18.504: FINITE MODEL THEORY
HOMEWORK 1

Due Thursday February 14, in class.

Problem 1. Recall that an *equivalence relation* on a set A is a binary relation that is reflexive, symmetric, and transitive. A subset C of A consisting of all the elements 'equivalent' to some fixed $a \in A$ is called an *equivalence class.* Let $\sigma = \{E\}$ be a vocabulary that contains a single binary relation symbol.

(a) Give a sentence ϕ_0 that defines the class of equivalence relations, that is, the class consisting of those σ-structures $\mathcal{A} = (A, E^{\mathcal{A}})$ such that $E^{\mathcal{A}}$ is an equivalence relation on A.
(b) Give a sentence ϕ_1 that defines the class of equivalence relations such that every equivalence class has at least 5 elements.
(c) Give a sentence ϕ_1 that defines the class of equivalence relations such that every equivalence class has exactly 5 elements.

Problem 2. Recall from class that given a σ-structure $\mathcal{A}$, we say that a subset $B \subseteq A^n$ is *definable* if there is a σ-formula $\phi(\overline{x})$, $\overline{x} = (x_1, \ldots, x_n)$, such that for any n-tuple $\overline{a} \in A^n$, $\overline{a} \in B$ if and only if $\mathcal{A} \models \phi(\overline{a})$.

Let $\mathcal{N} = (\mathbb{N}, +, \cdot, 0, 1, <)$.

(a) Show that the set $\{4\}$ is definable in $\mathcal{N}$. Then show that for any n, the set $\{n\}$ is definable in $\mathcal{N}$.
(b) Show that the set $B \subseteq \mathbb{N}$ of prime numbers is definable in $\mathcal{N}$.
(c) Show that the following subset of $\mathbb{N}^3$ is definable in $\mathcal{N}$.

$$\{(a, b, c) : c \text{ has exactly two prime divisors, } a \text{ and } b\}$$

Problem 3. Let σ be a finite vocabulary. Call a finite σ-structure $\mathcal{A}$ *standard* if the universe A equals $\{1, \ldots, n\}$, for some $n \in \mathbb{N}$.

(a) Prove that any finite structure is isomorphic to a standard structure.
(b) Prove that for any $n \in \mathbb{N}$, up to isomorphism, there are only finitely many σ-structures of size n. (In other words, there is a finite set of σ-structures of size n such that any other σ-structure of size n is isomorphic to one of the σ-structures in this finite set.)
(c) Give an example of a sentence ϕ such that, for any $n \in \mathbb{N}$, ϕ has a model of size n if and only if n is even.

Problem 4. An *automorphism* of a structure $\mathcal{A}$ is an isomorphism from $\mathcal{A}$ to itself. Let $\sigma = \{<\}$.

(a) Determine all the automorphisms of $\mathcal{A} = (\mathbb{Z}, <)$, with the usual ordering.
(b) Prove that any finite linear order $\mathcal{A}$ has exactly one automorphism, given by the identity function $i : A \to A$.