18.504: FINITE MODEL THEORY
HOMEWORK 2

Due Tuesday February 26, in class.

Problem 1. Say that a class $\mathcal{C}$ of σ-structures is *axiomatizable* if there is a set Φ of σ-sentences such that for any σ-structure $\mathcal{A}$, $\mathcal{A} \in \mathcal{C}$, if and only if, $\mathcal{A} \models \Phi$.

(a) Show that for any vocabulary σ, the class of finite σ-structures is not axiomatizable. [Hint: See the proof of Proposition 3.2. Suppose for contradiction that this class is axiomatizable, by some Φ. Consider

$$\Phi' = \Phi \cup \{\lambda_n : n \in \mathbb{N}\}$$

for λ_n as defined in the above mentioned proof.]

(b) Use Lemma 3.4 to show that, when we restrict attention to finite structures, every class $\mathcal{C}$ of finite σ-structures is axiomatizable. [Hint: Describe $\mathcal{C}$ in terms of all the structures *not* in $\mathcal{C}$.]

Problem 2. Give a sentence ϕ such that for all n, ϕ has a model of size n if and only if n is not prime. [Hint: Let $\sigma = \{E_1, E_2\}$ contain two binary relation symbols. Then let ϕ say that each of E_1 and E_2 is an equivalence relation, and that the two equivalence relations are 'cross-cutting' or 'perpendicular', so as to guarantee that all the E_1-equivalence classes in a single model of ϕ have the same size.]

Problem 3. For $n = 2, 3$, find the smallest non-isomorphic linear orders L_1 and L_2 such that $L_1 \equiv_n L_2$.

For arbitrary n, can you figure out the smallest distinct n_1, n_2 such that there are linear orders of size n_1 and n_2 that are n-equivalent (that is, so that the Duplicator wins the n-round Ehrenfeucht-Fraisse Game).

Problem 4. Let σ be a purely relational vocabulary. Given σ-structures $\mathcal{A}$ and $\mathcal{B}$, with $A \cap B = \emptyset$, one can define a new structure $\mathcal{A} \cup \mathcal{B}$ in an obvious way. (In particular, for each $R \in \sigma$, let $R^{\mathcal{A} \cup \mathcal{B}} = R^{\mathcal{A}} \cup R^{\mathcal{B}}$.) Fix n, and let $\mathcal{A}_1, \mathcal{A}_2, \mathcal{B}_1, \mathcal{B}_2$ be σ-structures such that $\mathcal{A}_1 \equiv_n \mathcal{B}_1$ and $\mathcal{A}_2 \equiv_n \mathcal{B}_2$. Show that

$$\mathcal{A}_1 \cup \mathcal{A}_2 \equiv_n \mathcal{B}_1 \cup \mathcal{B}_2$$

by describing the Duplicator's winning strategy in the n-round game on these structures.