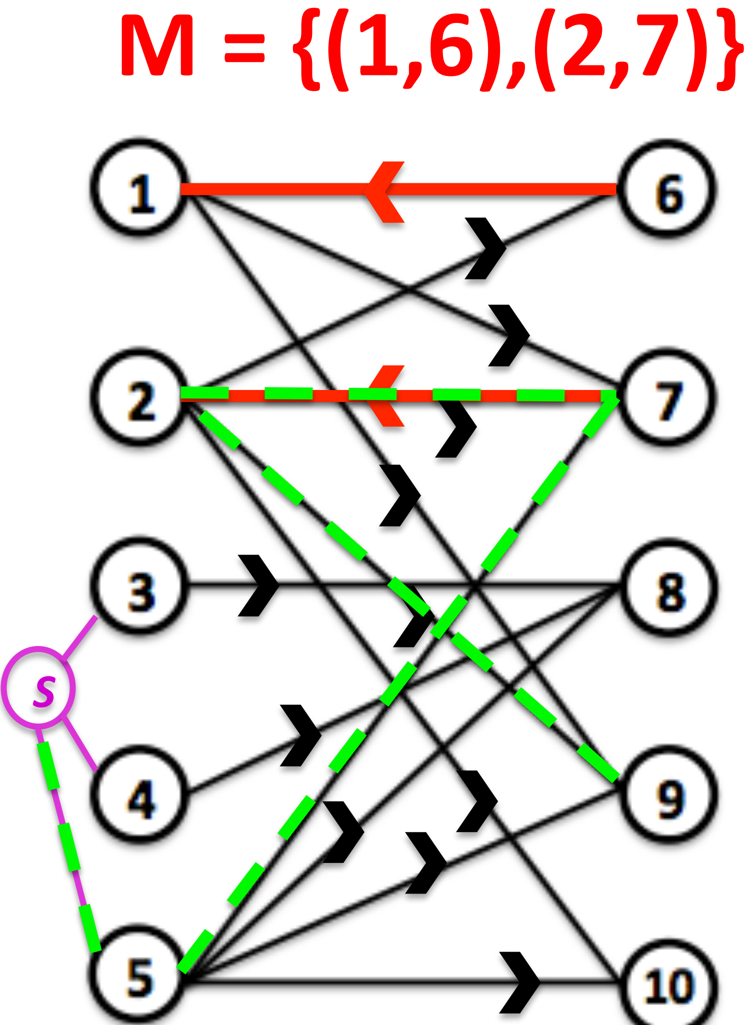
HOW???

Bipartite: Finding an Augmenting Path



1 – Direct all edges in the matching from B to A, and all edges not in the matching from A to B

2 – Create a node *s* that connects to all exposed vertices in set A

3 – Do a Breadth First Search to find an exposed vertex in set B from node *s*

Same

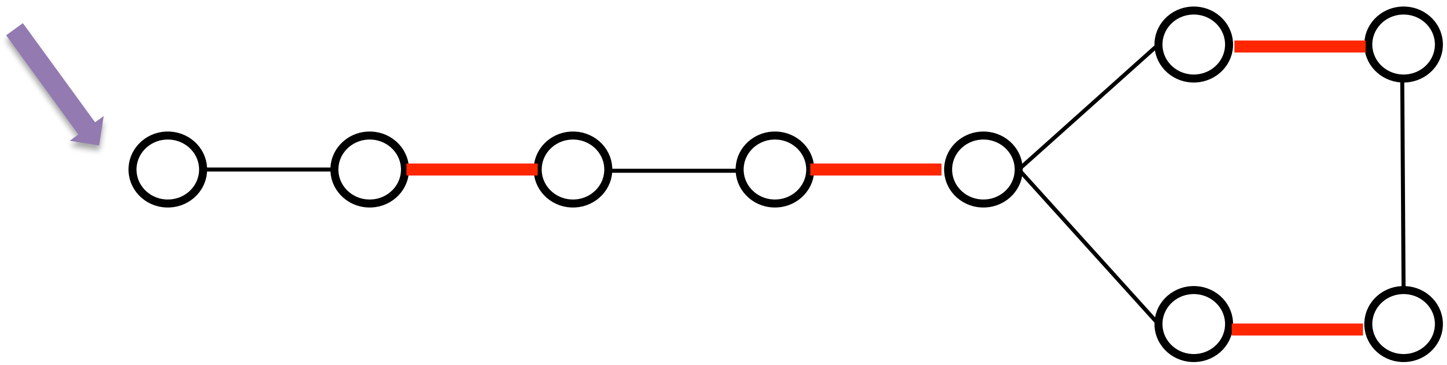
- A matching is maximum **if and only if** there are no augmenting paths
- *General Plan:* keep looking for augmenting paths to expand the matching

