# Generating Functions

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## Main Topics

- Definition of Generating Functions
- Operations on Generating Functions
- Famous Sequences

# Definition of Generating Functions

- Informally, a generating function F(x) is an infinite series
$$F(x) = f_0 + f_1 x + f_2 x^2 + f_3 x^3 + \cdots$$
- What are they used for?
- Analyzing the behavior of a sequence of numbers $f_0\ f_1\ f_2\ f_3 \cdots$
- Easy to think of $f_n$ as a coefficient of $x^n$ instead of enumerating the entire sequence up to $f_n$. (more examples to follow)

# Easy Example

- What is the generating function for the series 1, 1, 1, .... ?
- G(x)= $1 + x + x^2 + x^3 + \cdots.$
- What is a simplified form for writing it?
- G(x)= $\frac{1}{1 - x}$

# Good news…

- We can find more complex generating functions by performing algebraic operations on easier generating functions.

# Operations on Generating Functions

- 1) Scaling: Multiplying a generating function by a constant.
- Example: If we want the sequence 5, 5, 5, … we can multiply the sequence 1, 1, 1, … by 5. So we can conclude that the generating function we want is: $\frac{5}{1-x}$
- 2) Substitution
- 3) Addition: How else can we get the generating function above? (hint: addition)

# Operations on Generating Functions

4) Derivatives:

$$\frac{d}{dx}\frac{1}{1-x} = \frac{d}{dx}(1 + x + x^2 + x^3 + \cdots)$$

$$= 1 + 2x + 3x^2 + \cdots + nx^{n-1} + \cdots$$

- So we learn that $\frac{1}{(1-x)^2}$ is the generating function for the sequence 1, 2, 3, 4, …

# Operations on Generating Functions

- 5) Right Shifting: Multiply by x to shift one step to the right, by $x^n$ to shift by n steps.
- Using previous example:
- 1, 2, 3, 4, …. → 0, 1, 2, 3, ….
- 0, 1, 2, 3, …. → 0, 0, 1, 2, 3…

$$F(x) = \frac{1}{1+x}$$

# Operations on Generating Functions

- 6) Products:
- Let $A(x) = a_0 + a_1 x + a_2 x^2 + \cdots$
  and $B(x) = b_0 + b_1 x + b_2 x^2 + \cdots$
- Then $C(x) = A(x) * B(x)$
- Proposition: The n$^{th}$ coefficient of C(x) is

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \ldots + a_n b_0$$

# Famous Sequences

- The Fibonacci Sequence 1, 1, 2, 3, 5, 8, 13, … can be described by the recurrence relation:

$$f_0 = 1$$
$$f_1 = 1$$
$$f_n = f_{n-1} + f_{n-2}$$

We can use the recurrence relation to get the generating function $F(x) = \frac{x}{1 - x - x^2}$

- (Derivation of coefficients in "Lectures on Generating Functions" by S. K. Lando, chapter2)

# Famous Sequences

- The Catalan sequence 1, 1, 2, 5, 14, 42, 132, …

has the generating function

$$Cat(x) = \frac{1 - \sqrt{1 - 4x}}{2x}$$

(Derivation of coefficients in “Lectures on Generating Functions” by S. K. Lando, chapter 2)

# References


- Lando, S. K."2. Generating Functions for Well-known Sequences", *Lectures on Generating Functions*. Providence, RI: American Mathematical Society, 2003. STML v. 23.
- Lehman Eric, Leighton F Tom, Meyer Albert R. "15. Generating Functions", *Mathematics for Computer Science*, Creative Commons, 2013.