The Local Langlands Correspondence for Tamely Ramified Groups

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Abstract

This thesis generalizes the work of DeBacker and Reeder [16] to the case of reductive groups splitting over a tame extension of the field of definition. The approach is broadly similar and the restrictions on the parameter the same, but many of the details of the arguments differ.

Let $\mathbf{G}$ be a unitary group defined over a local field K and splitting over a tame extension E/K. Given a Langlands parameter $\varphi\colon \mathcal{W}_K \to {}^L\mathbf{G}$ that is tame, discrete and regular, we give a natural construction of an L-packet Π_φ associated to φ, consisting of representations of pure inner forms of $\mathbf{G}(K)$ and parameterized by the characters of the finite abelian group $A_\varphi = Z_{\hat{\mathbf{G}}}(\varphi)$.

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Chapter 1

Introduction

The Langlands correspondence connects Galois representations with representations of algebraic groups. The program has expanded over the past forty years to cover different choices of ground field on each side of the correspondence. In this thesis, we will focus on the classical local Langlands correspondence, which relates

(i) admissible complex representations of $\mathbf{G}(K)$ for a connected reductive group $\mathbf{G}$ over a local field K, and

(ii) representations of the absolute Galois group of K landing inside a complex algebraic group ${}^L\mathbf{G}$ determined by $\mathbf{G}$.

1.1 The Local Langlands Conjecture

Let K be a finite extension of $\mathbb{Q}_p$ and suppose that $\mathbf{G}$ is a quasi-split connected reductive group defined over K (note that Chapter 2 gives an exposition of some of the background used in this thesis). We say that a complex representation

$$\pi \colon \mathbf{G}(K) \to \mathrm{GL}(V)$$

is *admissible* if the stabilizer of every $v \in V$ is open in the p-adic topology on $\mathbf{G}(K)$, and for every open subgroup $H \subset \mathbf{G}(K)$, the set of vectors in V fixed by H is finite dimensional. Such a V must be either one-dimensional or infinite-dimensional. We will denote the set of isomorphism classes of irreducible admissible representations of $\mathbf{G}$ by $\Pi(\mathbf{G}/K)$.

On the other side of the correspondence, we define a connected complex algebraic group $\hat{\mathbf{G}}$ as a group whose root datum is dual to that of $\mathbf{G}$. Note that $\hat{\mathbf{G}}$ is defined up to isomorphism, and if $\mathbf{G}$ splits over some extension E of K then a choice of pinning yields an action of $\mathrm{Gal}(E/K)$ on $\hat{\mathbf{G}}$. We define

$$^L\mathbf{G} = \hat{\mathbf{G}} \rtimes \mathrm{Gal}(E/K).$$

We would like to consider representations of the absolute Galois group of K with image in $^L\mathbf{G}$, but we need to modify the absolute Galois group in order to get all of the representations of $\mathbf{G}(K)$ that we want. Recall that the Weil group $\mathcal{W}_K$ is the subgroup of the absolute Galois group consisting of those elements inducing an integral power of Frobenius on the residue field. We define the Weil-Deligne group WD_K by

$$\mathrm{WD}_K = \mathcal{W}_K \times \mathrm{SL}_2(\mathbb{C}).$$

Note that the Weil-Deligne group comes equipped with a canonical projection $\mathrm{WD}_K \to \mathrm{Gal}(E/K)$.

A *Langlands parameter* is a homomorphism

$$\varphi\colon\ \mathrm{WD}_K \to {}^L\mathbf{G}$$

satisfying constraints described in Section 2.8.2. We say that two parameters are equivalent if they differ by conjugation by an element of $\hat{\mathbf{G}}$, and write $\mathbb{L}(\mathbf{G}/K)$ for the set of equivalence classes of Langlands parameters for $\mathbf{G}$. We can now state the first part of the local Langlands conjecture.

Conjecture 1.1.1. *There is a natural surjective map*

$$\Pi(\mathbf{G}/K) \to \mathbb{L}(\mathbf{G}/K)$$

with finite fibers.

For general $\mathbf{G}$, this conjectured correspondence is not necessarily bijective. The fibers are known as *L-packets*, and partition the admissible representations of $\mathbf{G}(K)$ into finite sets with the same Langlands parameter. Many of the expected properties for the local Langlands correspondence are phrased in terms of this grouping into L-packets. For a more complete description of the expected properties of the correspondence see Vogan's overview [51].

The L-packet associated to φ should be parameterized as follows. We associate to φ a finite group A_φ, defined as the component group of the centralizer of φ:

$$A_\varphi = \pi_0(\mathrm{Z}_{\hat{\mathbf{G}}}(\varphi)).$$

Then the L-packet associated to φ should be in bijection with those irreducible representations of A_φ that give the trivial character of the center of ${}^L\mathbf{G}$. In fact, one can expand the L-packets so that they are parameterized by all irreducible representations of A_φ, and it will be these enlarged L-packets Π_φ that we will consider in this thesis. The representations in Π_φ are not all representations of $\mathbf{G}(K)$. Instead, the elements of Π_φ associated to representations of A_φ that give nontrivial characters of the center of ${}^L\mathbf{G}$ will be representations of $\mathbf{G}'(K)$ for some pure inner form $\mathbf{G}'$ of $\mathbf{G}$ [51, Def. 2.6].

1.2 Known Cases

We now summarize some known cases of the local Langlands correspondence.

With the benefit of hindsight, one can interpret local class field theory as the local Langlands correspondence for $\mathbf{G} = \mathbb{G}_m$. In this case, ${}^L\mathbf{G} = \mathbb{C}^\times$, and any Langlands parameter vanishes on $\mathrm{SL}_2(\mathbb{C}) \subset \mathrm{WD}_K$. Every homomorphism from $\mathcal{W}_K$ to $\mathbb{C}^\times$ factors through the abelianization of $\mathcal{W}_K$, and thus the Artin reciprocity isomorphism

$$\mathcal{W}_K^{\mathrm{ab}} \cong K^\times$$

gives a bijection between one dimensional complex representations of $\mathcal{W}_K$ and complex characters of $\mathbb{G}_m(K) = K^\times$.

Langlands generalized the case of $\mathbb{G}_m$ to a bijection

$$\mathrm{H}^1(K, \hat{\mathbf{T}}) \cong \mathrm{Hom}(\mathbf{T}(K), \mathbb{C}^\times)$$

for arbitrary tori $\mathbf{T}$ over K [35]; see Section 2.11 for more details.

Due to the work of Harris-Taylor [23, p. 2] and Henniart [25] we know that the local Langlands correspondence holds for $\mathbf{G} = \mathrm{GL}_n$. They give us a set of conditions that completely characterize the local Langlands correspondence for GL_n. However, neither Harris-Taylor's nor Henniart's proof constructs the correspondence directly, but instead uses an action of $\mathrm{GL}_n(K) \times \mathcal{W}_K$ on the cohomology of certain Shimura varieties to prove that the correspondence exists. Bushnell and Henniart have made progress on making this construction more explicit [24, 7, 8], but the story is not yet complete.

Another recent paper of Hiraga and Saito [26] gives a proof of the local Langlands correspondence for SL_n, where larger L-packets begin to arise.

Rather than fixing $\mathbf{G}$ and proving the full correspondence for that $\mathbf{G}$, one can instead work with a large class of $\mathbf{G}$ simultaneously and consider, for each $\mathbf{G}$, only a subset of $\mathbb{L}(\mathbf{G}/K)$. If one assumes that $\mathbf{G}$ is split over K and that φ is unramified, Vogan [51, Ex. 4.9] gives a construction of the associated L-packet Π_φ. In this case one obtains quotients of unramified principal series representations, and the relevant parameter class is determined by a semisimple conjugacy class in $\hat{\mathbf{G}}$ and a nilpotent element of $\hat{\mathfrak{g}}$.

1.2.1 The DeBacker-Reeder Case

DeBacker and Reeder [16] continue in this direction, analyzing a larger class of $\mathbf{G}$ and parameters. Without the full correspondence in hand, they aren't able to give a full set of conditions that characterize the correspondence uniquely. But they do prove that the L-packets they construct satisfy many of the expected properties. Since our methods owe much to theirs, we briefly outline their assumptions and how they construct the L-packet Π_φ.

They begin with a connected reductive group $\mathbf{G}$, and assume that $\mathbf{G}$ splits over an unramified extension E/K. Suppose that φ is a Langlands parameter for $\mathbf{G}$ that is trivial on $\mathrm{SL}_2(\mathbb{C})$, and assume that φ is

(i) *tame*: it factors through the quotient of $\mathcal{W}_K$ by wild inertia,

(ii) *discrete*: the centralizer of φ in $\hat{\mathbf{G}}$ is finite modulo the center of ${}^L\mathbf{G}$, and

(iii) *regular*: the image of inertia is generated by a semisimple element of $\hat{\mathbf{G}}$ whose centralizer is a maximal torus $\hat{\mathbf{S}}$.

Discrete parameters for which $\mathrm{SL}_2(\mathbb{C})$ acts trivially are expected to correspond to supercuspidal representations (see Section 2.16), and tame parameters should correspond to representations with depth zero (see Section 2.14.5); DeBacker and Reeder confirm both of these expectations.

Suppose that $\lambda \in X^*(\hat{\mathbf{S}})$. Given $\mathbf{G}, \varphi$ and λ they construct pairs (π_λ, F_λ), where F_λ is a twist of Frobenius and π_λ is a representation of $\mathbf{G}^{F_\lambda}$, the K-points of the pure inner form of $\mathbf{G}$ determined by F_λ. They then define a notion of equivalence of such pairs and prove that the equivalence class of (π_λ, F_λ) depends only on the class of λ in a finite quotient of $X^*(\hat{\mathbf{S}})$ isomorphic to $\mathrm{Irr}(A_\varphi)$.

The first step in the construction of π_λ is the construction of a point x_λ in the Bruhat-Tits building $\mathcal{B}(\mathbf{G})$ as the unique fixed point of a specific automorphism of the apartment $X^*(\hat{\mathbf{S}}) \otimes \mathbb{R}$. This point in the building determines a maximal compact subgroup $\mathbf{G}_\lambda$ used in the construction of π_λ. From x_λ they also obtain an unramified anisotropic maximal torus $\mathbf{S}_\lambda$ as a particular twist of a fixed maximal torus $\mathbf{S} \subset \mathbf{G}$.

The image of φ is contained within the normalizer of $\hat{\mathbf{S}}_\lambda$. If the image were in fact a semidirect product, then the local Langlands correspondence for tori would give a character of $\mathbf{S}_\lambda(K)$. In general there is no such semidirect product decomposition of the image, but DeBacker and Reeder are able to modify φ in a canonical way to obtain a new parameter whose image can be expressed as a semidirect product and thus defines a character on $\mathbf{S}_\lambda(K)$. They can then use Deligne-Lusztig theory to define a representation of the parahoric subgroup $\mathbf{G}_\lambda$, which compactly induces to the desired supercuspidal representation of $\mathbf{G}^{F_\lambda}$.

One benefit of the DeBacker-Reeder approach is that it explicitly constructs the representations in an L-packet from the data of a Langlands parameter. It also works for a broad class of groups $\mathbf{G}$.

1.3 Expanding Upon DeBacker-Reeder

In this thesis we will expand and modify the methods of DeBacker and Reeder to remove their condition that $\mathbf{G}$ splits over an unramified extension: this generalization constitutes our main result. We will continue to assume that φ factors through the tame Weil group, and this forces $\mathbf{G}$ to split over a tamely ramified extension of K. Moreover, such $\mathbf{G}$ are the most general admitting a tame parameter, since φ must project onto $\mathrm{Gal}(E/K)$ in the standard way. While our methods should apply to tame, discrete, regular parameters for arbitrary $\mathbf{G}$, we focus here on the case of unitary groups as a concrete example.

Suppose that K is a finite extension of $\mathbb{Q}_p$ with $p \neq 2$, E/K is a ramified quadratic extension of K, V is a Hermitian space over E, and $\mathbf{G}$ is the unitary group associated to V. We put conditions on $\varphi \in \mathbb{L}(\mathbf{G}/K)$ very similar to Debacker-Reeder's: we require that φ is

(i) tame,

(ii) discrete,

(iii) and regular. Since $\mathbf{G}$ splits over a tamely ramified extension, the image of tame inertia will be generated by an element of ${}^L\mathbf{G}$ rather than $\hat{\mathbf{G}}$, and we ask that the centralizer of this element in $\hat{\mathbf{G}}$ be a torus.

Our first two restrictions on φ are natural simplifying assumptions. The regularity condition is more technical; for an example illustrating the changes that occur for a non-regular parameter, see Section 5.5.3.

Any unitary group has a pure inner form that is quasi-split. A Langlands parameter for **G** will yield representations of all pure inner forms of **G**, so we may assume that **G** itself is quasi-split. Let **S** be the centralizer of a maximal K-split torus in **G**; since **G** is quasi-split **S** will be a maximal torus defined over K, uniquely determined up to conjugation within $\mathbf{G}(K)$. We use **S** in the construction of the Langlands dual group $\hat{\mathbf{G}}$, and thus $\hat{\mathbf{G}}$ comes equipped with a maximal torus $\hat{\mathbf{S}}$ over $\mathbb{C}$ dual to **S**.

Our strategy for constructing an L-packet from a tame, discrete, regular Langlands parameter φ is as follows.

(i) Use the image of Frobenius under φ to define an abstract torus **T** as an unramified twist of **S**; we prove that **T** is anisotropic. We give more detail on this construction in Chapter 3.

(ii) We have a Moy-Prasad filtration on $\mathbf{T}(K)$, where $\mathbf{T}(K)^0$ is the connected component of the identity in the Néron model of $\mathbf{T}(K)$ and $\mathbf{T}(K)^{0+}$ is the maximal pro-p subgroup of $\mathbf{T}(K)^0$. Using the parameter φ and the local Langlands correspondence for tori, we can construct a character χ_φ of $\mathbf{T}(K)^0$ that vanishes on $\mathbf{T}(K)^{0+}$. We give more detail on this construction in Chapter 4.

(iii) For each $\rho \in \mathrm{Hom}(A_\varphi, \mathbb{C}^\times)$, we embed the abstract torus **T** into a pure inner form $\mathbf{G}'$ of **G**. In fact, we embed **T** into a specific maximal compact subgroup H of $\mathbf{G}'$. By Bruhat-Tits theory, such a maximal compact also has a filtration coming from its structure as an O_K-scheme; the quotient H^0/H^{0+} is a connected reductive group over k, with maximal torus $\mathcal{T}(k) = \mathbf{T}(K)^0/\mathbf{T}(K)^{0+}$ and character $\chi_\varphi\colon \mathcal{T}(k) \to \mathbb{C}^\times$. In this situation, we can produce a Deligne-Lusztig representation π of H^0/H^{0+}. Pulling this representation back to H^0 and compactly inducing up to $\mathbf{G}'(K)$, we obtain the desired element of the L-packet. This process is described in Chapter 5.

Before proceeding to this construction, we give in Chapter 2 an exposition of some of the tools needed. The descriptions in the background chapter are intended to set notation, to provide a brief refresher on the relevant subjects, and to give references to more complete treatments.