Different

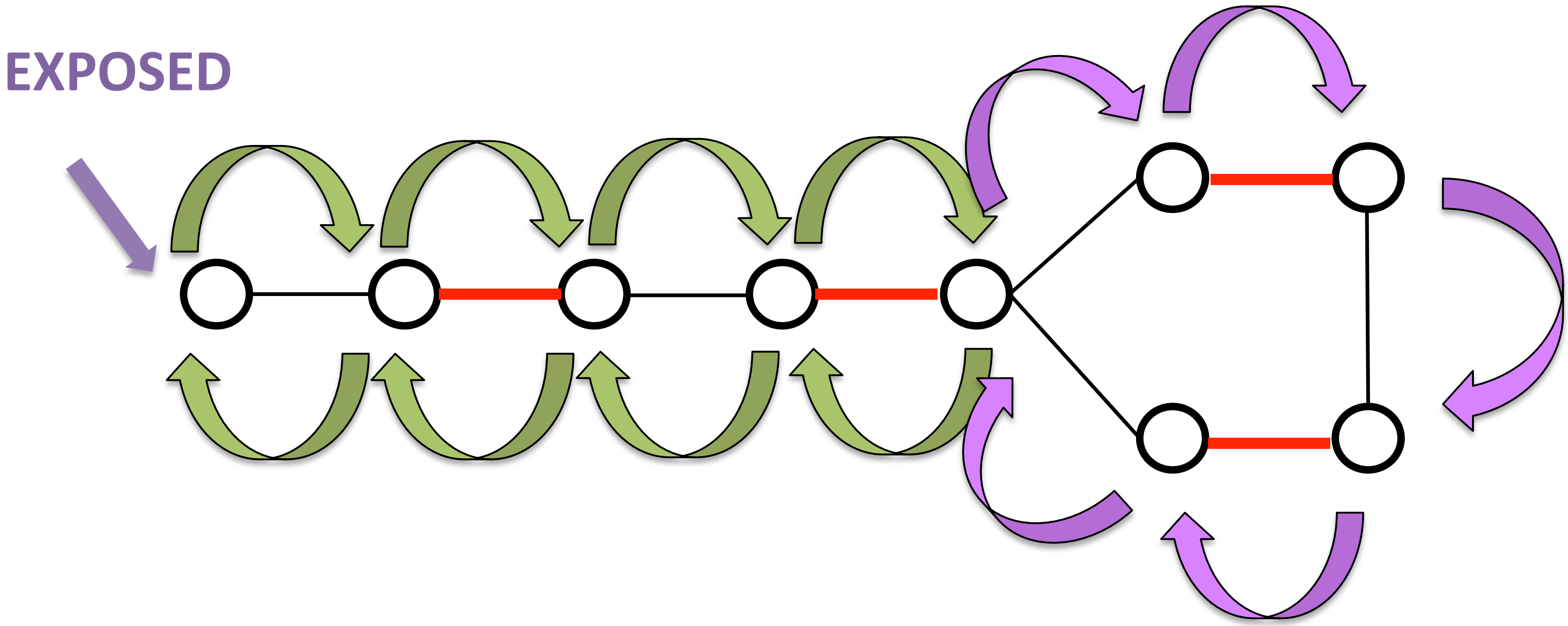
- We can't add direction to the edges to find augmenting paths
- We might find "fake" augmenting paths, called FLOWERS

