The Lucky Tickets Problem

Noor Doukmak

February 19, 2013

In Russia…

834012

Bus ticket

Lucky Tickets…

230113

942807

100100

123411

740623

004013

What makes them lucky?

Lucky Tickets Problem

Alexandre Kirillov in the early 70's

- How many lucky tickets exist?

- We would like to count the number of tickets

$d_1d_2d_3d_4d_5d_6$

for which

$$d_1+d_2+d_3=d_4+d_5+d_6$$

More generally,

- For 2*r*-digit numbers in base *q*,

$$\mathcal{T}(q,r) = \{(d_1, d_2 \ldots d_{2r}) : d_i \in \{0, .., q-1\}\}$$

$$T(q,r) = |\, \mathcal{T}(q,r) \,| = q^{2r}$$

$$\mathcal{L}(q,r) = \{(d_1, d_2 \ldots d_{2r}) \in \mathcal{T}(q,r) : d_1 + \ldots + d_r = d_{r+1} + \ldots + d_{2r}\}$$

$$\boxed{L(q,r) = |\, \mathcal{L}(q,r) \,|}$$

Lucky Sum

- Each lucky ticket has a lucky sum *n*:

$$n = d_1 + ... + d_r = d_{r+1} + ... + d_{2r}$$

$$n \in \{0, ..., r(q-1)\}$$

Let a_n be the number of r-digit numbers whose digits sum to n:

$$a_n = | \{(d_1, ..., d_r) : d_1 + ... + d_r = n\} |$$

n	$d_1 d_2 d_3$	a_n
0	000	1
1	100, 010, 001	3
2	200, 020, 002, 101, 110, 011	6
…	…	…
27	999	1

Number of lucky tickets with lucky sum *n* is

$$a_n * a_n = a_n^2$$

The number of lucky tickets in general is

$$L(q,r) = a_o^2 + a_1^2 + ... + a_{r(q-1)}^2$$

Let

$$A_q(s) = 1 + s + s^2 + s^3 + \dots + s^{q-1}$$

Claim

L(q,r) is the sum of the squares of the coefficients of $A_q^r(s)$

Proof

$$A_q^r(s) = \left(\sum_{d_1=0}^{q-1} s^{d_1}\right)\left(\sum_{d_2=0}^{q-1} s^{d_2}\right)\dots\left(\sum_{d_r=0}^{q-1} s^{d_r}\right)$$

$$= \sum_{d_1=0}^{q-1}\sum_{d_2=0}^{q-1}\dots\sum_{d_r=0}^{q-1} s^{d_1+d_2+..+d_r}$$

$$= \sum_{n=0}^{r(q-1)} \mid \{(d_1,\dots,d_r) : d_1+\dots+d_r = n\} \mid s^n$$

$$= \sum_{n=0}^{r(q-1)} a_n(q,r)s^n$$

$\square$

Conclusion

- In order to find *L(q,r)*, we must find the coefficients of the polynomial

$$A_q^r(s) = (1 + s + s^2 + s^3 + ... + s^{q-1})^r$$

To solve **our** problem…

- We want to find

$$\sum_{n=0}^{27} a_n^2$$

- Thus, we need to know a_0, a_1, a_2, a_3, … a_{27} which are the coefficients of

$$A_{10}^3(s) = (1 + s + s^2 + s^3 + s^4 + s^5 + s^6 + s^7 + s^8 + s^9)^3$$

Two steps

1. $A_{10}^{2}(s) = A_{10}(s) \cdot A_{10}(s)$

2. $A_{10}^{3}(s) = A_{10}^{2}(s) \cdot A_{10}(s)$

1	s	s^2	s^3	s^4	s^5	s^6	s^7	s^8	s^9	s^{10}	s^{11}	s^{12}	s^{13}	s^{14}	s^{15}	s^{16}	s^{17}	s^{18}
1	1	1	1	1	1	1	1	1	1									
	1	1	1	1	1	1	1	1	1	1								
		1	1	1	1	1	1	1	1	1	1							
			1	1	1	1	1	1	1	1	1	1						
				1	1	1	1	1	1	1	1	1	1					
					1	1	1	1	1	1	1	1	1	1				
						1	1	1	1	1	1	1	1	1	1			
							1	1	1	1	1	1	1	1	1	1		
								1	1	1	1	1	1	1	1	1	1	
									1	1	1	1	1	1	1	1	1	1
1	2	3	4	5	6	7	8	9	10	9	8	7	6	5	4	3	2	1

1	s	s^2	s^3	s^4	s^5	s^6	s^7	s^8	s^9	s^{10}	s^{11}	s^{12}	s^{13}	s^{14}	s^{15}	s^{16}	s^{17}	s^{18}	s^{19}	s^{20}	s^{21}	s^{22}	s^{23}	s^{24}	s^{25}	s^{26}	s^{27}
1	2	3	4	5	6	7	8	9	10	9	8	7	6	5	4	3	2	1									
	1	2	3	4	5	6	7	8	9	10	9	8	7	6	5	4	3	2	1								
		1	2	3	4	5	6	7	8	9	10	9	8	7	6	5	4	3	2	1							
			1	2	3	4	5	6	7	8	9	10	9	8	7	6	5	4	3	2	1						
				1	2	3	4	5	6	7	8	9	10	9	8	7	6	5	4	3	2	1					
					1	2	3	4	5	6	7	8	9	10	9	8	7	6	5	4	3	2	1				
						1	2	3	4	5	6	7	8	9	10	9	8	7	6	5	4	3	2	1			
							1	2	3	4	5	6	7	8	9	10	9	8	7	6	5	4	3	2	1		
								1	2	3	4	5	6	7	8	9	10	9	8	7	6	5	4	3	2	1	
									1	2	3	4	5	6	7	8	9	10	9	8	7	6	5	4	3	2	1
1	3	6	10	15	21	28	36	45	55	63	69	73	75	75	73	69	63	55	45	36	28	21	15	10	6	3	1

1	3	6	10	15	21	28	36	45	55	63	69	73	75	75	73	69	63	55	45	36	28	21	15	10	6	3	1
1	9	36	100	225	441	784	1296	2025	3025	3969	4761	5329	5625	5625	5329	4761	3969	3025	2025	1296	784	441	225	100	36	9	1

55252

$$\frac{55252}{10^6} = 5.5252\%$$

References

- Lando, S. K. "1.1 The Lucky Tickets Problem." *Lectures on Generating Functions*. Providence, RI: American Mathematical Society, 2003. 1-3. Print.
- Jonathon Novak, MIT (Notes)

Thank you.

Questions?