Chapter 2

Background

In this chapter we outline some of the background material needed for the rest of the thesis. In the interest of brevity, we provide general references at the beginning of each section, and then proceed to outline only the parts of the theory that we use. Most of the material covered in this chapter is standard.

2.1 Local Fields

Serre's *Local Fields* [45, Part 1-2] provides a standard reference for local fields in algebraic number theory. See also Fröhlich's chapter on local fields in Cassels-Fröhlich [13, Ch. 1] and Robert's book on p-adic analysis [44].

Let K be a finite extension of $\mathbb{Q}_p$ for some p; we will also refer to such K as *p-adic fields*. We will write v_K for the valuation on K (normalized so that the valuation of a uniformizer π_K is 1), $\mathcal{O}_K$ for the subring consisting of elements of non-negative valuation, k for the residue field, p for the characteristic of k and q for the cardinality of k. We fix an algebraic closure $\bar{K}$ of K, and denote by Γ the absolute Galois group $\mathrm{Gal}(\bar{K}/K)$. We will implicitly assume that all extensions of K are contained in $\bar{K}$. If M is any Galois extension of K, we will write Γ_M for the absolute Galois group $\mathrm{Gal}(\bar{K}/M)$ of M.

The choice of $\bar{K}$ determines an algebraic closure $\bar{k}$ of k. The absolute Galois group Γ_k of k is isomorphic to $\hat{\mathbb{Z}}$, topologically generated by the Frobenius automorphism F. We will use the arithmetic Frobenius throughout this thesis: F is the map on $\bar{k}$ sending x to x^q.

For any finite field extension E/K, there is a unique extension of v_K to E given by setting

$$v_E(x) = \frac{1}{[E:K]} v_K(\mathrm{Nm}_{E/K}(x)).$$

The smallest positive valuation of any element of E is of the form $\frac{1}{e}$, where e is a positive integer called the *ramification degree*. We also define an integer f as the degree of the residue field of E over k. These two integers satisfy $[E : K] = ef$.

We say that E/K is *unramified* if $e = 1$ and *totally ramified* if $f = 1$. Moreover, if e is relatively prime to p, we say that E/K is *tamely ramified*. The compositum of two unramified extensions is again unramified, and similarly for tamely ramified extensions. We will denote the maximal unramified extension of K by K_{nr}, the maximal tame extension of K by K_t, the Galois group $\mathrm{Gal}(K_{nr}/K)$ by Γ_{nr} and the Galois group $\mathrm{Gal}(K_t/K)$ by Γ_t.

Any automorphism of $\bar{K}$ induces an automorphism of $\bar{k}$, so we get a homomorphism

$$\Gamma \to \Gamma_k; \tag{2.1.1}$$

the kernel of this map is called the *inertia subgroup* and denoted $\mathcal{I}_K$ or $\mathcal{I}$. The fixed field of $\mathcal{I}$ is precisely K_{nr}.

Write $\mathcal{I}_t$ for the image of $\mathcal{I}$ in Γ_t; $\mathcal{I}_t = \mathrm{Gal}(K_t/K_{nr})$. We can determine the structure of K_t [13, Cor. 1 of Prop. I.8.1] and $\mathcal{I}_t$ [13, Cor. 3 of Thm. I.8.1]:

$$K_t = \bigcup_{p \nmid e} K_{nr}(\pi_K^{1/e}),$$

$$\mathcal{I}_t \cong \prod_{\ell \neq p} \mathbb{Z}_\ell.$$

Since $\mathcal{I}_t$ is pro-cyclic, we may choose a topological generator $\tilde{\tau}$, which will remain fixed for the rest of this thesis.

The map of (2.1.1) induces an isomorphism of Γ_{nr} with Γ_k, and yields a well defined Frobenius element of Γ_{nr}. We pick a lift of Frobenius to Γ_t; we will denote the Frobenius elements of Γ_k, Γ_{nr} and Γ_t by F. The element $F \in \Gamma_t$ acts by conjugation on the normal subgroup $\mathcal{I}_t$, and in fact Γ_t is topologically generated by $\tilde{\tau}$ and F [30, §3: Thm. 2], subject only to the relation

$$F\tilde{\tau}F^{-1} = \tilde{\tau}^q.$$

Another characterization of tamely ramified fields will prove useful to us. For any extension M/E of local fields, the trace pairing

$$\begin{aligned} M \times M &\to E \\ x, y &\mapsto \mathrm{Tr}_{M/E}(xy) \end{aligned}$$

defines a nondegenerate symmetric bilinear form on M. The dual

$$\{x \in M \mid \mathrm{Tr}_{M/E}(xy) \in \mathcal{O}_E \forall y \in \mathcal{O}_M\}$$

of $\mathcal{O}_M$ is an $\mathcal{O}_E$-module containing $\mathcal{O}_M$, and thus it is the inverse of an ideal $\mathfrak{D}(M/E)$ of M, called the *different*. If e is the ramification index of M/E,

$$v_M(\mathfrak{D}(M/E)) \geq e - 1,$$

and equality holds if and only if M/E is tamely ramified [13, Thm. I.5.2]. In particular, $\mathfrak{D}(M/E) = \mathcal{O}_M$ if and only if M/E is unramified.

We now define a subgroup of Γ, called the *Weil group* of K, whose importance will be highlighted by Theorem 2.3.1. Define $\mathcal{W}_K$ to be the subgroup of elements that induce an *integral* power of Frobenius on $\bar{k}$. We give $\mathcal{W}_K$ not the subspace topology, but instead the subspace topology on $\mathcal{I}_K$ and the discrete topology on $\langle F \rangle$. Finally, we define the Weil-Deligne group

$$\mathrm{WD}_K = \mathcal{W}_K \times \mathrm{SL}_2(\mathbb{C}),$$

which will play a role in the definition of Langlands parameters. See [49] for further discussion of the Weil and Weil-Deligne groups.

2.2 Galois Cohomology

See Serre's *Local Fields* [45, Part 3] or Cassels-Fröhlich [13, Ch. 4] for a summary of the parts of group cohomology particularly useful to number theorists, Cartan-Eilenberg [9] for more detail, or Serre's *Galois Cohomology* [46] for more details about the cohomology of Galois groups.

Recall that, for a group G, group cohomology $\mathrm{H}^r(G, A)$ is the right derived functor of the functor

$$A \mapsto A^G = \{a \in A \mid g \cdot a = a \; \forall g \in G\}$$

from the category of G-modules to the category of abelian groups, mapping A to its G-invariants. Group homology $\mathrm{H}_r(G, A)$ is similarly defined as the left derived functor of

$$A \mapsto A_G = A/\langle g \cdot a - a \rangle,$$

again from the category of G-modules to the category of abelian groups, mapping A to its G-coinvariants. We will be mainly interested in the case that $G = \mathrm{Gal}(L/K)$ for some Galois extension L/K.

We will need two variants on these functors. The first is nonabelian cohomology (outlined in Serre [45, Appx. to Ch. 7]), where the G-module A is no longer abelian; now we can only define the group $\mathrm{H}^0(G, A) = A^G$ and the pointed set $\mathrm{H}^1(G, A)$.

The second variant is called *Tate cohomology*, which merges group cohomology and group homology in the case that G is finite [45, §8.1]. If G is finite, then we can define the norm map

$$a \mapsto \sum_{g \in G} g \cdot a$$

from A to A. This induces a map from A_G to A^G, and we define the Tate cohomology groups $\hat{\mathrm{H}}^0(G, A)$ and $\hat{\mathrm{H}}^{-1}(G, A)$ by the sequence

$$0 \to \hat{\mathrm{H}}^{-1}(G, A) \to A_G \xrightarrow{\mathrm{Nm}} A^G \to \hat{\mathrm{H}}^0(G, A) \to 0.$$

We then define, for $i \in \mathbb{Z}_{>0}$,

$$\hat{\mathrm{H}}^i(G, A) = \mathrm{H}^i(G, A)$$
$$\hat{\mathrm{H}}^{-i-1}(G, A) = \mathrm{H}_i(G, A)$$

The benefits of this definition include:

- A long exact sequence gluing the long exact sequences for cohomology and homology. If $0 \to A \to B \to C \to 0$ is an exact sequence of G-modules, then

$$\cdots \to \hat{\mathrm{H}}^{-2}(G, C) \to \hat{\mathrm{H}}^{-1}(G, A) \to \hat{\mathrm{H}}^{-1}(G, B) \to \hat{\mathrm{H}}^{-1}(G, C) \to \hat{\mathrm{H}}^0(G, A) \to \cdots$$

  is exact; the map from $\hat{\mathrm{H}}^{-1}(G, C)$ to $\hat{\mathrm{H}}^0(G, A)$ comes from the snake lemma.

- The cup product pairing on cohomology extends to Tate cohomology (see [45, §8.3]). Namely, for any G-modules A and B there is a cup product pairing

$$\hat{\mathrm{H}}^i(G,A) \otimes \hat{\mathrm{H}}^j(G,B) \to \hat{\mathrm{H}}^{i+j}(G,A\otimes B).$$

Finally, we will find the following interpretation of $\mathrm{H}^1(G,A)$ useful. We can give a bijection between the group $\mathrm{H}^1(G,A)$ and equivalence classes of homomorphisms

$$\varphi\colon G \to A \rtimes G$$

such that the composition of φ with the projection $A \rtimes G \to G$ is the identity. Here we consider two such homomorphisms equivalent if they differ by conjugation by an element of $A \subset A \rtimes G$. Indeed, suppose that $g \mapsto a_g$ is a 1-cocycle. I claim that

$$\varphi\colon g \mapsto (a_g, g)$$

is a homomorphism: $(a_{gh}, gh) = (a_g + ga_h, gh) = (a_g, g)\cdot(a_h, h)$ by the cocycle condition and the definition of a semidirect product. Moreover, conjugating by a fixed $a \in A$ changes (a_g, g) to $(a_g + a - g\cdot a, g)$, precisely corresponding to the addition of a 1-coboundary. If we start with such a homomorphism, we can recover the standard 1-cocyle by just projecting onto $A \subset A \rtimes G$.

Alternatively, if G acts on A through a quotient G/H, we can replace $A \rtimes G$ by $A \rtimes G/H$ and require that the composition of φ with the projection $A \rtimes G/H$ be the standard projection $G \to G/H$. This interpretation of $\mathrm{H}^1(G,A)$ will arise in our definition of Langlands parameters. See Serre [45, §7.3] for more details.

2.3 Local Class Field Theory

Serre's *Local Fields* [45, Part 4] once more provides a good reference for this material, as does Cassels-Fröhlich [13, Ch. 6]. Milne's course notes [37] include exercises and more detail.

As noted in the introduction, the Langlands correspondence can be thought of as a generalization of class field theory. More importantly for us, local class field theory is used crucially in the Langlands correspondence for tori and thus in our construction of L-packets. In this section I will only scratch the surface of local class field theory, touching only on those theorems that will be of use to us.

Theorem 2.3.1 (Local Reciprocity; c.f. [37, Thm. 1.1])**.** *There is a unique isomorphism of topological groups*

$$\mathrm{rec}_K \colon K^\times \to \mathcal{W}_K^{\mathrm{ab}}$$

such that

(i) For any uniformizer $\pi \in K^\times$, $\mathrm{rec}_K(\pi)$ acts as Frobenius on $\bar{k}$.

(ii) For any finite abelian extension L/K, $\mathrm{rec}_K(\mathrm{Nm}_{L/K} L^\times)$ acts trivially on L, and rec_K induces an isomorphism

$$\mathrm{rec}_{L/K} \colon K^\times / \mathrm{Nm}_{L/K} L^\times \to \mathrm{Gal}(L/K).$$

Note that rec_K does depend on our choice of arithmetic Frobenius instead of geometric

Frobenius. We choose a uniformizer π_K for K so that $\mathrm{rec}_K(\pi_K)$ has the same image in $\mathrm{Gal}(K_t/K)^{\mathrm{ab}}$ as our chosen Frobenius $F \in \mathrm{Gal}(K_t/K)$.

As a consequence we have the following description of the abelian extensions of K:

Theorem 2.3.2 (Existence theorem; see [37, Cor. 1.2 & Thm. 1.4])**.** *Let K be a non-archimedean local field. Then*

(i) The map $L \to \mathrm{Nm}(L^\times)$ is a bijection from the set of finite abelian extensions of K onto the set of open finite-index subgroups of $K^\times$.

(ii) For extensions L and L' of K,

$$L \subset L' \quad \Leftrightarrow \quad \mathrm{Nm}(L^\times) \supset \mathrm{Nm}(L'^\times).$$

Theorem 2.3.2 allows us to describe the quadratic extensions of a p-adic field K. Suppose for simplicity that $p \neq 2$ and thus the extensions are all tamely ramified. Then the quadratic extensions of K are in bijection with index 2 subgroups of $K^\times$. We have that

$$K^\times \cong \mathbb{Z} \times k^\times \times (1 + \pi_K \mathcal{O}_K).$$

Since all subgroups of $(1 + \pi_K \mathcal{O}_K)$ have index a power of p and $p \neq 2$, there are three quadratic extensions of K, corresponding to a choice of nontrivial element of $(\mathbb{Z}/2\mathbb{Z}) \times (k^\times/(k^\times)^2)$. The extension corresponding to $2\mathbb{Z} \times k^\times$ is unramified, while the other two are tamely ramified.

2.4 Induction

For a treatment of induction in finite groups see Isaacs [29, Ch. 5], for parabolic induction consult Casselman [12, §3], and for smooth and compact induction see Bushnell-Henniart [7, Ch. 2].

Suppose that H is a subgroup of a group G. If V is a vector space and $\rho_G : G \to \mathrm{GL}(V)$ a representation of G, then restriction to H defines a representation of H, known as the *restriction*. The various types of induction give ways of producing a representation of G from a representation of H and are adjoint to restriction in an appropriate category.

In the simplest case that G is a finite group and $\rho\colon H \to \mathrm{GL}(V)$ is a representation of H, we define $\mathrm{Ind}_H^G(V)$ to be the vector space of functions $f\colon G \to V$ such that

$$f(hg) = \rho(h)f(g) \text{ for } h \in H \text{ and } g \in G.$$

The group G acts on $\mathrm{Ind}_H^G(V)$ by $(gf)(g') = f(g'g)$; we write $\mathrm{Ind}_H^G \rho$ for this representation. Frobenius reciprocity states that for any vector space W with an action of G,

$$\mathrm{Hom}_H(V, \mathrm{Res}_H^G(W)) = \mathrm{Hom}_G(\mathrm{Ind}_H^G(V), W).$$

In fact, induction for finite groups is both a left and a right adjoint.

We will be working with rational points of algebraic groups $\mathbf{G}$ over K, which are generally not finite. We thus need to restrict the types of representations we allow. A *smooth* representation of $\mathbf{G}$ is a pair (π, V) where V is a vector space over $\mathbb{C}$ and π is a homomorphism from $\mathbf{G}(K)$ to $\mathrm{GL}_{\mathbb{C}}(V)$ such that the stabilizer of any vector $v \in V$ is open in the p-adic topology on $\mathbf{G}(K)$. We say that (π, V) is *admissible* if for any open subgroup

$H \subset \mathbf{G}(K)$, the space V^H of H-fixed vectors has finite dimension.

There are two primary ways to construct admissible representations from simpler objects: parabolic induction, which applies for algebraic groups over an arbitrary field, and compact induction, using the p-adic topology on $\mathbf{G}(K)$.