Flowers: Stems & Blossoms

EXPOSED



Flowers: Stems & Blossoms



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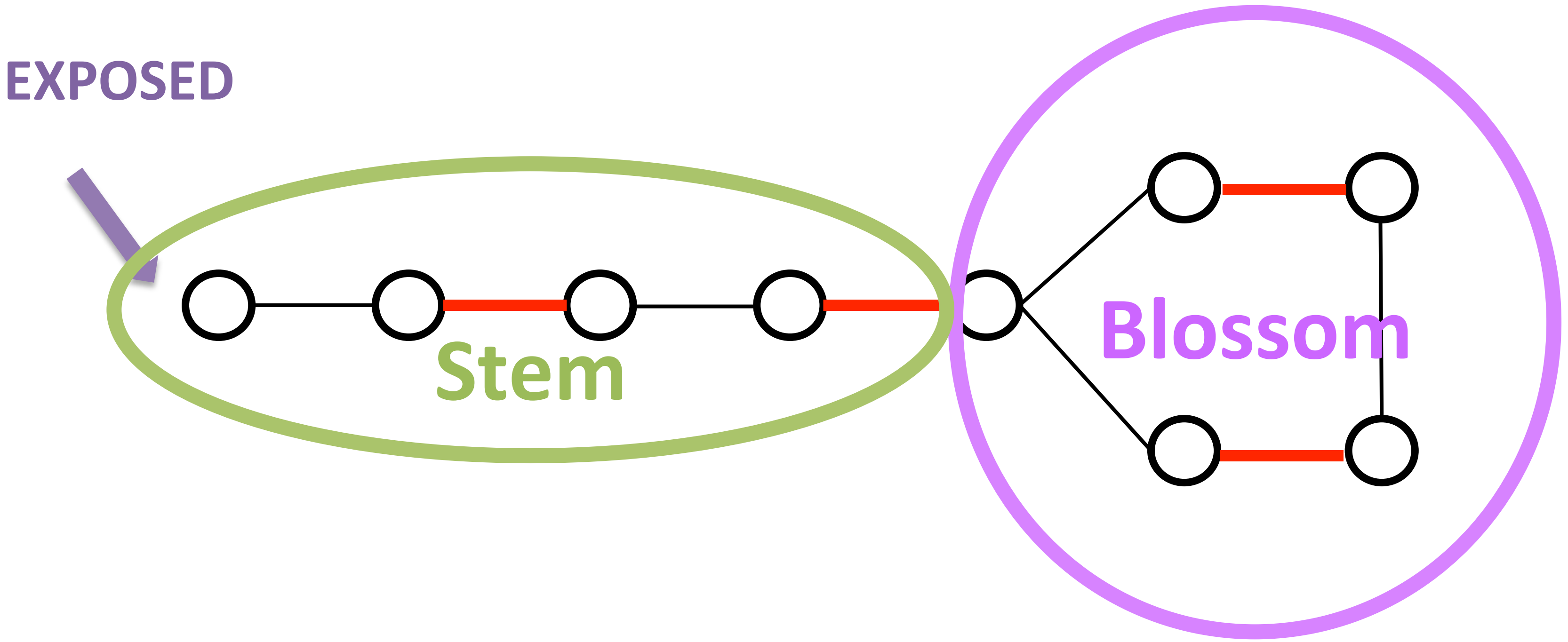
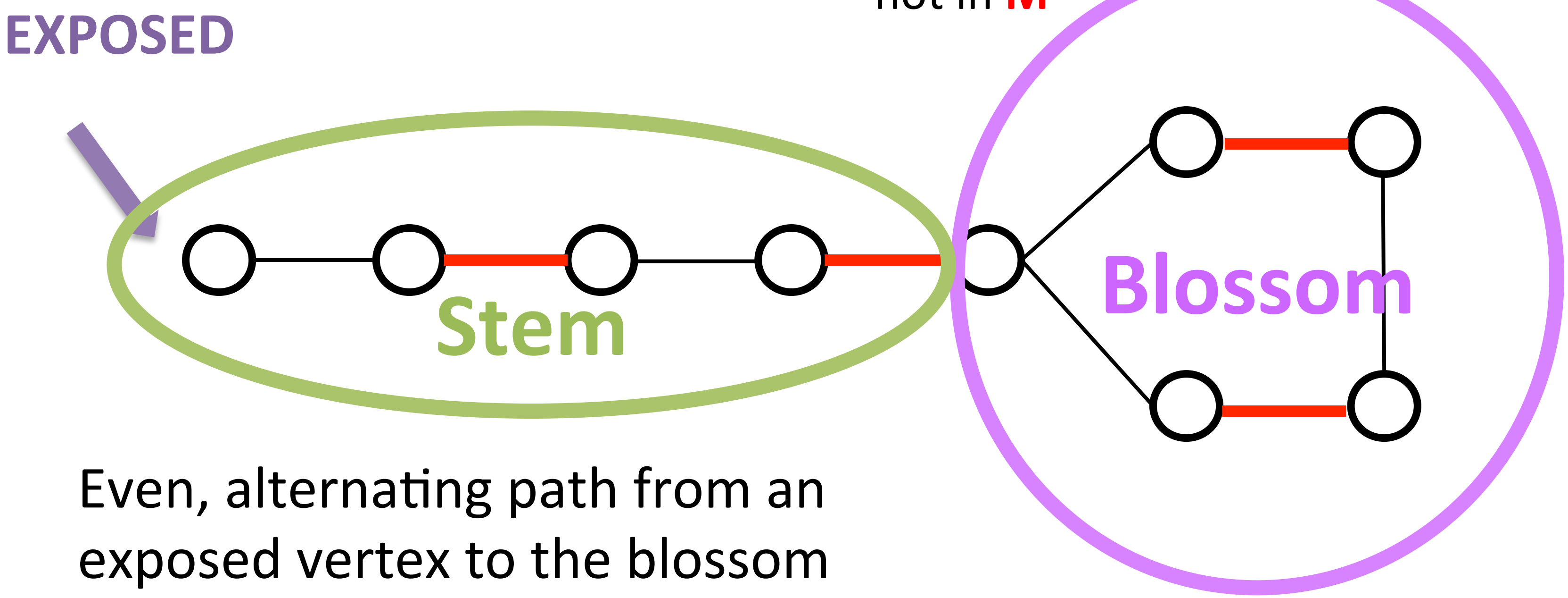


Flowers: Stems & Blossoms

Odd, alternating cycle with two edges adjacent to the stem and not in **M**



Even, alternating path from an exposed vertex to the blossom

Revised Algorithm

1 – Start with any matching **M** (let's say M = {})

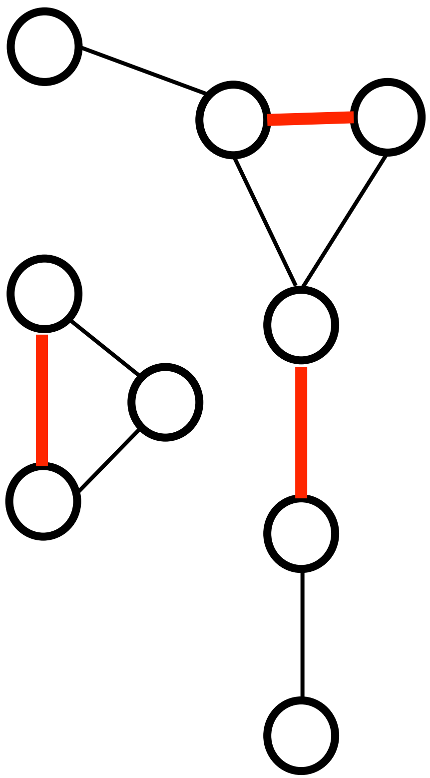
2 – Find a **flower**, **augmenting path** or **neither**:

   3 – If **neither**: We're done!

