# Combinatorial Games

Daniel Heins

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- Fairness

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- Fairness
- Games can be played as miserè or normal, which set the conditions for victory

# Subtraction Game

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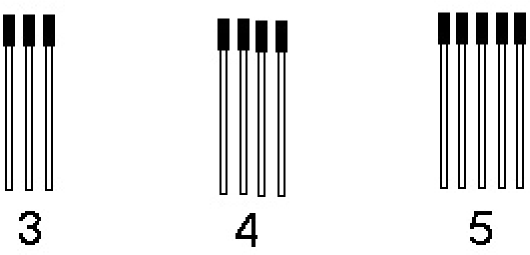
- Start with $n$ objects
- On each turn, a player may remove anywhere from 1 to $k$ objects
- What is the winning strategy?
- Always leave $m$ objects, where $m \equiv 0 \pmod{k+1}$

# Nim

- Each turn, remove any number of objects from ONE pile



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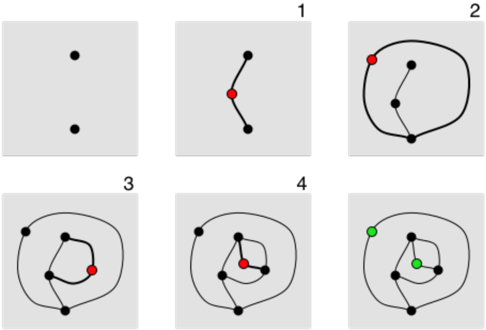
- To win a game of Nim, at the end of each turn, the Nim Sum of the heaps should be 0

# Sprouts

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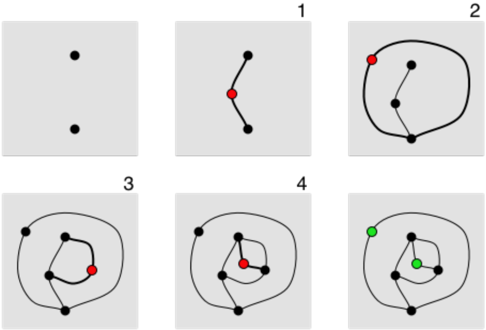


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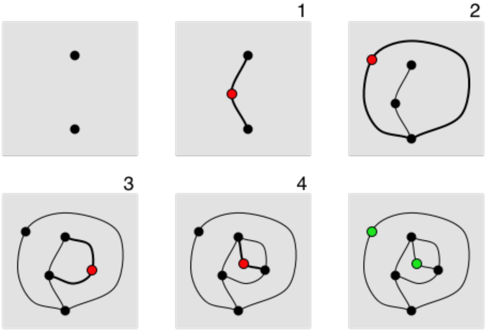


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- Each spot can only have up to 3 lines attached
- If a player cannot connect two open spots without crossing, that player loses

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| Spots | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Normal Outcome | Loss | Loss | Win | Win | Win | Loss |

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- Zernelo's Theorem - Chess must have a winning strategy.

# Go

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- Game-tree complexity $\approx 10^{700}$

# Conway's Game of Life

- Any live cell with fewer than two live neighbors dies, as if caused by under-population
- Any live cell with two or three live neighbors lives on to the next generation
- Any live cell with more than three live neighbors dies, as if by overcrowding
- Any dead cell with exactly three live neighbors becomes a live cell, as if by reproduction