2.4.1 Parabolic Induction

Let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$ (see Section 2.5.6) with decomposition $\mathbf{P} = \mathbf{MU}$ into a Levi subgroup $\mathbf{M}$ and unipotent radical $\mathbf{U}$. The *modulus character* $\delta_{\mathbf{P}} \colon \mathbf{P}(K) \to \mathbb{C}^\times$ is defined by $\delta_{\mathbf{P}}(p) = |\det \mathrm{Ad}_{\mathfrak{u}}(p)|$, where $\mathfrak{u}$ is the Lie algebra of $\mathbf{U}$ (or as a ratio of left and right Haar measures). Start with an admissible representation (π, V) of $\mathbf{M}$; this determines a representation of $\mathbf{P}$ since $\mathbf{M} = \mathbf{P}/\mathbf{U}$. The *parabolic induction* $\mathrm{Ind}_{\mathbf{P}}^{\mathbf{G}} \pi$ of π is the right regular representation of $\mathbf{G}(K)$ on the space $\mathrm{Ind}_{\mathbf{P}}^{\mathbf{G}} V$ of locally constant functions $f \colon \mathbf{G}(K) \to V$ satisfying

$$f(pg) = \delta_{\mathbf{P}}(p)^{1/2}\pi(p)f(g) \qquad \text{for } p \in \mathbf{P}(K) \text{ and } g \in \mathbf{G}(K).$$

The reason for the inclusion of the modulus character is that parabolic induction then takes unitary representations to unitary representations [12, Prop. 3.1.4], though we will not focus on this feature. More importantly for us, the parabolic induction of an admissible representation is once again admissible (see Section 2.16).

The process of parabolic induction allows us to build representations of $\mathbf{G}$ from representations of its proper Levi subgroups. We will call a representation *supercuspidal* if it is not a subquotient of a representation parabolically induced from any proper Levi subgroup (c.f. [12, §5]). The representations we construct will all be supercuspidal, and thus we need another tool to create them.

2.4.2 Smooth and Compact Induction

Suppose that H is a closed subgroup of $\mathbf{G}(K)$ and (π, V) is a smooth representation of H. We define the *smooth induction* of π to $\mathbf{G}(K)$ as the right regular representation of $\mathbf{G}(K)$ on the space of functions $f\colon \mathbf{G}(K) \to V$ satisfying

- $f(hg) = \pi(h)f(g)$ for $h \in H$ and $g \in \mathbf{G}(K)$, and
- for some compact open subgroup $B \subset \mathbf{G}(K)$, we have $f(gb) = f(g)$ for all $b \in B$ and $g \in \mathbf{G}(K)$.

If we impose the additional restriction that each function $f\colon \mathbf{G}(K) \to V$ be compactly supported modulo H, we still get a right regular representation of $\mathbf{G}(K)$. This representation is known as the compact induction of π to $\mathbf{G}(K)$ and denoted by $\mathrm{ind}_H^G \pi$; we write $\mathrm{ind}_H^G V$ for the vector space of such functions. Note that if $\mathbf{G}(K)/H$ is compact, then smooth and compact induction coincide.

2.5 Algebraic Groups

A complete discussion of algebraic groups would take us too far afield, so in this section we focus on the structure theory of reductive groups and the definitions that appear in our constructions. We begin by considering algebraic groups over algebraically closed fields, and then pass to other fields using Galois descent. For comprehensive treatments of linear algebraic groups, see Springer [47] and Waterhouse [52].

For our purposes, an algebraic group is a finite-type reduced affine group scheme $\mathbf{G}$ defined over a field K. For any K-algebra R, an R-point of $\mathbf{G}$ is a morphism $\operatorname{Spec} R \to \mathbf{G}$; the set of R-points form a group, which we will denote $\mathbf{G}(R)$. In fact, $\mathbf{G}$ is characterized by this functor from the category of K-algebras to the category of groups, and one can take "representable functor from K-algebras to groups" as the definition of an affine group scheme. By representable we mean that there is some K-algebra $A_{\mathbf{G}}$ so that $\mathbf{G}(R) = \mathrm{Hom}_K(A_{\mathbf{G}}, R)$. For example, if $\mathbf{G}$ is the algebraic group $\mathbb{G}_m$ defined by $\mathbb{G}_m(R) = R^\times$, then $\mathbf{G}$ is represented by the algebra $K[X, X^{-1}]$ since a K-algebra homomorphism from $K[X, X^{-1}]$ is determined by the image of X in R, which must be a unit. Similarly, the algebraic group $\mathbb{G}_a$ defined by $\mathbb{G}_a(R) = R$ is represented by the K-algebra $K[X]$. Another classic example of an algebraic group is GL_n, whose R points are the $n \times n$ matrices with entries in R and determinant a unit in R.

We will be concerned primarily with reductive groups. Any algebraic group embeds as a closed subgroup of GL_n for some n [47, Thm. 2.3.7]. Using such an embedding we can define the semisimple and unipotent parts of an element of $\mathbf{G}$. A subgroup of $\mathbf{G}$ is said to be unipotent if every element has trivial semisimple part, and the *unipotent radical* $R_u(\mathbf{G})$ of $\mathbf{G}$ is the maximal connected, normal unipotent subgroup. We say that $\mathbf{G}$ is *reductive* if it has trivial unipotent radical. Since every unipotent subgroup is solvable, the unipotent radical lies within the *radical* $R(\mathbf{G})$: the maximal connected, normal solvable subgroup. Those reductive groups with trivial radical are said to be *semisimple*. A *torus* is an algebraic group that becomes isomorphic to $\mathbb{G}_m^n$ over the algebraic closure of K; tori are reductive but not semisimple.

Suppose now that $\mathbf{G}$ is a connected reductive group over $\bar{K}$. A *maximal torus* in $\mathbf{G}$ is a subtorus not strictly contained in any other subtorus; any two maximal tori are conjugate by an element of $\mathbf{G}$. The dimension of a maximal torus is called the *rank* of $\mathbf{G}$. We attach various structures to a maximal torus.

2.5.1 Character and Cocharacter groups

We can define abelian groups

$$X^*(\mathbf{T}) = \mathrm{Hom}(\mathbf{T}, \mathbb{G}_m) \qquad \text{and} \qquad X_*(\mathbf{T}) = \mathrm{Hom}(\mathbb{G}_m, \mathbf{T})$$

for any algebraic group $\mathbf{T}$, called the *character group* and *cocharacter group* of $\mathbf{T}$. If $\mathbf{T}$ is a torus they are free abelian groups. The rank of $X^*(\mathbf{T})$ is equal to the dimension of $\mathbf{T}$. Composition defines a perfect pairing

$$\langle , \rangle : X^*(\mathbf{T}) \times X_*(\mathbf{T}) \to \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z}. \tag{2.5.1}$$

We can give explicit bases for $X^*(\mathbb{G}_m^n)$ and $X_*(\mathbb{G}_m^n)$. Let χ_i be the character

$$(\alpha_1, \ldots, \alpha_n) \mapsto \alpha_i,$$

and let λ_j be the cocharacter

$$\alpha \mapsto (1, \ldots, \alpha, \ldots, 1),$$

with the α in the j^{th} position. These bases are dual under the pairing given above.

We can use the cocharacter group to give another interpretation of the $\bar{K}$ points of a torus $\mathbf{T}$: evaluation gives a canonical isomorphism

$$\mathbf{T}(\bar{K}) \cong (X_*(\mathbf{T}) \otimes_{\mathbb{Z}} \bar{K}^\times).$$

This description will prove useful when we want to consider tori over non-algebraically closed fields.

While the character and cocharacter groups are intrinsic to $\mathbf{T}$, we now consider other structures attached to $\mathbf{T}$ that depend on the group $\mathbf{G}$ in which we embed it.

2.5.2 Roots

For any torus $\mathbf{S}$ (not necessarily maximal), and any rational representation $r\colon \mathbf{S} \to \mathrm{GL}(V)$, V breaks up as a direct sum of spaces V_χ, for some set of characters χ of $\mathbf{S}$, where

$$V_\chi = \{v \in V \mid r(s)v = \chi(s)v \text{ for all } s \in \mathbf{S}\}.$$

The characters χ that appear are called the *weights* of $\mathbf{S}$ in V, and the V_χ are called the *weight spaces* [47, §7.1.1]. In particular, we can obtain a rational representation of a maximal torus $\mathbf{T} \subset \mathbf{G}$ by restricting the adjoint representation of $\mathbf{G}$ on its Lie algebra $\mathfrak{g}$ (c.f. [47, Prop. 4.4.5]) to $\mathbf{T}$. In this case we call the nonzero weights *roots* and the weight spaces $\mathfrak{g}_\alpha$ for roots α the *root spaces*. We write $\Phi(\mathbf{G}, \mathbf{T})$ for the set of roots.

In a reductive group, the nontrivial root spaces are one dimensional. We set $\mathfrak{t}$ to be the trivial root space, which we may identify with the Lie algebra of $\mathbf{T}$. We get a decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi(\mathbf{G},\mathbf{T})} \mathfrak{g}_\alpha. \tag{2.5.2}$$

This decomposition of the Lie algebra is matched in $\mathbf{G}$ by the following proposition:

Proposition 2.5.1 (c.f. [47, 8.1.1])**.** *For any $\alpha \in \Phi(\mathbf{G}, \mathbf{T})$, there is an isomorphism u_α from $\mathbb{G}_a$ onto a unique closed subgroup $\mathbf{U}_\alpha$ of $\mathbf{G}$ such that $tu_\alpha(x)t^{-1} = u_\alpha(\alpha(t)x)$ for $t \in \mathbf{T}$ and $x \in \bar{K}$. The image of the differential of u_α is the root space $\mathfrak{g}_\alpha$, and $\mathbf{G}$ is generated by $\mathbf{T}$ and* *the* $\mathbf{U}_\alpha$.

These groups $\mathbf{U}_\alpha$ will be crucial in the construction of group schemes attached to points on the Bruhat-Tits building of $\mathbf{G}$.

As an example, consider $\mathbf{G} = \mathrm{GL}_n$ with $\mathbf{T}$ consisting of diagonal matrices. The character group $X^*(\mathbf{T})$ has as basis the functions χ_i which pick out the i^{th} entry. The Lie algebra $\mathfrak{g}$ is the space of all $n \times n$ matrices, invertible or not. We choose as a basis for $\mathfrak{g}$ the matrices $A_{i,j}$ with a 1 in the (i, j) position and zeros elsewhere. $\mathbf{G}$ acts by conjugation on $\mathfrak{g}$, and a diagonal matrix t acts on $A_{i,j}$ as multiplication by $\chi_i(t)\chi_j(t)^{-1}$. We can identify the span of the $A_{i,i}$ with the Lie algebra of $\mathbf{T}$, we have $\Phi(\mathbf{G}, \mathbf{T}) = \{\chi_i - \chi_j \mid i \neq j\}$, and the root space attached to the root $\alpha = \chi_i - \chi_j$ is just the line spanned by $A_{i,j}$. Finally, the group $\mathbf{U}_\alpha$ of Proposition 2.5.1 is the subgroup of matrices differing from the identity only in the (i, j) position.

2.5.3 Coroots

In this section we attach a coroot $\alpha^\vee \in X_*(\mathbf{T})$ to each root α. We begin with the map $u_\alpha \colon \mathbb{G}_a \to \mathbf{G}$ of Proposition 2.5.1. We can identify $\mathbb{G}_a$ with the subgroup $\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$ of SL_2. The Jacobson-Morozov theorem [14, Thm. 3.3.1] tells us that we can extend u_α to a homomorphism

$$\mathrm{SL}_2 \to \mathbf{G},$$

so that the diagonal torus maps into $\mathbf{T}$. Composition with the standard cocharacter $a \mapsto \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ gives us a cocharacter $\alpha^\vee \colon \mathbb{G}_m \to \mathbf{T}$. From the definition of u_α we have $\langle \alpha, \alpha^\vee \rangle = 2$.

For $\mathbf{G} = \mathrm{GL}_n$, the inclusion of SL_2 into GL_n associated to a root $\chi_i - \chi_j$ maps a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ to the matrix with a in the (i, i) entry, b in the (i, j) entry, c in the (j, i) entry, d in the

(j, j) entry, ones elsewhere on the diagonal and zeros everywhere else. We get the coroot $\lambda_i - \lambda_j$, where λ_i is the cocharacter mapping x to a diagonal matrix with an x in the (i, i) position and ones elsewhere.

2.5.4 The Weyl Group

If $\mathbf{T}$ is a maximal torus in $\mathbf{G}$, its normalizer $\mathbf{N}$ contains $\mathbf{T}$ with finite index. The quotient $\mathbf{W} = \mathbf{N}/\mathbf{T}$ is called the *Weyl group*, and conjugation within $\mathbf{G}$ induces isomorphisms between the Weyl groups associated to different maximal tori. The natural action of $\mathbf{W}$ on $\mathbf{T}$ induces a faithful action on $X^*(\mathbf{T})$; we identify $\mathbf{W}$ with this group of automorphisms.

For any root $\alpha \in \Phi(\mathbf{G}, \mathbf{T})$, we can define a reflection on $X^*(\mathbf{T})$ by

$$s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha \qquad \text{for } x \in X^*(\mathbf{T}).$$

In fact, these automorphisms are induced by an element $n_\alpha \in \mathbf{N}$ [47, Lem. 8.1.4], and they generate $\mathbf{W}$ as a group of automorphisms of $X^*(\mathbf{T})$. The action of $\mathbf{W}$ on $X^*(\mathbf{T})$ stabilizes the root system Φ.

In the case of $\mathbf{G} = \mathrm{GL}_n$ and $\mathbf{T}$ the group of diagonal matrices, the normalizer is the group of monomial matrices (each row and column has exactly one nonzero entry). The Weyl group is isomorphic to the symmetric group Σ_n, and it acts on $X^*(\mathbf{T})$ via permutation of the basis vectors χ_i. Reflection in the root $\chi_i - \chi_j$ is induced by conjugation by a standard permutation matrix.

2.5.5 Positive and Simple Systems of Roots

The Weyl group is an example of a Coxeter group, and much of the structure theory for Coxeter groups is useful for understanding Weyl groups, and thus reductive groups as well (see Humphreys [27] for an introduction to Coxeter groups and root systems).

The *root hyperplane* H_α associated to a root $\alpha \in \Phi$ is the set of vectors $v \in X_*(\mathbf{T})_\mathbb{R} = X_*(\mathbf{T}) \otimes \mathbb{R}$ such that $\langle \alpha, v \rangle = 0$. A *Weyl chamber* is a connected component of $X_*(\mathbf{T})_\mathbb{R} - \bigcup_\alpha H_\alpha$. Given a Weyl chamber C, the set of roots α with $\langle \alpha, v \rangle > 0$ is independent of the choice of $v \in C$; this set is called the associated *positive system* and denoted Φ_C^+ or just Φ^+. We write Φ^- for those roots with $\langle \alpha, v \rangle < 0$.

For a given choice of positive system Φ^+, we say that a root $\alpha \in \Phi^+$ is *simple* if

$$\#\left(s_\alpha(\Phi^+) \cap \Phi^+\right) = \#\left(\Phi^+\right) - 1.$$

We call the set of simple roots in Φ^+ the associated *simple system* and denote it by Δ.