   4 – If **augmenting path**: augment to **M' = M Δ P**

   5 – If **flower**: find a larger matching or decide that **M** is maximum…

<u>Theorem</u>: M is maximum in G if and only if M-B is maximum in G-B



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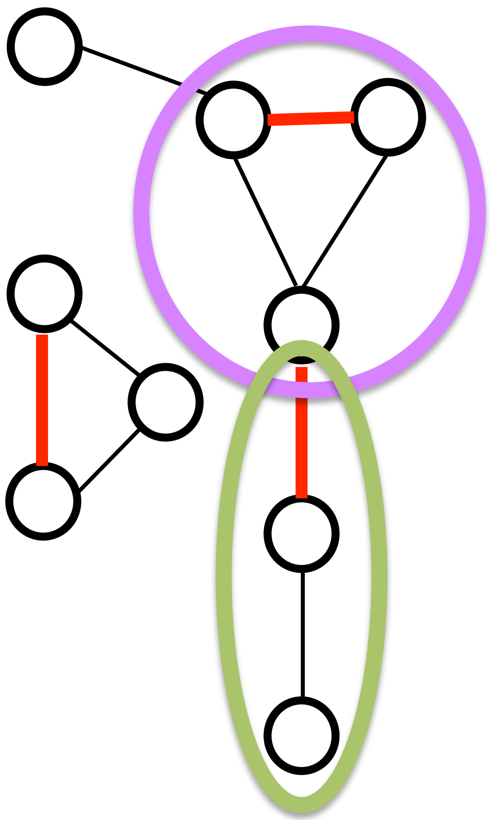

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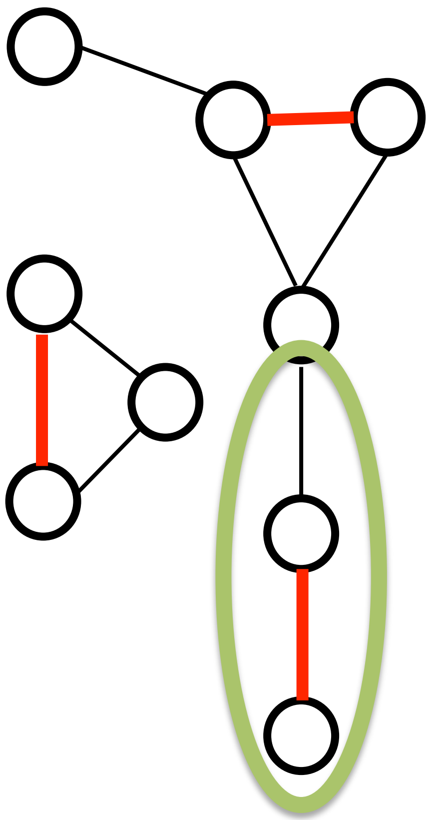

1- Flip the stem

- -The matching is still the same size
- -The blossom has an exposed vertex

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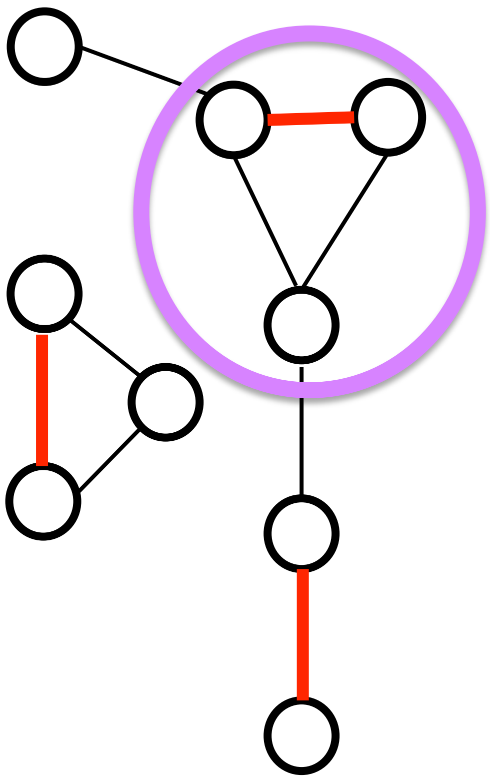

1- Flip the stem
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2- Shrink the blossom to one single vertex
- -All vertices in B combine into ß
- -Edges into any vertex in B go into ß

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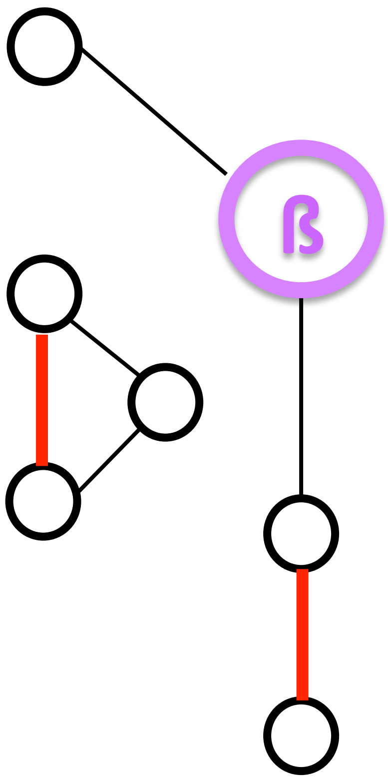

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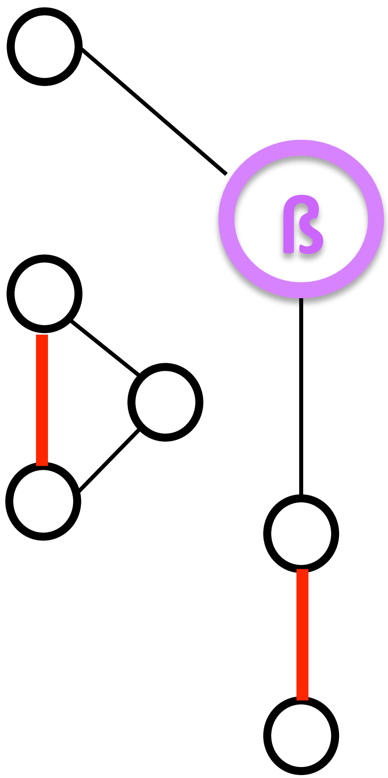

