Combinatorial Games

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   - Sprouts

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Definitions

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- Fairness
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- "Impartial games" - the moves available depend on the position, and not on which player has the turn
- Fairness
- Games can be played as miserè or normal, which set the conditions for victory
Subtraction Game

Assume we are playing a normal game (if a player can’t move, that player loses)

- Start with $n$ objects
- On each turn, a player may remove anywhere from 1 to $k$ objects
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- What is the winning strategy?
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- Start with $n$ objects
- On each turn, a player may remove anywhere from 1 to $k$ objects
- What is the winning strategy?
- Always leave $m$ objects, where $m \equiv 0 \pmod{k+1}$
Nim

- Each turn, remove any number of objects from ONE pile
Nim Sum

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To win a game of Nim, at the end of each turn, the Nim Sum of the heaps should be 0
Sprouts

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- Each spot can only have up to 3 lines attached.
- If a player cannot connect two open spots without crossing, that player loses.
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<table>
<thead>
<tr>
<th>Spots</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normal Outcome</td>
<td>Loss</td>
<td>Loss</td>
<td>Win</td>
<td>Win</td>
<td>Win</td>
<td>Loss</td>
</tr>
</tbody>
</table>
Chess

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- Game-tree complexity $\approx 10^{123}$
- Zernelo’s Theorem - Chess must have a winning strategy.
Go

- Number of legal game positions $\approx 10^{170}$
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- Number of legal game positions $\approx 10^{170}$
- Game-tree complexity $\approx 10^{700}$
Conway’s Game of Life

- Any live cell with fewer than two live neighbors dies, as if caused by under-population
- Any live cell with two or three live neighbors lives on to the next generation
- Any live cell with more than three live neighbors dies, as if by overcrowding
- Any dead cell with exactly three live neighbors becomes a live cell, as if by reproduction