- The simple roots in Φ^+ are precisely those that are not expressible as a combination of other roots in Φ^+ with nonnegative integer coefficients.

- The simple roots are linearly independent, and every root in Φ^+ can be written as a linear combination of simple roots with non-negative coefficients; every root in Φ^- can thus be written as a linear combination of simple roots with non-positive coefficients.

- As α ranges over a simple system, the reflections s_α generate $\mathbf{W}$.

- $\mathbf{W}$ acts transitively on positive systems, mapping simple roots to simple roots.

- If α and β are two roots in simple system, then there is an integer $m_{\alpha,\beta}$ so that the angle between α and β defined using a **W**-invariant inner product is $\pi(1 - \frac{1}{m_{\alpha,\beta}})$. In fact, $m_{\alpha,\beta}$ must be $2, 3, 4$, or 6.

- Root systems are classified by *Dynkin diagrams*: graphs whose vertices correspond to simple roots. Two vertices corresponding to $\alpha, \beta \in \Delta$ are connected with

  (i) an edge if $m_{\alpha,\beta} = 3$,

  (ii) a double edge pointing to the shorter root if $m_{\alpha,\beta} = 4$, or

  (iii) a triple edge pointing to the shorter root if $m_{\alpha,\beta} = 6$.

2.5.6 Borel and Parabolic Subgroups

A *Borel subgroup* **B** of **G** is a maximal closed connected solvable subgroup. Since any torus is connected and solvable, any maximal torus is contained in a Borel subgroup, and conversely any Borel contains a maximal torus. The fact that any two maximal tori are conjugate is strengthened by the following theorem:

Theorem 2.5.2. *Suppose* $\mathbf{B}_1 \supset \mathbf{T}_1$ *and* $\mathbf{B}_2 \supset \mathbf{T}_2$ *are Borel subgroups in* **G***, each containing a maximal torus. Then there is a* $g \in \mathbf{G}$ *such that*

$$\mathbf{T}_2 = g\mathbf{T}_1 g^{-1} \qquad\qquad \mathbf{B}_2 = g\mathbf{B}_1 g^{-1}.$$

For example, the subgroup of diagonal matrices is a maximal torus in GL_n and the subgroup of upper triangular matrices is a Borel containing it; every other maximal torus and Borel subgroup in GL_n are conjugate to these.

One reason to consider Borel subgroups is that we can use them to produce representations of $\mathbf{G}$ from characters, using the process of parabolic induction (see Section 2.4.1). This process works for a more general class of subgroups: We say that $\mathbf{P} \subset \mathbf{G}$ is *parabolic* if the quotient $\mathbf{G}/\mathbf{P}$ is a projective variety. In fact, a subgroup is parabolic if and only if it contains a Borel subgroup.

For a fixed maximal torus $\mathbf{T}$, the Borel subgroups containing $\mathbf{T}$ are classified by systems of positive roots $\Phi^+ \subset \Phi(\mathbf{G}, \mathbf{T})$: the group generated by $\mathbf{T}$ and the $\mathbf{U}_\alpha$ for $\alpha \in \Phi^+$ is a Borel subgroup, and all Borel subgroups are of this form for some torus and some choice of Φ^+. Let $\Delta \subset \Phi^+$ be the system of simple roots. Then parabolic subgroups $\mathbf{P} \supset \mathbf{B}$ are classified by arbitrary subsets $I \subseteq \Delta$ [47, Thm. 8.4.3]: the empty set $I = \emptyset$ corresponds to $\mathbf{P} = \mathbf{B}$, whereas $I = \Delta$ corresponds to $\mathbf{P} = \mathbf{G}$.

Suppose $\mathbf{P} \supset \mathbf{B} \supset \mathbf{T}$. Then there is a unique closed subgroup $\mathbf{L}$ of $\mathbf{P}$ containing $\mathbf{T}$ such that multiplication induces an isomorphism $R_u(\mathbf{P}) \rtimes \mathbf{L} \to \mathbf{P}$. Such a subgroup is known as a *Levi subgroup*; when $\mathbf{P}$ is a Borel then the Levi subgroups are precisely the maximal tori in $\mathbf{B}$. The Lie algebra of $\mathbf{P}$ and $\mathbf{L}$ are determined by the set $I \subseteq \Delta$ mentioned above. Let $\Phi_I \subseteq \Phi$ consist of those roots that are a linear combination of roots in I, and define $\Phi_I^+ \subseteq \Phi$ by

$$\Phi_I^+ = \left\{ \sum_{\alpha \in \Delta} m_\alpha \alpha \;\middle|\; m_\alpha \geq 0 \text{ for } \alpha \in \Delta - I \right\}.$$

Then in the notation of the root decomposition of $\mathfrak{g}$ from (2.5.2),

$$\mathrm{Lie}(\mathbf{L}) = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi_I} \mathfrak{g}_\alpha$$

$$\mathrm{Lie}(\mathbf{P}) = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi_I^+} \mathfrak{g}_\alpha$$

2.5.7 Root Data

The structures we've attached to maximal tori allow us to classify algebraic groups over algebraically closed fields. We define a *root datum* for $\mathbf{G}$ to be the following quadruple:

$$\Psi(\mathbf{G},\mathbf{T}) = (X^*(\mathbf{T}), \Phi(\mathbf{G},\mathbf{T}), X_*(\mathbf{T}), \Phi^\vee(\mathbf{G},\mathbf{T})).$$

One can also define the notion of an abstract root datum [47, §7.4] and morphisms between them; up to isomorphism, $\Psi(\mathbf{G},\mathbf{T})$ does not depend on the choice of $\mathbf{T}$ and thus we may write $\Psi(\mathbf{G})$. We have the following theorems, classifying algebraic groups over algebraically closed fields.

Theorem 2.5.3 (c.f. [47, Thm. 9.6.2])**.** *Suppose* $\mathbf{G}$ *and* $\mathbf{G}'$ *are connected reductive algebraic groups over* $\bar{K}$*, and that* $j\colon \Psi(\mathbf{G}',\mathbf{T}') \to \Psi(\mathbf{G},\mathbf{T})$ *is an isomorphism. Then* j *is induced by an isomorphism* $J\colon \mathbf{G} \to \mathbf{G}'$ *of algebraic groups mapping* $\mathbf{T}$ *to* $\mathbf{T}'$*, which is unique up to conjugation by* $\mathbf{T}$.

In addition to this uniqueness theorem, we have an existence theorem that guarantees the existence of an algebraic group whose root datum is any given quadruple satisfying certain axioms.

Theorem 2.5.4 (c.f. [47, Thm. 10.1.1])**.** *Suppose* Ψ *is an abstract root datum. Then there is a connected reductive group* $\mathbf{G}$ *over* $\bar{K}$ *such that* $\Psi(\mathbf{G}) \cong \Psi$.

We will only need one more property of these axioms: given an abstract root datum $\Psi = (X, \Phi, X^\vee, \Phi^\vee)$, the *dual root datum*

$$\hat{\Psi} = (X^\vee, \Phi^\vee, X, \Phi)$$

is also an abstract root datum. This symmetry of root data allows us to define the Langlands dual group $\hat{\mathbf{G}}$ in the following section.

2.5.8 The Connected Langlands Dual Group

Definition 2.5.5. For an algebraic group $\mathbf{G}$ over any field K, we define a *connected Langlands dual group* $\hat{\mathbf{G}}$ for $\mathbf{G}$ to be a connected reductive group $\hat{\mathbf{G}}$ defined over $\mathbb{C}$, together with an isomorphism

$$\Psi(\hat{\mathbf{G}}) \cong \widehat{\Psi(\mathbf{G})}.$$

The simplest example of this duality is for tori. If $\mathbf{T}$ is a torus over K, then the dual torus $\hat{\mathbf{T}}$ is given by

$$\hat{\mathbf{T}} = X^*(\mathbf{T}) \otimes \mathbb{C}^\times;$$

we have isomorphisms $X_*(\hat{\mathbf{T}}) \cong X^*(\mathbf{T})$ and $X^*(\hat{\mathbf{T}}) \cong X_*(\mathbf{T})$. Moreover, the action of Γ on $X^*(\mathbf{T})$ makes $\hat{\mathbf{T}}$ into a Γ-module.

For groups with nontrivial semisimple part, taking the connected dual has the effect of exchanging long and short roots, as well as exchanging adjoint and simply connected forms. For example, we have the following dualities:

$\mathbf{G}$	GL_n	SL_n	PGL_n	Sp_{2n}	SO_{2n}
$\hat{\mathbf{G}}$	GL_n	PGL_n	SL_n	SO_{2n+1}	SO_{2n}

2.5.9 Semisimple Groups

Using the following result, we can obtain a semisimple group from a reductive group.

Proposition 2.5.6 (c.f. [47, Cor. 8.1.6 and Prop. 7.3.1])**.**

(i) *The commutator subgroup* $(\mathbf{G}, \mathbf{G})$ *is semisimple,*

(ii) *the product map yields a surjection* $R(\mathbf{G}) \times (\mathbf{G}, \mathbf{G}) \to \mathbf{G}$ *with finite kernel, and*

(iii) *the radical* $R(\mathbf{G})$ *is the identity component of the center of* $\mathbf{G}$ *(and hence is a torus).*

Suppose for the remainder of this section that $\mathbf{G}$ is semisimple. Then $\mathbb{Z}\Phi$ has finite index in $X^*(\mathbf{T})$ and is called the *root lattice*, and $\mathbb{Z}\Phi^\vee$ has finite index in $X_*(\mathbf{T})$ and is called the *coroot lattice*. Define the *weight lattice* as

$$P = \{x \in X^*(\mathbf{T}) \otimes \mathbb{R} \mid \langle x, \Phi^\vee \rangle \subseteq \mathbb{Z}\};$$

we have $\mathbb{Z}\Phi \subseteq P$. The quotient $P/\mathbb{Z}\Phi$ is known as the *fundamental group* of the root system Φ. Subgroups of this fundamental group are in bijection with algebraic groups with root system Φ (all such groups are isogenous). On one end of the spectrum are *adjoint groups*, where $X^*(\mathbf{T}) = \mathbb{Z}\Phi$ and the center of $\mathbf{G}$ is trivial. Examples include PGL_n and PSO_n. On the other end are *simply connected groups*, where $X^*(\mathbf{T}) = P$ and the center of $\mathbf{G}$ is dual to the fundamental group of Φ (c.f. [47, Ex. 8.1.12.8]). Examples include SL_n and Spin_n for $n \geq 3$.

2.6 Unitary Groups

We assume in Chapter 3 that our group $\mathbf{G}$ is a unitary group, so in this section we give an introduction to Hermitian spaces and unitary groups.

Let K be a field and fix an algebraic closure $\bar{K}$. All finite extensions of K are considered to be subfields of $\bar{K}$. Let E/K be a (separable) quadratic extension of K and let τ be the nontrivial element of $\mathrm{Gal}(E/K)$. Suppose that V is a free E-module of rank n, and $\phi\colon V \times V \to E$ a nondegenerate Hermitian form:

(i) for all $v_1, v_2, v' \in V$ and $a, b \in E$, $\phi(av_1 + bv_2, v') = a\phi(v_1, v') + b\phi(v_2, v')$;

(ii) for all $v, v' \in V$, $\phi(v, v') = \tau\phi(v', v)$;

(iii) if $\phi(v, v') = 0$ for all $v' \in V$ then $v = 0$.

For any K-algebra B, set $V_B = V \otimes_K B$. Define an action of τ on $E_B = E \otimes_K B$ by setting it to act trivially on B; ϕ then extends to a Hermitian form $V_B \times V_B \to E_B$, which we will also denote by ϕ. We define an algebraic group $\mathrm{U}(V)$ over K by setting

$$\mathrm{U}(V)(B) = \{g \in \mathrm{GL}(V_B) \ \mid \phi(gv, gw) = \phi(v, w) \text{ for all } v, w \in V_B\}.$$

If M/K is Galois, we can define an action of $\mathrm{Gal}(M/K)$ on $\mathrm{U}(V)(M)$ as follows. Since $\mathrm{Gal}(M/K)$ acts on V_M by $\sigma(v \otimes \alpha) = v \otimes \sigma(\alpha)$, we get an action on $\mathrm{GL}(V_M)$ via $\sigma.g = \sigma \circ g \circ \sigma^{-1}$. Because $\sigma \circ g \circ \sigma^{-1}$ is M-linear, $\phi(\sigma.g(v), \sigma.g(w)) = \phi(v, w)$ and we get an action of $\mathrm{Gal}(M/K)$ on $\mathrm{U}(V)(M)$ as desired. For any algebraic extension M/K we have

$$\mathrm{U}(V)(M) = \mathrm{U}(V)(\bar{K})^{\Gamma_M}.$$

2.6.1 Discriminants

Suppose V is a nondegenerate Hermitian space associated to E/K with basis $v_1, \ldots, v_n$. The determinant of the matrix $(\phi(v_i, v_j))$ lies in $K^\times$, and changing basis scales this determinant by an element of $\mathrm{Nm}_{E/K} E^\times$. We thus attach an element $dV \in K^\times / \mathrm{Nm}_{E/K} E^\times \cong \mathbb{Z}/2\mathbb{Z}$ to V, known as the *discriminant*.

Proposition 2.6.1 (c.f. [20, Thm 2.67, §4.4])**.** *Two Hermitian spaces V and W associated to E/K are isometric if and only if they have the same dimension and same discriminant.*

Suppose now that K is a p-adic field. Since $K^\times / \operatorname{Nm}_{E/K} E^\times$ has order 2, there are two isometry classes of Hermitian space in each dimension. Moreover, the discriminant clearly satisfies the following property:

$$d(V \oplus W) = dV \cdot dW, \tag{2.6.1}$$

where by $V \oplus W$ we mean the orthogonal sum of V and W.

2.6.2 One dimensional unitary groups

If $V \cong E$, then the discriminant is given by the class of

$$\phi(1, 1) \in K^\times / \operatorname{Nm}_{E/K} E^\times \simeq \mu_2.$$

Hence there are two equivalence classes of one-dimensional unitary spaces. Nevertheless, the two unitary groups are isomorphic. An element $\alpha \in \mathrm{GL}(V) \cong E^\times$ will preserve ϕ if and only if $\operatorname{Nm}_{E/K}(\alpha) = 1$, regardless of the value of $\phi(1, 1)$. We refer to this group as $\mathrm{U}(E/K)$, or U_1 if the extension E/K is fixed.

2.6.3 Two dimensional unitary groups

Let V be a two dimensional Hermitian space. There are two possibilities:

(i) Suppose that there is an isotropic vector in V, namely some $v \in V$ with $v \neq 0$ and

$\phi(v, v) = 0$. Since ϕ is nondegenerate, for $w \in W$ not a multiple of v we have $\phi(v, w) \neq 0$. By adjusting $\alpha \in K$, we can force

$$\phi(w + \alpha v, w + \alpha v) = \phi(w, w) + \mathrm{Tr}_{E/K}(\alpha\phi(v, w))$$

to be zero, since $\mathrm{Tr}_{E/K}$ is surjective. By rescaling the resulting vector, we can find a $w \in V$ with $\phi(w, w) = 0$ and $\phi(v, w) = \phi(w, v) = 1$. Therefore, any two Hermitian spaces with an isotropic vector are isometric; we will call this space the *hyperbolic plane* associated to E/K and denote it by $\mathbb{H}$. The discriminant of the hyperbolic plane is clearly $d\mathbb{H} = -1$.