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3- The matching M-B in graph G-B is |B-1|/2 smaller than matching M in G

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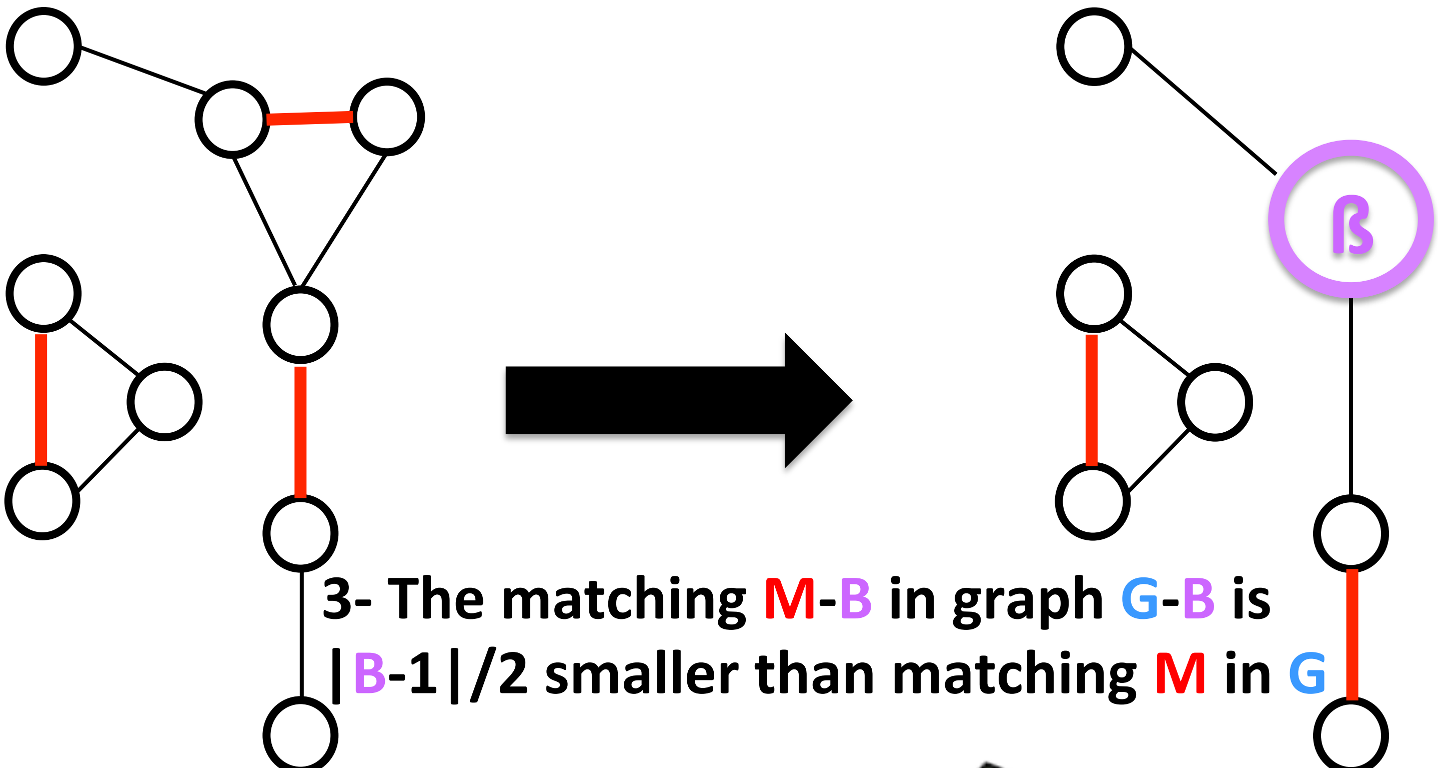