(ii) The other isometry class of two dimensional Hermitian spaces has no isotropic vector. One method for constructing it takes advantage of the fact that we know that its discriminant must be different from -1 modulo $\mathrm{Nm}_{E/K} E^\times$. For any $\alpha, \beta \in K^\times$ with $\alpha\beta \neq -1 \pmod{\mathrm{Nm}_{E/K} E^\times}$, we can give a basis v, w for V and define

$$\begin{aligned} \phi(v, v) &= \alpha \\ \phi(v, w) &= 0 \\ \phi(w, w) &= \beta \end{aligned} \tag{2.6.2}$$

Any two such spaces are isometric, and any Hermitian space not isometric to a hyperbolic plane will have bases $\{v, w\}$ satisfying (2.6.2) for any appropriate choice of α and β. We will refer to this other isometry class of two dimensional Hermitian space as the *anisotropic plane* and denote it by $\mathbb{B}$.

One can also start with a quaternion algebra $\mathbb{B}$ containing E and put the structure of a Hermitian space on it: see Gross [21, §5] and Springer [47, §17.1.4] for details. If this quaternion algebra is split we get an isomorphism with $\mathbb{H}$; the non-split case yields the anisotropic planes, and the associated group of unitary transformations can be identified with $\mathbb{B}^\times$.

Unlike the one dimensional case, the unitary groups associated to these two classes of Hermitian space are not isomorphic. We can prove this fact by noting that the unitary group of the hyperbolic plane has a K-split subtorus of dimension 1, while the unitary group of the anisotropic plane has no nontrivial K-split subtorus.

2.6.4 Higher dimensional unitary groups

We can express any higher dimensional Hermitian space as an orthogonal sum of the one and two dimensional spaces we've already defined. For our purposes, the following decomposition will be most useful:

Proposition 2.6.2. *Suppose V is a Hermitian space of dimension n. Then*

(i) if $n = 2m$, then either

$$V \cong \mathbb{H}^m \qquad dV \equiv (-1)^m$$

or

$$V \cong \mathbb{H}^{m-1} \oplus \mathbb{B} \qquad dV \not\equiv (-1)^m;$$

we will call such unitary groups even.

(ii) If $n = 2m + 1$*, then*

$$V \cong \mathbb{H}^m \oplus L \qquad\qquad dV \equiv (-1)^m dL$$

for some one dimensional Hermitian space L. We will call such unitary groups odd*.*

Proof. We obtain each possible discriminant and thus each isometry class of Hermitian space. □

Once again, the maximal K-split tori in the even-dimensional unitary groups have different dimensions, and thus there are two isomorphism classes of unitary group in even dimension. But when the dimension of V is odd, scaling the Hermitian form by an element of $K^\times - \mathrm{Nm}_{E/K} E^\times$ changes the discriminant but leaves the notion of unitary transformation invariant. So the two different isometry classes of Hermitian space in odd dimensions yield isomorphic unitary groups.

It will be useful to specify a basis for V in each case.

- When $V \cong \mathbb{H}^m$, let $\{v_i, v_{-i}\}$ be the standard basis for the i^{th} hyperbolic plane.

- When $V \cong \mathbb{H}^m \oplus L$, let $\{v_i, v_{-i}\}$ be the standard basis for the i^{th} hyperbolic plane, and $\{v_0\}$ a basis for L. We require $\phi(v_0, v_0) \in \mathcal{O}_K^\times$ if E/K is ramified, and $\phi(v_0, v_0) \in \{1, \pi_K\}$ if E/K is unramified.

- When $V \cong \mathbb{H}^{m-1} \oplus \mathbb{B}$, the notation is somewhat less ideal. We let $\{v_i, v_{-i}\}$ be the standard basis for the i^{th} hyperbolic plane, and then choose two orthogonal vectors $v_0, v_0' \in \mathbb{B}$; we can normalize the choice of v_0 and v_0' by imposing the same conditions as on v_0 above.

2.6.5 Special Unitary Groups

The determinant map $\mathrm{GL}(V) \to \mathbb{G}_m$ restricts to a homomorphism

$$\mathrm{U}(V) \to \mathrm{U}_1(E/K).$$

Just as we can define a semisimple group $\mathrm{SL}(V)$ as the kernel of the determinant, we define $\mathrm{SU}(V)$ as the subgroup of $\mathrm{U}(V)$ consisting of matrices with determinant 1.

Two special unitary groups $\mathrm{SU}(V)$ and $\mathrm{SU}(V')$ are isomorphic if and only if the corresponding unitary groups $\mathrm{U}(V)$ and $\mathrm{U}(V')$ are. We introduce special unitary groups as an example of a non-split semisimple group. Moreover, the fact that $\mathrm{SU}(V)$ is simply connected allows for the construction of non-regular parameters: see Section 5.5.3.

2.7 Reductive groups over p-adic fields

We can use our understanding of reductive groups over algebraically closed fields to study phenomena arising over non-algebraically closed fields. We assume now that K is a finite extension of $\mathbb{Q}_p$. Our goal in this section will be to use the action of Γ on $\mathbf{G}(\bar{K})$ and related structures to study algebraic structures defined over K. In particular, Galois cohomology will serve as one of our primary tools. See Springer [47, Ch. 11-17] for a treatment of these kinds of rationality questions.

2.7.1 Changing base fields

Now that we are working over K, we introduce two processes to change the field of definition of an algebraic group: base change and Weil restriction of scalars.

Suppose $\mathbf{G}$ is an affine group scheme over an arbitrary ring R, and that S is an R algebra. The base change of $\mathbf{G}$ to S is the fiber product of $\mathbf{G}$ with $\operatorname{Spec} S$, or in other words the group scheme over S represented by the S-algebra $R[\mathbf{G}] \otimes_R S$. When we want to emphasize the base ring, we will write $\mathbf{G}/S$ for this base change.

Restriction of scalars is the right adjoint to base change: if $\mathbf{G}$ is a group scheme over S then $\operatorname{Res}_{S/R} \mathbf{G}$ is a group scheme over R satisfying

$$\operatorname{Res}_{S/R} \mathbf{G}(A) = \mathbf{G}(A \otimes_R S)$$

for any R-algebra A. This functor is not always representable, but it is if S/R is a separable field extension and $\mathbf{G}$ is affine. In this case, we have the following result:

Proposition 2.7.1 (c.f. [47, Prop. 11.4.22]). *Let E/K be a separable field extension.*

(i) Suppose that $\mathbf{G}$ is an algebraic group over E. Then there is an isomorphism of Γ-modules

$$\operatorname{Res}_{E/K} \mathbf{G}(K_s) \cong \operatorname{Ind}_{\Gamma_E}^{\Gamma} \mathbf{G}(K_s),$$

where K_s is the separable closure of K.

(ii) Suppose that $\mathbf{T}$ is a torus defined over E. Then there are isomorphisms of Γ-modules

$$X^*(\operatorname{Res}_{E/K} \mathbf{T}) \cong \operatorname{Ind}_{\Gamma_E}^{\Gamma} X^*(\mathbf{T}),$$

$$X_*(\operatorname{Res}_{E/K} \mathbf{T}) \cong \operatorname{Ind}_{\Gamma_E}^{\Gamma} X_*(\mathbf{T}).$$

For example, we can interpret the unitary group $\mathrm{U}(V)$ associated to a Hermitian space V as a subgroup of the restriction of scalars $\operatorname{Res}_{E/K} \mathrm{GL}(V)$. This point of view will be useful in defining tori in $\mathrm{U}(V)$, since we can use the previous proposition to understand their character and cocharacter groups.

2.7.2 Tori

While there is a unique torus of each dimension over $\bar{K}$ (up to isomorphism), tori proliferate when we pass to K. For example, each quadratic extension E/K yields a one dimensional torus $\mathrm{U}_1(E/K)$; these groups all become isomorphic to $\mathbb{G}_m$ over $\bar{K}$, but different E yield non-isomorphic tori over K. The following proposition reduces the classification of tori over K to the classification of integral Galois representations:

Proposition 2.7.2. *There is an anti-equivalence of categories between*

(i) the category of tori $\mathbf{T}$ *over* K *and*

(ii) the category of finitely-generated free abelian groups X *over* K *equipped with a continuous action of* Γ,

given by

$$\mathbf{T} \mapsto X^*(\mathbf{T}).$$

Note that the elements of $X^*(\mathbf{T})$ are defined over $\bar{K}$, but not necessarily over K. Of course, we also have an action of Γ on $X_*(\mathbf{T})$, and the pairing between $X^*(\mathbf{T})$ and $X_*(\mathbf{T})$ is Γ-equivariant. We can read off many of the properties of $\mathbf{T}$ from $X^*(\mathbf{T})$ and $X_*(\mathbf{T})$: for example we can use (2.5.1) and the general fact that $\mathbf{G}(K) = \mathbf{G}(\bar{K})^\Gamma$ to show

$$\mathbf{T}(K) = (X_*(\mathbf{T}) \otimes \bar{K}^\times)^\Gamma.$$

This proposition allows us to determine that $\mathbb{G}_m$ and the $\mathrm{U}_1(E/K)$ exhaust the one dimensional tori over K: since $\mathrm{Aut}(\mathbb{Z}) = \{\pm 1\}$ any homomorphism from Γ must factor through $\mathrm{Gal}(E/K)$ for some quadratic extension E.

We say that $\mathbf{T}$ is *split* if Γ acts trivially on $X_*(\mathbf{T})$ and $X^*(\mathbf{T})$, or equivalently if there is a K-isomorphism of $\mathbf{T}$ with $\mathbb{G}_m^n$ for some n. At the other extreme, we say that $\mathbf{T}$ is *anisotropic* if Γ fixes no nonzero element of $X_*(\mathbf{T})$ or of $X^*(\mathbf{T})$, or equivalently if $\mathbf{T}$ contains no nontrivial split subtorus [47, Prop. 13.2.2].

While there is always a maximal torus in $\mathbf{G}$ that is defined over K [47, Thm. 13.3.6], for a general reductive group there may be no maximal torus in $\mathbf{G}$ that is also split. If $\mathbf{G}$ contains a K-split maximal torus, we say that $\mathbf{G}$ itself is *split*. More generally, we define a *maximal K-split torus* in $\mathbf{G}$ to be a subtorus that is maximal among those that are split over K. Not all maximal tori in $\mathbf{G}$ are conjugate over K, but any two maximal K-split tori are [47, Thm. 15.2.6]. Moreover, we can find a maximal K-torus containing any given maximal K-split torus $\mathbf{A} \subset \mathbf{G}$, since $\mathrm{Z}_{\mathbf{G}}(\mathbf{A})$ contains a maximal torus defined over K.

Consider an n-dimensional Hermitian space V associated to E/K. By Proposition 2.6.2, we need to examine three cases:

(i) First consider $V = \mathbb{H}$ with the standard basis $\{v_{-1}, v_1\}$. Let $\mathbf{S}'$ be the maximal torus in $\mathrm{GL}(V)$ consisting of those matrices with v_{-1} and v_1 as eigenvectors. Define $\mathbf{S} \subset \mathrm{U}(V)$ as the intersection of $\mathrm{Res}_{E/K}\,\mathbf{S}'$ with $\mathrm{U}(V) \subset \mathrm{Res}_{E/K}\,\mathrm{GL}(V)$. Then $\mathbf{S}$ is isomorphic to $\mathrm{Res}_{E/K}\,\mathbb{G}_m$, with K points consisting of those matrices scaling v_1 by $\alpha \in E^\times$ and v_{-1} by $\tau(\alpha)^{-1}$. We now define $\mathbf{A}$ as the maximal K-split subtorus of $\mathbf{S}$, which is also a maximal K-split subtoris in $\mathrm{U}(V)$. The K-points of $\mathbf{A}$ consist of those matrices scaling v_1 by $\alpha \in K^\times$ and v_{-1} by α^{-1}.

By Proposition 2.7.1, we can choose a basis $\{\chi_1, \chi_{-1}\}$ of $X^*(\mathbf{S})$ so that $\tau \in \mathrm{Gal}(E/K)$ acts by $\tau(\chi_1) = -\chi_{-1}$ (the sign is a convenience to unify the notation for odd and even dimensional unitary groups), and a dual basis $\{\lambda_1, \lambda_{-1}\}$ of $X_*(\mathbf{S})$. Note that χ_1 and χ_{-1} are not the characters that pick out the eigenvalues of v_1 and v_{-1}. Rather, $X_*(\mathbf{A})$ is spanned by $\lambda_1 - \lambda_{-1}$, and restriction induces an orthogonal projection $X^*(\mathbf{S}) \to X^*(\mathbf{A})$ with kernel spanned by $\chi_1 + \chi_{-1}$ and leaving $\chi_1 - \chi_{-1}$ fixed. We identify $X^*(\mathbf{A})$ with the span of $\chi_1 - \chi_{-1}$. The complementary subspace of $X^*(\mathbf{S})$ spanned by $\chi_1 + \chi_{-1}$ also corresponds to a subtorus $\mathbf{A}'$ of dimension 1; it is defined over K since the span of $\chi_1 + \chi_{-1}$ is stabilized by $\mathrm{Gal}(E/K)$. Since $\tau \in \mathrm{Gal}(E/K)$ negates $\chi_1 + \chi_{-1}$, $\mathbf{A}'$ is isomorphic to U_1. In fact $\mathbf{A}'$ is the maximal anisotropic K-torus in $\mathbf{S}$, $\mathbf{S} = \mathbf{A} \cdot \mathbf{A}'$ and $\mathbf{A} \cap \mathbf{A}' = \{\pm 1\}$ in accordance with [47, Prop. 13.2.4].

Passing to $V = \mathbb{H} \oplus \cdots \oplus \mathbb{H}$ of dimension $2m$ now only involves more indices. For the analogous torus $\mathbf{S}$, we can write down a basis $\{\chi_{-m}, \ldots, \chi_{-1}, \chi_1, \ldots, \chi_m\}$ of $X^*(\mathbf{S})$ with dual basis $\{\lambda_{-m}, \ldots, \lambda_{-1}, \lambda_1, \ldots, \lambda_m\}$ for $X_*(\mathbf{T})$ so that Γ acts through its quotient $\mathrm{Gal}(E/K)$, with the nontrivial element of $\mathrm{Gal}(E/K)$ mapping χ_i to $-\chi_{-i}$ and λ_i to $-\lambda_{-i}$. All of the analogues of the results for $V = \mathbb{H}$ hold in this case as well, as these tori are just the sums of m copies of the tori in the $V = \mathbb{H}$ case. We can identify $\mathbf{A}$ and $\mathbf{S}$:

$$\mathbf{A} \cong (\mathbb{G}_m)^m,$$

$$\mathbf{S} \cong (\mathrm{Res}_{E/K} \mathbb{G}_m)^m.$$

(ii) $V \cong \mathbb{H} \oplus \cdots \oplus \mathbb{H} \oplus L$ of dimension $2m+1$. We construct $\mathbf{S}$ in this case similarly. Let

v_i, v_{-i} be standard basis vectors for the hyperbolic planes in V, and let v_0 span L. Then $\mathbf{S}'$ is again the torus in $\mathrm{GL}(V)$ whose elements have our basis as eigenvectors, and $\mathbf{S}$ is the intersection of $\mathrm{Res}_{E/K}\mathbf{S}'$ with $\mathrm{U}(V)$. We may choose a basis $\{\chi_{-m}, \ldots, \chi_m\}$ for $X^*(\mathbf{S})$, with Γ again acting through $\mathrm{Gal}(E/K)$ with τ mapping χ_i to $-\chi_{-i}$ and λ_i to $-\lambda_{-i}$. The maximal K-split torus $\mathbf{A}$ is the same as the $\mathbb{H} \oplus \cdots \oplus \mathbb{H}$ case, but $\mathbf{S}$ has an extra U_1 coming from χ_0. We can identify $\mathbf{A}$ and $\mathbf{S}$ again:

$$\mathbf{A} \cong (\mathbb{G}_m)^m,$$

$$\mathbf{S} \cong (\mathrm{Res}_{E/K}\mathbb{G}_m)^m \times \mathrm{U}_1(E/K).$$

(iii) Now consider $V = \mathbb{B}$. The associated unitary group is anisotropic, and there is now no canonical choice of K-conjugacy class of maximal tori. Similarly, there are multiple K-conjugacy classes of maximal tori in the unitary group associated to $\mathbb{H} \oplus \cdots \oplus \mathbb{H} \oplus \mathbb{B}$. See Section 2.7.5 for more discussion of these cases.

We can use this analysis of these maximal tori in $\mathrm{U}(V)$ to give a description of the maximal tori in $\mathrm{SU}(V)$ that lie within a rational Borel. Suppose first that V has dimension $2m+1$, and let $\mathbf{S} \cong (\mathrm{Res}_{E/K}\mathbb{G}_m)^m \times \mathrm{U}_1(E/K)$ be the maximal torus in $\mathrm{U}(V)$ described above. The determinant restricts to a homomorphism $\mathbf{S} \to \mathrm{U}_1(E/K)$ that is surjective even on K-points, and thus the intersection of $\mathbf{S}$ with $\mathrm{SU}(V)$ is just $(\mathrm{Res}_{E/K}\mathbb{G}_m)^m$. This intersection is a quasi-split maximal torus in $\mathrm{SU}(V)$.

If V has dimension $2m$, then the best description of this maximal torus is just the set of elements in $(\mathrm{Res}_{E/K}\mathbb{G}_m)^m$ with determinant 1. The character group of this torus is the quotient of $X^*(\mathbf{S})$ by the subgroup spanned by $\chi_{-m} + \cdots \chi_{-1} + \chi_1 + \cdots \chi_m$.

2.7.3 Rational forms

The general approach of classifying objects over an algebraic closure $\bar{K}$ of K and then determining how these isomorphism classes break up upon passing to K is known as *Galois descent*. See Conrad's notes [15] and Springer [47, §11.1] for expositions of Galois descent for vector spaces and algebras, and Springer [47, §12.3] and Serre [46, §1.5, 3.1] for Galois descent for schemes and algebraic groups. See Vogan [51, §2] for a description of rational forms of algebraic groups, including pure inner forms.

Given a connected reductive algebraic group $\mathbf{G}$ defined over an arbitrary field K, we say that another algebraic group $\mathbf{G}'$ over K is a *rational form* of $\mathbf{G}$ if $\mathbf{G}$ and $\mathbf{G}'$ become isomorphic after base changing to $\bar{K}$. If we start with $\mathbf{G}$, we can parameterize all rational forms of $\mathbf{G}$ using Galois descent:

Theorem 2.7.3. *The rational forms of* $\mathbf{G}$ *are in bijection with the cohomology set* $\mathrm{H}^1(K, \mathrm{Aut}(\mathbf{G}))$. *Given a cocycle* ξ *representing a class in* $\mathrm{H}^1(K, \mathrm{Aut}(\mathbf{G}))$*, the associated form of* $\mathbf{G}$ *is given by defining a new action of* Γ *on* $\mathbf{G}(K)$ *by setting*

$$\sigma \cdot_\xi g = \xi(\sigma)(\sigma \cdot g)$$

for $\sigma \in \Gamma$ *and* $g \in \mathbf{G}(\bar{K})$.

Conversely, Γ *acts on the set of isomorphisms from* $\mathbf{G}$ *to* $\mathbf{G}'$*:* $\psi^\sigma(g) = \sigma^{-1} \cdot \psi(\sigma \cdot g)$. *Given a rational form* $\mathbf{G}'$ *of* $\mathbf{G}$ *and an isomorphism* $\psi \colon \mathbf{G} \to \mathbf{G}'$*, we associate to* $\mathbf{G}'$ *the class in* $\mathrm{H}^1(K, \mathrm{Aut}(\mathbf{G}))$ *of the cocycle* ξ *defined by*

$$\xi(\sigma) = \psi^\sigma \circ \psi^{-1}.$$

Among all rational forms of $\mathbf{G}$, the *inner forms* are the ones for which the associated cocycle takes values in the inner subgroup $\mathrm{Inn}(\mathbf{G}) \subseteq \mathrm{Aut}(\mathbf{G})$. The inner automorphisms in $\mathrm{Aut}(\mathbf{G})$ are naturally isomorphic to $\mathbf{G}_{\mathrm{ad}}$, so the inner forms are precisely those rational forms corresponding to elements in the image of

$$\mathrm{H}^1(K, \mathbf{G}_{\mathrm{ad}}) \to \mathrm{H}^1(K, \mathrm{Aut}(\mathbf{G})).$$

Let $\mathrm{Out}(\mathbf{G}) = \mathrm{Aut}(\mathbf{G})/\mathbf{G}_{\mathrm{ad}}$. The inner forms of $\mathbf{G}$ will be parameterized by $\mathrm{H}^1(K, \mathbf{G}_{\mathrm{ad}})$ if and only if the map

$$\mathrm{Aut}(\mathbf{G})^\Gamma \to \mathrm{Out}(\mathbf{G})^\Gamma$$

is surjective. This condition holds for quasi-split $\mathbf{G}$ [17, §3.10].

We can also ask for a more concrete description of $\mathrm{Out}(\mathbf{G})$. Let $\mathcal{D}$ be the Dynkin diagram of $\mathbf{G}$ associated to $\mathbf{T}$ and $\mathbf{B}$. An automorphism of $\mathbf{G}$ stabilizing $\mathbf{T}$ and $\mathbf{B}$ induces an automorphism of $\mathcal{D}$; the resulting homomorphism $\mathrm{Aut}(\mathbf{G}) \to \mathrm{Aut}(\mathcal{D})$ factors through $\mathrm{Out}(\mathbf{G})$ [17, §3.5]. For semisimple $\mathbf{G}$, $\mathrm{Out}(\mathbf{G})$ injects into $\mathrm{Aut}(\mathcal{D})$, and for simply connected or adjoint $\mathbf{G}$ we get an isomorphism [47, Lem. 16.3.8]. For general reductive groups, there are far too many automorphisms for the same results to hold: $\mathrm{Aut}(\mathbf{G})$ and $\mathrm{Out}(\mathbf{G})$ are not even finite type group schemes over K. For example, in the case of a torus $\mathbf{T} \cong (\mathbb{G}_m)^n$, every automorphism is outer and $\mathrm{Aut}(\mathbf{T}) \cong \mathrm{GL}_n(\mathbb{Z})$. In general, any automorphism of $\mathbf{G}$ induces an automorphism of the derived subgroup, and the map $\mathrm{Aut}(\mathbf{G}) \to \mathrm{Aut}(\mathcal{D})$ of group schemes will be surjective if the derived subgroup of $\mathbf{G}$ is simply connected or adjoint.

A *pure inner form* of $\mathbf{G}$ is a continuous homomorphism

$$\delta\colon \Gamma \to \mathbf{G} \rtimes \Gamma$$

whose projection onto Γ is the identity. The pure inner forms of a group $\mathbf{G}$ are parameterized by $\mathrm{H}^1(K, \mathbf{G})$ [51, Prop. 2.7].

We can associate a rational form to each pure inner form by defining a map $r(\delta)\colon \Gamma \to \mathrm{Aut}(\mathbf{G}(\bar{K}))$ by letting $r(\delta)(\sigma)$ be conjugation by $\delta(\sigma)$. In fact, the resulting rational form is an inner form of $\mathbf{G}$ [51, Prop. 2.7], and this association corresponds to the map

$$\mathrm{H}^1(K, \mathbf{G}) \to \mathrm{H}^1(K, \mathbf{G}_{\mathrm{ad}}) \tag{2.7.1}$$

induced by the quotient $\mathbf{G} \to \mathbf{G}_{\mathrm{ad}}$.

Note that in general the map in (2.7.1) need not be either injective or surjective. Consider the case that $\mathbf{G} = \mathrm{U}(V)$ is the unitary group associated to a Hermitian space over a local field K. Note that $\mathrm{U}(V)$ is that automorphism group of V with its Hermitian form; thus $\mathrm{H}^1(K, \mathrm{U}(V))$ classifies K-isomorphism classes of Hermitian spaces. There are two isomorphism classes of Hermitian space over K, so there are two isomorphism classes of pure inner forms of $\mathbf{G}$. Yet when the dimension of V is odd, there is only a single isomorphism class of unitary group.

For an example where not every inner form comes from a pure inner form, consider GL_n. Since $\mathrm{H}^1(K, \mathrm{GL}_n) = 0$, GL_n has no pure inner forms, but it does have inner forms arising from quaternion algebras [47, §17.1.1].

Despite the failure of (2.7.1) to be a bijection, we will frequently refer to the rational form associated a pure inner form of $\mathbf{G}$ as just a pure inner form of $\mathbf{G}$.

2.7.4 Root systems over K

Once again, let K be a p-adic field, let $\mathbf{A}$ be a maximal K-split torus in $\mathbf{G}$ and let $\mathbf{S}$ be a maximal K-torus containing $\mathbf{A}$. The character and cocharacter groups of $\mathbf{A}$ and $\mathbf{S}$ are related. The inclusion $\mathbf{A} \subset \mathbf{S}$ induces a natural injection of cocharacter groups $X_*(\mathbf{A}) \hookrightarrow X_*(\mathbf{S})$. Let

$$X_*(\mathbf{A})^\perp = \{\chi \in X^*(\mathbf{S}) \mid \langle \chi, \lambda \rangle = 0 \ \ \forall \lambda \in X_*(\mathbf{A})\}.$$

Restriction of characters induces a projection map $X^*(\mathbf{S}) \to X^*(\mathbf{A})$, orthogonal with respect to a $\mathbf{W}$-invariant inner product on $X^*(\mathbf{S})$. The kernel of this map is precisely $X_*(\mathbf{A})^\perp$ [47, §15.3].

We call the weights of the torus $\mathbf{A}$ the *K-roots* and denote them $\Phi(\mathbf{G}, \mathbf{A})$, or $\Phi_{\mathbf{A}}$. Restriction of characters maps $\Phi = \Phi(\mathbf{G}, \mathbf{S})$ to $\Phi_{\mathbf{A}} \cup \{0\}$; let $\Phi_\perp$ be the subset of Φ mapping to $0 \in X^*(\mathbf{A})$. For any choice of positive system $\Phi_{\mathbf{A}}^+$ in $\Phi_{\mathbf{A}}$, there is a positive system Φ^+ in Φ so that an element of $\Phi - \Phi_\perp$ lies in Φ^+ if and only if its restriction to $\mathbf{A}$ lies in $\Phi_{\mathbf{A}}^+$. Moreover, every simple root in $\Delta \subset \Phi^+$ projects to either a simple root in $\Delta_{\mathbf{A}} \subset \Phi_{\mathbf{A}}^+$ or 0; let $\Delta_\perp$ denote the set of simple roots mapping to 0.

Γ acts on Φ, and each $\gamma \in \Gamma$ maps Φ^+ to another positive system. There is a unique $w_\gamma \in \mathbf{W}$ with $w_\gamma(\gamma.\Phi^+) = \Phi^+$, and thus $w_\gamma(\gamma.\Delta) = \Delta$. Define a homomorphism $\epsilon\colon \Gamma \to \mathrm{Aut}(\Delta)$ by

$$\epsilon(\gamma)(\chi) = w_\gamma(\gamma.\chi).$$

$\epsilon(\gamma)$ in fact yields an automorphism of the Dynkin diagram with vertices Δ, and stabilizes

$\Delta_\perp$ and $\Delta - \Delta_\perp$.

We consider the root systems of low-dimensional unitary groups in order to illustrate these definitions. Figure 2.1 shows the root system of U_3, which has type A_2, and the projection from $X^*(\mathbf{S})$ to $X^*(\mathbf{A})$.

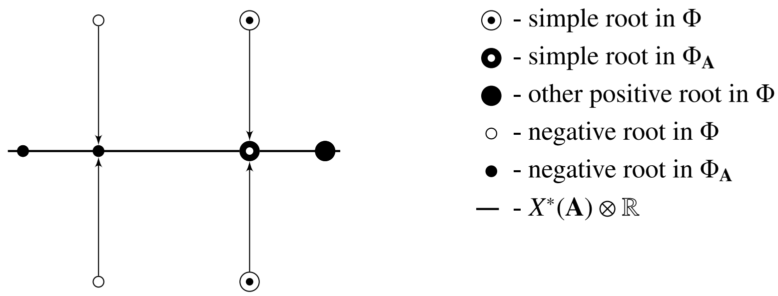

Figure 2.1: The root system of SU_3

The nontrivial element of $\mathrm{Gal}(E/K)$ acts by reflection across the x-axis, and $X^*(\mathbf{A})\otimes\mathbb{R}$ is precisely the subspace fixed by this action. Note that the choices of positive system match, and the simple roots in Φ map to the simple root in $\Phi_{\mathbf{A}}$. In this case, there is a unique positive system in $X^*(\mathbf{S})$ projecting onto each positive system of $X^*(\mathbf{A})$. The automorphism of the simple roots defined by τ induces the nontrivial automorphism of the Dynkin diagram A_2.

The root system of $X^*(\mathbf{A})$ in this case is of type BC_1 and is not reduced; see Springer [47, §15.3.9] for a brief overview of the differences between reduced and non-reduced root systems.

We next consider the two different isomorphism classes of four dimensional unitary groups. First suppose that $V = \mathbb{H} \oplus \mathbb{H}$, $\mathbf{A}$ is a maximal K-split torus as usual and $\mathbf{S}$ is the maximal K-torus containing $\mathbf{A}$. The root system of $\mathbf{S}$ is of type A_3, which we can visualize as consisting of the midpoints of the edges of a cube: see Figure 2.2.