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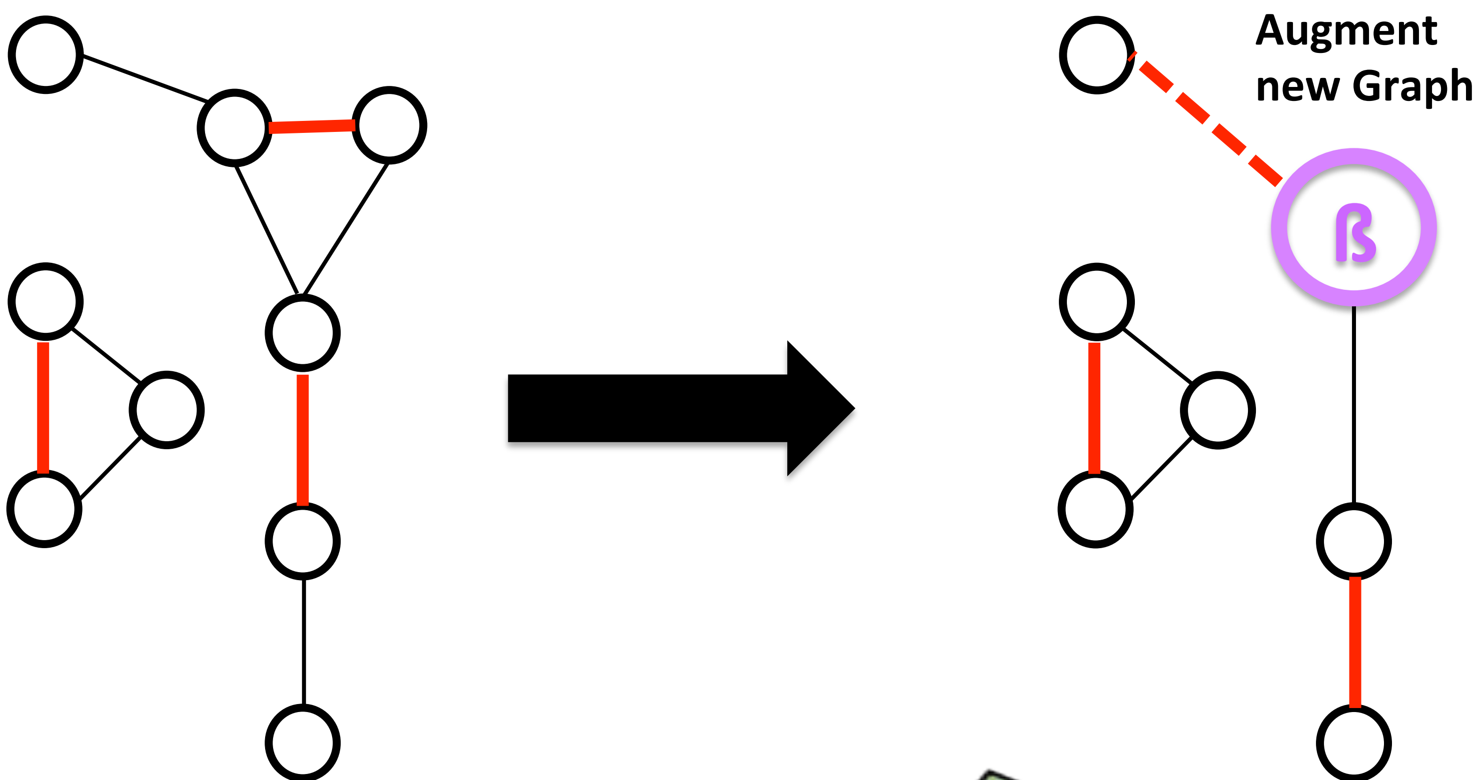
Increasing a Matching from a Flower



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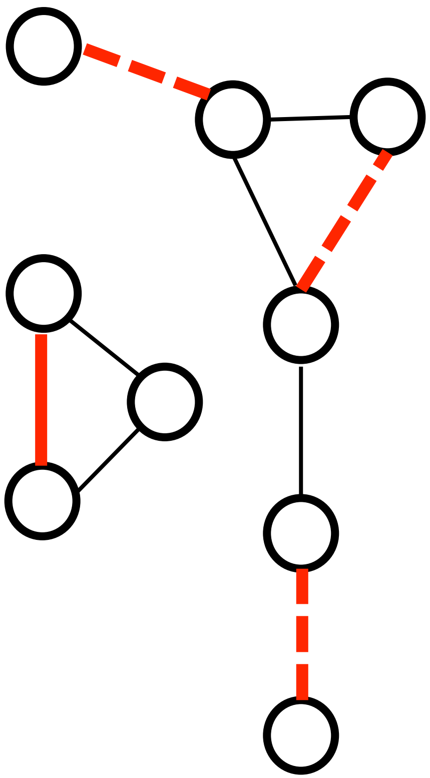
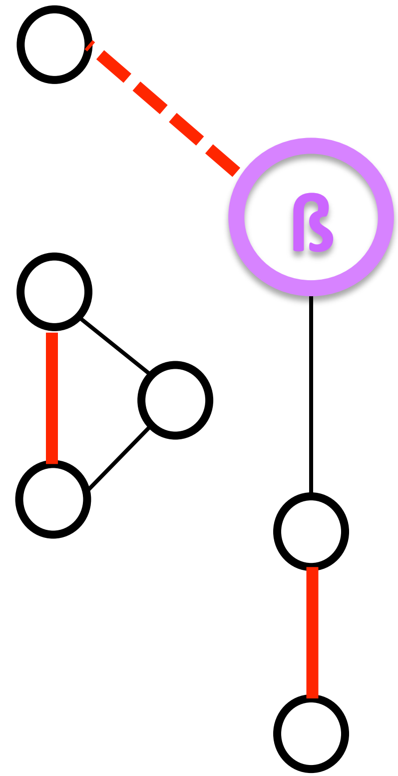


Add back in the blossom with extra edges from the new matching



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- 5 – If **flower**: (recursively…)
  - a. Flip the stem
  - b. Shrink the blossom to a single vertex
  - c. Increase M or decide that it is Maximum