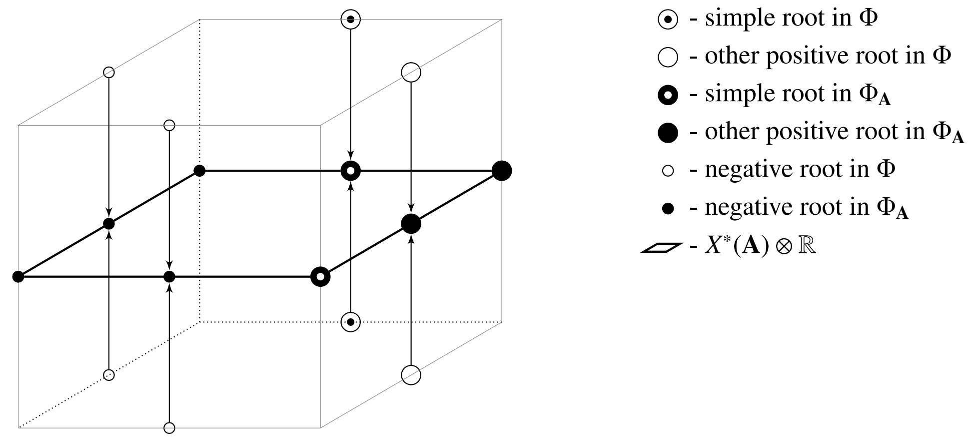

Figure 2.2: The root system of a quasi-split SU_4

This time $\mathbf{A}$ has dimension 2, and the root system $\Phi(\mathbf{G}, \mathbf{A})$ has type C_2. Here the nontrivial element of $\mathrm{Gal}(E/K)$ acts by reflection across the horizontal plane in Figure 2.2. Once again the positive system Φ^+ is stabilized by this action and is the unique positive system projecting onto $\Phi^+_{\mathbf{A}}$. The homomorphism ϵ is given just by the induced permutation action on the simple roots Δ.

Finally, consider the other four dimensional unitary group, associated to the Hermitian space $V = \mathbb{H} \oplus \mathbb{B}$. There are now multiple conjugacy classes of maximal tori to choose; we pick an $\mathbf{S}$ containing a maximal K-split torus $\mathbf{A}$ and isomorphic to $\mathbf{A} \times \mathrm{U}_1 \times \mathrm{U}_1$. Figure 2.3 shows the projection from $X^*(\mathbf{S})$ to $X^*(\mathbf{A})$, as well as the action of Γ, which acts on $X^*(\mathbf{S})$ through $\mathrm{Gal}(E/K)$.

The root system $\Phi(\mathbf{G}, \mathbf{A})$ is once again of type BC_1, but this time $\Phi_\perp$ is nonempty. As a consequence, the positive system Φ^+ is not fixed by Γ, but is instead mapped by τ to another potential choice of positive system. In order to return to Φ^+, we can compose the action of τ with the element of $\mathbf{W}$ defined by reflection in the plane shown on the lower cube. This composition stabilizes Δ (as well as $\Delta_\perp$ and $\Delta - \Delta_\perp$), and defines the homomorphism ϵ, which again maps τ to the nontrivial automorphism of the Dynkin diagram of A_3.

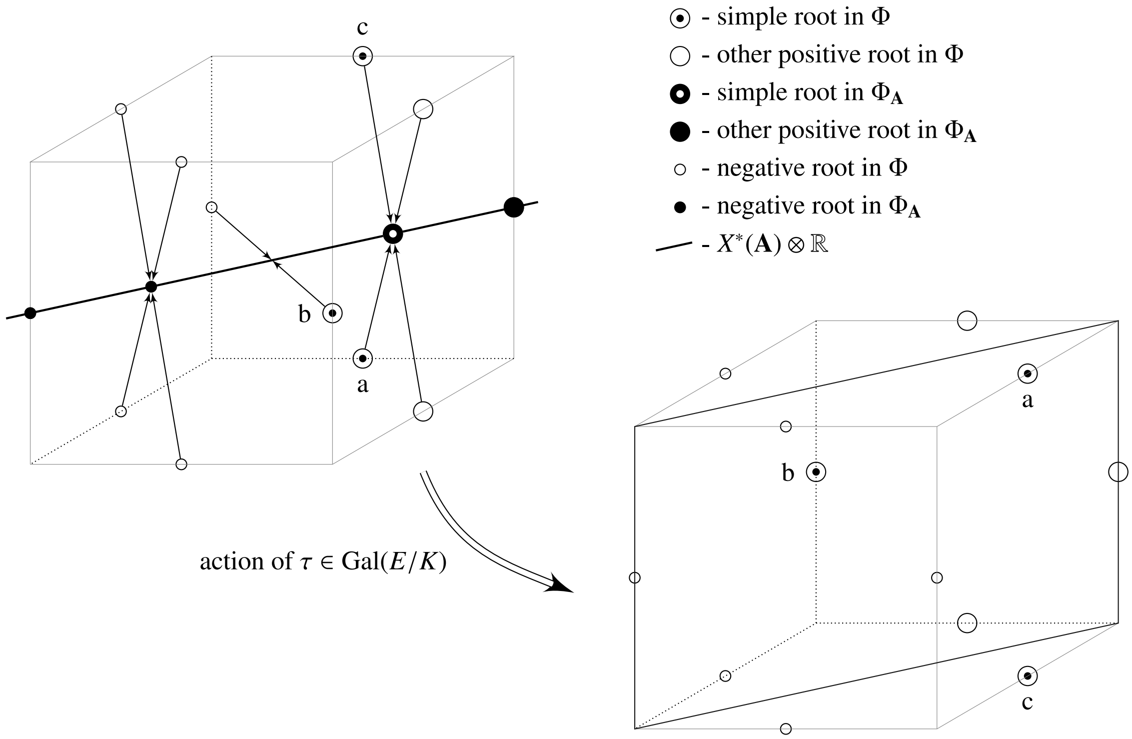

Figure 2.3: The root system of a non-quasi-split SU_4

In general, the root system $\Phi_{\mathbf{A}}$ for $U(V)$ is of type C_m when $V \cong \mathbb{H}^m$, and is of type BC_m when $V \cong \mathbb{H}^m \oplus L$ or $V \cong \mathbb{H}^m \oplus \mathbb{B}$ [47, §15.3.10].

We close this section with a definition of the indexed root datum attached to a reductive group, which will play a similar role for groups over K that the root datum plays for groups over $\bar{K}$. For an algebraic group $\mathbf{G}$ defined over K with maximal torus $\mathbf{S}$ (defined over K and containing a maximal K-split torus), the *indexed root datam* of $\mathbf{G}$ is a sextuple

$$(X^*(\mathbf{S}), \Delta, X_*(\mathbf{S}), \Delta^\vee, \Delta_\perp, \epsilon).$$

Here Δ is a basis of simple roots in $X^*(\mathbf{S})$, and $\Delta_\perp$ and ϵ are as defined previously in this section. See [47, Ch. 16-17] for a classification of reductive groups over K using indexed root data.

2.7.5 Quasi-split groups

None of our unitary groups are split, but some of them satisfy a weaker condition that will still prove useful. If $\mathbf{G}$ were split then in its indexed root datum we would have a trivial ϵ and $\Delta_\perp = \emptyset$. We say that a group $\mathbf{G}$ over K is *quasi-split* if one of the following equivalent conditions hold:

Proposition 2.7.4 (c.f. [47, Prop. 16.2.2]). *The following are equivalent:*

(i) the set $\Delta_\perp$ *is empty,*

(ii) the centralizer $\mathrm{Z}_{\mathbf{G}}(\mathbf{A})$ *of any maximal* K*-split torus* $\mathbf{A}$ *is a maximal torus,*

(iii) there is a Borel subgroup of $\mathbf{G}$ *that is defined over* K*.*

Note that since the conjugacy class of maximal K-split tori is uniquely determined, the second criteria for $\mathbf{G}$ to be quasi-split allows us to pick out a $\mathbf{G}(K)$ conjugacy class of maximal tori, which we will refer to as the *quasi-split maximal torus* in $\mathbf{G}$. At the other extreme, we say that $\mathbf{G}$ is *anisotropic* if $\Delta_\perp = \Delta$, or equivalently if $\mathbf{G}$ contains no nontrivial K-split tori.

We can always obtain a quasi-split form by twisting:

Proposition 2.7.5 (c.f. [47, Prop. 16.4.9]). *Any connected reductive group* $\mathbf{G}$ *over* K *has an inner form which is quasi-split.*

Referring back to our description in Section 2.7.2 of the maximal K-split tori in unitary groups for K a p-adic field, we see that the odd-dimensional unitary group is quasi-split, and the unitary group associated to an even-dimensional Hermitian space V is quasi-split if and only if V has an isotropic subspace of half its dimension (or equivalently, if V is the sum of hyperbolic planes). In the case that $V = \mathbb{H} \oplus \cdots \oplus \mathbb{H} \oplus \mathbb{B}$, the centralizer of a maximal K-split torus $\mathbf{A}$ is isomorphic to $\mathbf{A} \times \mathrm{U}(\mathbb{B})$, and thus contains many different maximal tori. For unitary groups the map

$$\mathrm{H}^1(K, \mathbf{G}) \to \mathrm{H}^1(K, \mathbf{G}_{\mathrm{ad}})$$

is surjective since $\mathrm{H}^1(K, \mathrm{U}_1) = 0$ by Tate duality (see Proposition 2.9.1), and thus any unitary group has a pure inner form that is quasi-split.

The first diagram in Figure 2.3 gives some insight into the equivalence of (i) and (iii) in Proposition 2.7.4. The existence of roots projecting to 0 allows us to find multiple positive systems whose intersection with $X^*(\mathbf{A})$ is the same positive system $\Phi^+_{\mathbf{A}}$. The action of Γ on $X^*(\mathbf{S})$ permutes these systems. On the other hand, if every root of $X^*(\mathbf{S})$ projects to a root in $\Phi(\mathbf{G}, \mathbf{A})$, then there is a unique positive system Φ^+ projecting to $\Phi^+_{\mathbf{A}}$; since Φ^+ is stabilized by Γ, the associated Borel is defined over K.

2.7.6 Maximal Tori over K

We close our discussion of reductive groups over K by returning to tori and considering what we can say about maximal tori in a given reductive group $\mathbf{G}$. We have seen in Section 2.7.2 that there is a unique conjugacy class of maximal K-split tori $\mathbf{A}$; we pick one arbitrarily. We now assume that $\mathbf{G}$ is quasi-split, so that the centralizer $\mathbf{S} = \mathrm{Z}_{\mathbf{G}}(\mathbf{A})$ will be a maximal torus. Our first goal in this section will be to describe the other K-tori in $\mathbf{G}$ as twists of this fixed maximal torus. See Reeder [43, §6] for a more extensive discussion of this question.

We say that two tori $\mathbf{S}_1$ and $\mathbf{S}_2$ are *rationally conjugate* if there is an element of $\mathbf{G}(K)$ conjugating $\mathbf{S}_1(K)$ to $\mathbf{S}_2(K)$, and *stably conjugate* if there is an element of $\mathbf{G}(\bar{K})$ conjugating $\mathbf{S}_1(K)$ to $\mathbf{S}_2(K)$. These notions partition the K-tori in $\mathbf{G}$ into stable conjugacy classes, and each stable conjugacy class into rational conjugacy classes. The normalizer $\mathbf{N}$ of $\mathbf{S}$ in $\mathbf{G}$ acts by conjugation on $\mathbf{S}$, and the quotient $\mathbf{W} = \mathbf{N}/\mathbf{S}$ is the Weyl group of $\mathbf{S}$ (and of $\mathbf{A}$). We have maps

$$\mathrm{H}^1(K, \mathbf{N}) \to \mathrm{H}^1(K, \mathbf{W}),$$

induced by the projection $\mathbf{N} \to \mathbf{W}$, and

$$\mathrm{H}^1(K, \mathbf{N}) \to \mathrm{H}^1(K, \mathbf{G}),$$

induced by the inclusion $\mathbf{N} \to \mathbf{G}$. These cohomology groups give us a parameterization of the K-conjugacy classes of maximal tori in $\mathbf{G}$ and its pure inner forms.

Proposition 2.7.6 (c.f. [43, Prop 6.1] and [46, Cor. 2 of Prop. I.36])**.** *Let* $\mathbf{G}$ *be a quasi-split group over* K*.*

(i) The rational classes of maximal tori in the twist of $\mathbf{G}$ *corresponding to a cocycle* $\xi \in \mathrm{Z}^1(K, \mathbf{G})$ *are in bijection with the set* R_ξ *of cohomology classes in* $\mathrm{H}^1(K, \mathbf{N})$ *mapping to the class of* ξ *in* $\mathrm{H}^1(K, \mathbf{G})$*. In particular, the rational classes of maximal tori in* $\mathbf{G}$ *are in bijection with the kernel of* $\mathrm{H}^1(K, \mathbf{N}) \to \mathrm{H}^1(K, \mathbf{G})$*.*

(ii) The stable classes of maximal tori in the twist of $\mathbf{G}$ *corresponding to* ξ *are in bijection with the image in* $\mathrm{H}^1(K, \mathbf{W})$ *of* R_ξ*.*

(iii) *The stable classes of maximal tori in* $\mathbf{G}$ *are in bijection with* $\mathrm{H}^1(K, \mathbf{W})$.

We will use this proposition to understand the tori in unitary groups in Chapter 3. Toward this end, we will also find it useful to have a description of the action of Γ on $\mathbf{W}$ in the case that $\mathbf{G}$ is a quasi-split unitary group of dimension $n = 2m$ or $n = 2m + 1$. Recall from Section 2.7.2 that $X^*(\mathbf{S})$ has basis $\{\chi_{-m}, \ldots, \chi_m\}$ (including χ_0 if n is odd), and that Γ acts through $\mathrm{Gal}(E/K)$, with τ mapping χ_i to $-\chi_{-i}$. If we consider $\mathbf{W}$ as a group of automorphisms of $X^*(\mathbf{S})$, then Γ acts on $\mathbf{W}$ by conjugation. In fact, this action is inner: the action of τ on $\mathbf{W}$ is induced by conjugation by an element of $\mathbf{W}$.

Let ω_i be the reflection in the root $\chi_i - \chi_{-i}$. Define $\omega \in \mathbf{W}$ as the product of these commuting reflections:

$$\omega = \prod_{i=1}^{m} \omega_i.$$

A straightforward computation shows that the action of $\mathrm{Gal}(E/K)$ on $\mathbf{W}$ is given by

$$\tau.\sigma = \omega\sigma\omega$$

for any $\sigma \in \mathbf{W}$.

As an application, we see that the rational Weyl group $\mathbf{W}(K) = \mathbf{W}^\Gamma$ is given by the centralizer of ω. We will see in Proposition 3.3.1 that this centralized is isomorphic to the Weyl group of the root system C_m, lining up with the fact mentioned in Section 2.7.4 that $\Phi(\mathbf{G}, \mathbf{A}) \cong \mathrm{C}_m$.

2.8 L-Groups and Langlands Parameters

In this section we generalize the notion of a Galois representation to reductive groups $\mathbf{G}$ other than GL_n. As the first step in this process, we replace the image $\mathrm{GL}_n(\mathbb{C})$ of an n-dimensional Galois representation with a different algebraic group ${}^L\mathbf{G}$.

2.8.1 The L-group

Recall from Section 2.5.8 that $\hat{\mathbf{G}}$ is the group over $\mathbb{C}$ defined from $\mathbf{G}$ by exchanging the roles of characters and cocharacters, roots and coroots.

Note that the connected Langlands dual group depends only on $\mathbf{G}(\bar{K})$ and not on the rational form of $\mathbf{G}$. In order to see the rational structure, we define an L-group associated to $\mathbf{G}$. In order to avoid dealing with non-finite-type group schemes, we take a slightly different approach than Vogan [51, §3].