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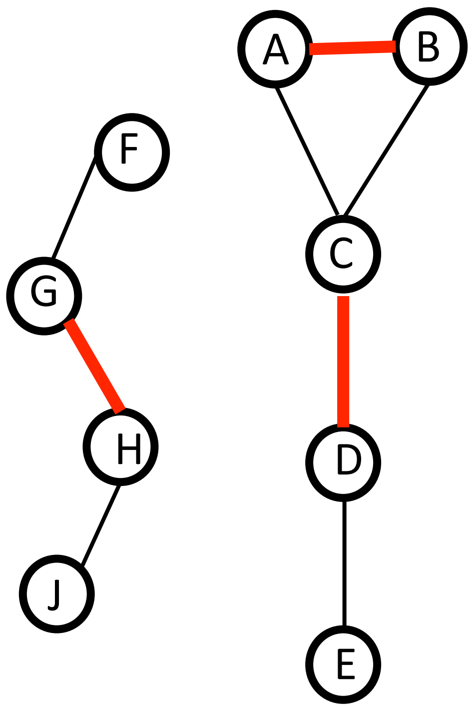
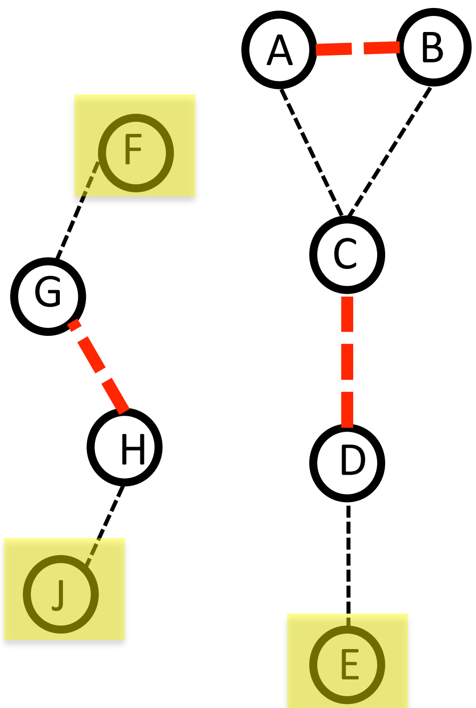
HOW???

Finding Flowers and Augmenting Paths



Creating Alternating Forests

1. Label all exposed vertices as SQUARE, start a new tree in our alternating forest for each one



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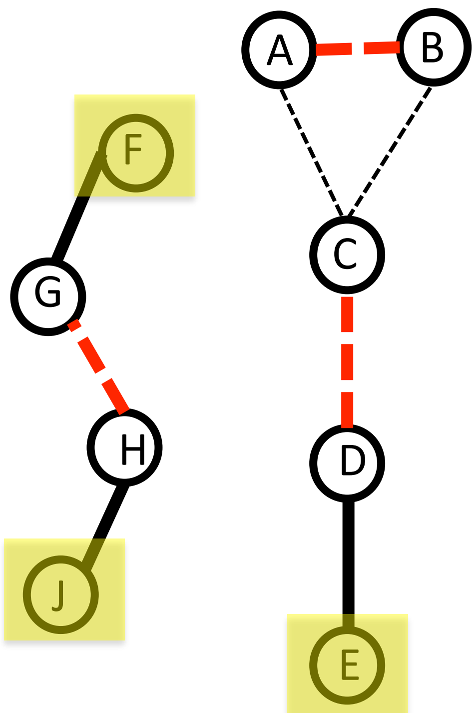

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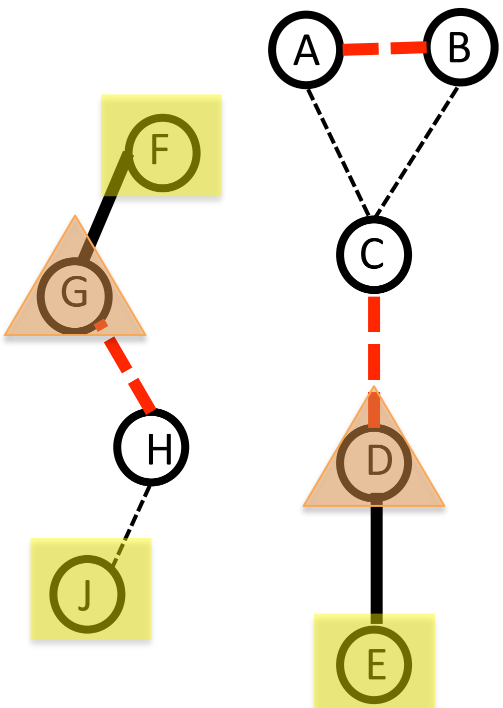

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2. Add edges (u, v) from u in the forest to v
3. If an edge (*u*, *v*) has *v* unlabelled, label it TRIANGLE. Label its "mate" (across an edge in the matching) as a SQUARE

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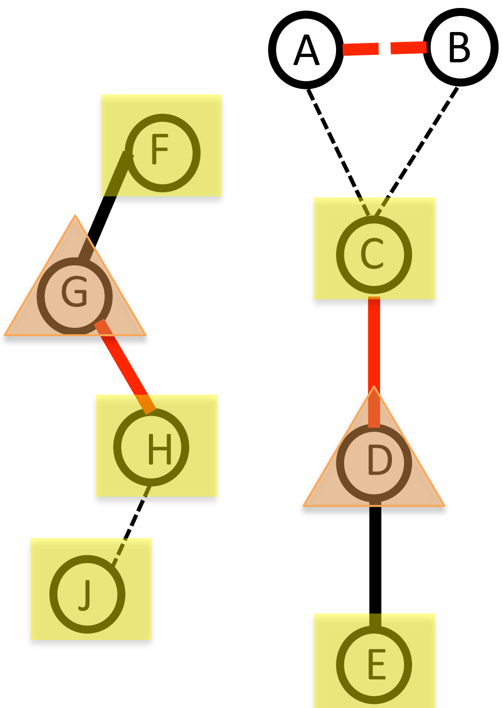

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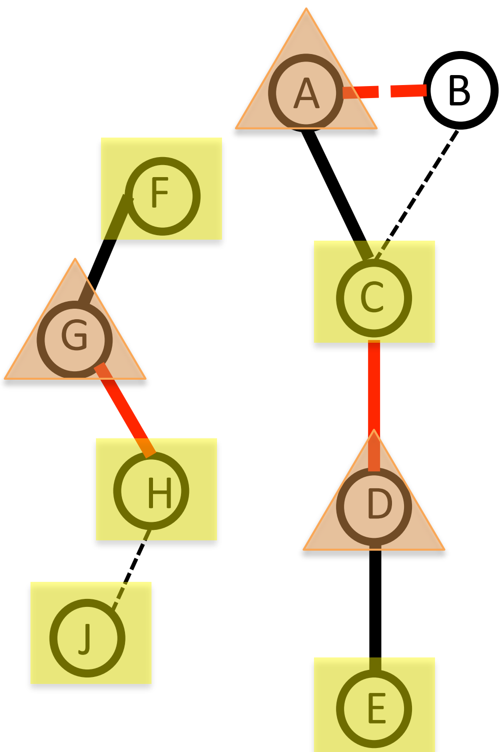

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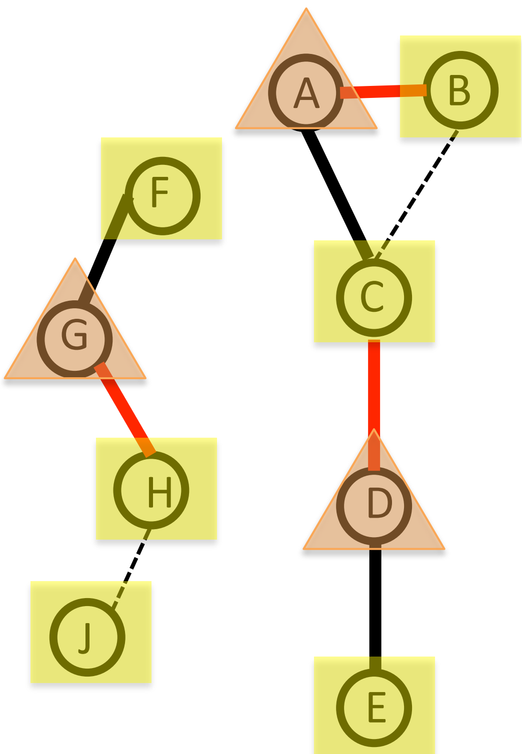

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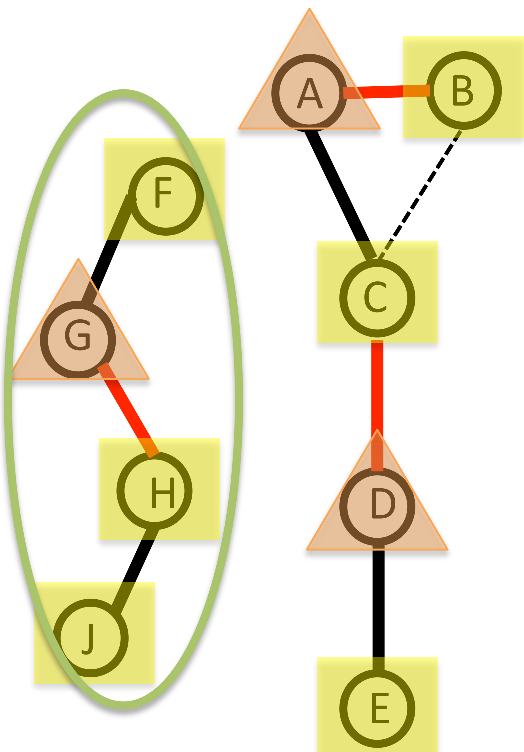

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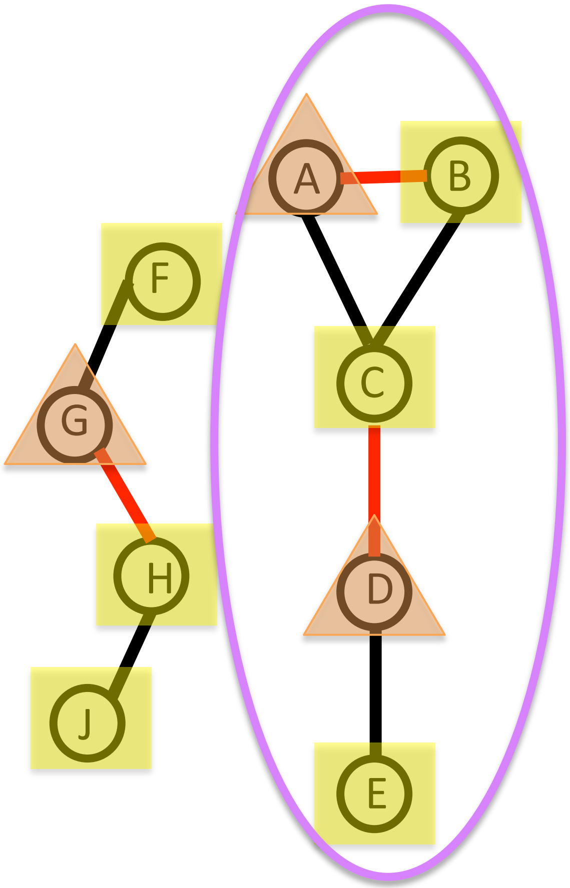

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5. If an edge (*u, v*) already has *v* labelled SQUARE and *v* belongs to the same alternating tree, then we have found a flower

Questions?