Suppose that $\mathbf{G}$ is quasi-split and $\mathbf{A}$ is a maximal K-split torus in $\mathbf{G}$. Then $\mathbf{S} = \mathrm{Z}_{\mathbf{G}}(\mathbf{A})$ is a maximal torus of $\mathbf{G}$ defined over K. Since $\mathbf{G}$ is quasi-split, both $\mathbf{A}$ and $\mathbf{S}$ are uniquely defined up to conjugacy. We can thus define a canonical splitting field E for $\mathbf{G}$: E is the extension of K so that Γ acts on $X^*(\mathbf{S})$ through the quotient $\mathrm{Gal}(E/K)$. We may also choose a Borel $\mathbf{B}$ defined over K, corresponding to a $\mathrm{Gal}(E/K)$ stable basis Δ of $\Phi(\mathbf{G}, \mathbf{S})$.

Our objective is to give an action of $\mathrm{Gal}(E/K)$ on $\hat{\mathbf{G}}$. We define a *pinning* as a choice of basis vector for each simple root space in the Lie algebra $\hat{\mathfrak{g}}$ of $\hat{\mathbf{G}}$. For a given choice of pinning, each element of $\mathrm{Gal}(E/K)$ acts on Δ, defines a permutation of the corresponding simple root spaces and thus gives an automorphism of the pinning. This automorphism induces an automorphism of $\hat{\mathbf{G}}$, yielding an action of $\mathrm{Gal}(E/K)$ on $\hat{\mathbf{G}}$. For an exposition of these pinned automorphisms see [42, §3.1]. Note that our choice of maximal torus $\mathbf{S}$ and

Borel $\mathbf{B}$ also defines the corresponding structures $\hat{\mathbf{S}}$ and $\hat{\mathbf{B}}$ in $\hat{\mathbf{G}}$.

Definition 2.8.1. Fix $\mathbf{B} \subset \mathbf{G}$ and a choice of pinning. Then the *L-group* of $\mathbf{G}$ is defined by

$$^L\mathbf{G} = \hat{\mathbf{G}} \rtimes \mathrm{Gal}(E/K).$$

We say that an element $g \in {}^L\mathbf{G}$ is *semisimple* if $g^{[E:K]} \in \hat{\mathbf{G}}$ is semisimple.

2.8.2 Langlands Parameters

For the rest of this section, assume that K is a p-adic field. For split groups, we can generalize Galois representations by considering homomorphisms from WD_K to $\hat{\mathbf{G}}$. The appearance of WD_K rather than $\mathcal{W}_K$ comes from the need for more Langlands parameters to match those coming from parabolic induction on the representation theoretic side. However, the representations we will construct in this thesis will restrict trivially to the $\mathrm{SL}_2(\mathbb{C})$ component of WD_K.

For split $\mathbf{G}$ such as GL_n, we can think of $\mathcal{W}_K$ as acting trivially on $\hat{\mathbf{G}}$. So there's a bijection between homomorphisms up to conjugacy from $\mathcal{W}_K$ to $\hat{\mathbf{G}}$ and $\mathrm{H}^1(\mathcal{W}_K, \hat{\mathbf{G}})$. This provides a model for our definition of a Langlands parameter in the non-split case.

Definition 2.8.2. A *Langlands parameter* is a homomorphism

$$\varphi\colon \mathrm{WD}_K \to {}^L\mathbf{G}$$

such that

(i) the image of any element of $\mathcal{W}_K$ is semisimple;

(ii) the restriction of φ to $\mathrm{SL}_2(\mathbb{C}) \subset \mathrm{WD}_K$ is a morphism of varieties;

(iii) the composition of φ with the projection ${}^L\mathbf{G} \to \mathrm{Gal}(E/K)$ is the standard projection $\mathrm{WD}_K \to \mathrm{Gal}(E/K)$.

We say that two parameters are equivalent if they differ by conjugation by an element of $\hat{\mathbf{G}}$.

Let $\mathbb{L}(\mathbf{G}/K)$ be the set of equivalence classes of Langlands parameters with values in ${}^L\mathbf{G}$. We will call elements of $\mathbb{L}(\mathbf{G}/K)$ *parameter classes*.

Gross and Reeder [22, §2.1] prove that such homomorphisms from WD_K are equivalent to the more standard notion of a Weil-Deligne representation giving a homomorphism $\mathcal{W}_K \to {}^L\mathbf{G}$ and an appropriate nilpotent element of $\hat{\mathfrak{g}}$ separately.

We will need various conditions on our Langlands parameters.

Definition 2.8.3.

(i) We say that a Langlands parameter φ is *discrete* if $\mathrm{Z}_{\hat{\mathbf{G}}}(\varphi)$ is finite modulo the center of ${}^L\mathbf{G}$.

(ii) We call a Langlands parameter φ *tame* if it factors through $\mathcal{W}_t \times \mathrm{SL}_2(\mathbb{C})$.

For any discrete parameter φ, the image $\varphi(\mathcal{W}_K)$ is finite [22, Lem. 3.1]. We assume for the moment that φ is trivial on the $\mathrm{SL}_2(\mathbb{C})$ component of WD_K, a condition that we will shortly replace by a stronger one. If such a Langlands parameter is also tame then it is determined by its values on our chosen Frobenius F and on our generator $\tilde{\tau}$ of tame inertia, since these generate $\mathcal{W}_t = \mathcal{W}_K/\mathcal{I}_w$. Both $\varphi(F)$ and $\varphi(\tilde{\tau})$ will have finite order, and by conjugating by an element of $\hat{\mathbf{G}}$ we can constrain the image of φ.

If E/K is unramified, then $\varphi(\tilde{\tau})$ is a semisimple element in $\hat{\mathbf{G}}$, which we can take to lie in our fixed maximal torus $\hat{\mathbf{S}}$. The image of Frobenius under φ must normalize $\hat{\mathbf{S}}$ and thus

$\varphi(F) \in \mathrm{N}_{\hat{\mathbf{G}}}(\mathbf{S}) \rtimes \mathrm{Gal}(E/K)$. In order to get a finite centralizer for φ, the centralizer of $\varphi(\tilde{\tau})$ can't be too large: we must have

$$\hat{\mathbf{S}} \subseteq \mathrm{Z}_{\hat{\mathbf{G}}}(\varphi(\tilde{\tau})) \subseteq \mathrm{N}_{\hat{\mathbf{G}}}(\hat{\mathbf{S}}).$$

If E/K is totally ramified, similar constraints apply. Now $\varphi(\tilde{\tau}) \in \hat{\mathbf{S}} \rtimes \mathrm{Gal}(E/K)$ and $\varphi(F) \in \mathrm{N}_{\hat{\mathbf{G}}}(\hat{\mathbf{S}})$. This time not all of $\hat{\mathbf{S}}$ will centralize $\varphi(\tilde{\tau})$: only $\hat{\mathbf{S}}^{\tau}$. But once again the centralizer can't be too large: we have

$$\hat{\mathbf{S}}^{\tau} \subseteq \mathrm{Z}_{\hat{\mathbf{G}}}(\varphi(\tilde{\tau})) \subset \mathrm{N}_{\hat{\mathbf{G}}}(\hat{\mathbf{S}}),$$

and in fact the centralizer contains $\hat{\mathbf{S}}^{\tau}$ with finite index. See Section 3.2 for more details.

In either case, a generic value for $\varphi(\tilde{\tau})$ will have the smallest possible centralizer.

Definition 2.8.4. A Langlands parameter φ is *regular* if $\mathrm{Z}_{\hat{\mathbf{G}}}(\varphi(\tilde{\tau}))$ is a torus.

This condition is a technical one, but important. It implies that the corresponding L-packet has the expected size (see Section 5.4 for a discussion of the sizes of L-packets). Non-regular parameters lead to larger L-packets: see Section 5.5.3 for an example. Note that a discrete regular parameter must be trivial on $\mathrm{SL}_2(\mathbb{C})$.

2.9 Tate Duality and Embeddings of Tori

For a description of some related duality theorems, see Milne's book [36].

Proposition 2.9.1 (c.f. [36, Cor. 2.4])**.** *Let* $\mathbf{T}$ *be a torus over* K*. The cup product defines dualities between*

- *The compact group* $\mathrm{H}^0(K, \mathbf{T})^\wedge$ *(completion with respect to open subgroups of finite index) and the discrete group* $\mathrm{H}^2(K, X^*(\mathbf{T}))$*;*
- *the finite groups* $\mathrm{H}^1(K, \mathbf{T})$ *and* $\mathrm{H}^1(K, X^*(\mathbf{T}))$*;*
- *the discrete group* $\mathrm{H}^2(K, \mathbf{T})$ *and the compact group* $\mathrm{H}^0(K, X^*(\mathbf{T}))^\wedge$.

Corollary 2.9.2. *Suppose that* $\mathbf{T}$ *is an anisotropic torus. Then*

$$\mathrm{H}^1(K, \mathbf{T}) \cong \mathrm{Hom}(\hat{\mathbf{T}}^\Gamma, \mathbb{C}^\times).$$

Proof. From the definition of $\hat{\mathbf{T}}$ we have an exact sequence

$$1 \to X^*(\mathbf{T}) \to X^*(\mathbf{T}) \otimes \mathbb{C} \xrightarrow{\exp} \hat{\mathbf{T}} \to 1.$$

Since $\mathbf{T}$ is anisotropic $\mathrm{H}^0(K, X^*(\mathbf{T}) \otimes \mathbb{C}) = 0$, and since $X^*(\mathbf{T}) \otimes \mathbb{C}$ is divisible $\mathrm{H}^1(K, X^*(\mathbf{T}) \otimes \mathbb{C}) = 0$. The corollary now follows from the associated long exact sequence and the finiteness of

$$\hat{\mathbf{T}}^\Gamma = \mathrm{H}^0(K, \hat{\mathbf{T}}) \cong \mathrm{H}^1(K, X^*(\mathbf{T})).$$

□

We can also interpret the group $\mathrm{H}^1(K, \mathbf{T})$ as parameterizing the different embeddings of $\mathbf{T}$ into twists of $\mathbf{G}$. Suppose we fix a single embedding $\iota_0 \colon \mathbf{T} \hookrightarrow \mathbf{G}$ defined over K. Recall from Section 2.7.3 that $\mathrm{H}^1(K, \mathbf{G})$ parameterizes the pure inner forms of $\mathbf{G}$. The map ι_0 induces a map

$$\iota \colon \mathrm{H}^1(K, \mathbf{T}) \to \mathrm{H}^1(K, \mathbf{G}).$$

Proposition 2.9.3. *The kernel of ι may be identified with equivalence classes of embeddings $\mathbf{T} \hookrightarrow \mathbf{G}$ defined over K, where we consider two such embeddings equivalent if they differ by conjugation by an element of $\mathbf{G}(K)$.*

Proof. By Corollary 1 of [46, Prop. 36], we may identify the kernel with the quotient of $(\mathbf{G}/\mathbf{T})^\Gamma$ by the action of $\mathbf{G}(K)$. Since $\mathbf{T}$ is a maximal torus, the centralizer of $\mathbf{T}$ in $\mathbf{G}$ is just $\mathbf{T}$ and we may identify $(\mathbf{G}/\mathbf{T})$ with the embeddings of $\mathbf{T}$ into $\mathbf{G}$; the subset of such embeddings that are fixed by Γ are precisely those that are defined over K. $\square$

Using similar methods, we can identify other fibers of ι:

Proposition 2.9.4. *Suppose $\alpha \in \mathrm{H}^1(K, \mathbf{T})$ and let $\mathbf{G}'$ be a pure inner form of $\mathbf{G}$ associated to $\iota(\alpha) \in \mathrm{H}^1(K, \mathbf{G})$. Then the elements of $\mathrm{H}^1(K, \mathbf{T})$ with the same image as α under ι are in bijection with the equivalence classes of K-embeddings $\mathbf{T} \hookrightarrow \mathbf{G}'$, where we consider two such embeddings equivalent if they differ by conjugation by an element of $\mathbf{G}'(K)$.*

Proof. This follows from Corollary 2 of [46, Prop. 36], using the argument in Proposition 2.9.3 and the fact that twisting $\mathbf{T}$ by an element of $\mathrm{H}^1(K, \mathbf{T})$ leaves it fixed. $\square$

2.10 L-packets

A recent paper of Gan, Gross and Prasad [18, §9-10] gives some expected properties of L-packets, and goes through examples for specific classical groups. Another paper of Gross and Reeder [22, §8] gives a description of the central character of an L-packet. The material in this section can be found in greater depth in these two sources.

To each Langlands parameter φ we want to associate a set Π_φ, called an *L-packet*; each element of Π_φ should be an isomorphism class of admissible representations of $\mathbf{G}'(K)$, where $\mathbf{G}'$ is a pure inner form of $\mathbf{G}$ (see Section 2.16 for the definition of admissibility). As usual, $\mathbf{G}$ is a quasi-split group over K with center $\mathbf{Z}$, $\mathbf{A}$ is a maximal K-split torus, $\mathbf{S} = \mathrm{Z}_{\mathbf{G}}(\mathbf{A})$ is a maximal torus, $\mathbf{B}$ is a Borel subgroup of $\mathbf{G}$ defined over K and containing $\mathbf{S}$, and $\mathbf{U}$ is the unipotent radical of $\mathbf{B}$.

The following are some of the expected properties of the L-packets Π_φ.

- For each pure inner form $\mathbf{G}'$ of $\mathbf{G}$, every admissible representation of $\mathbf{G}'(K)$ lies in some L-packet associated to a Langlands parameter for $\mathbf{G}$.

- The L-packet Π_φ can be parameterized by irreducible representations of the finite group

  $$A_\varphi = \pi_0(\mathrm{Z}_{\hat{\mathbf{G}}}(\varphi)).$$

  The parameterization is not canonical, but is determined by a choice of generic character of $\mathbf{U}$. We say that a character

  $$\theta\colon \mathbf{U}(K) \to \mathbb{C}^\times$$

  is *generic* if its stabilizer in $\mathbf{S}(K)$ is $\mathbf{Z}(K)$. For each generic character θ, there is a bijection

  $$J(\theta)\colon \Pi_\varphi \to \mathrm{Irr}(A_\varphi)$$

- Suppose $\pi \in \Pi_\varphi$ is a representation of $\mathbf{G}'(K)$. We can determine $\mathbf{G}'$ by the image of π in $\mathrm{Irr}(A_\varphi)$ under any of the bijections $J(\theta)$. The injection

  $$\mathrm{Z}(\hat{\mathbf{G}}^\Gamma) \hookrightarrow \mathrm{Z}_{\hat{\mathbf{G}}}(\varphi)$$

induces a map from $\pi_0(\mathrm{Z}(\hat{\mathbf{G}}^\Gamma))$ to the center of A_φ. We now use the following theorem, due to Kottwitz [33, Prop. 6.4]:

Theorem 2.10.1.

$$\mathrm{H}^1(K, \mathbf{G}) \cong \mathrm{Hom}(\pi_0(\mathrm{Z}(\hat{\mathbf{G}})^\Gamma), \mathbb{C}^\times).$$

So for any $\chi \in \mathrm{Irr}(A_\varphi)$, restriction yields an element of $\mathrm{H}^1(K, \mathbf{G})$, which determines a pure inner form $\mathbf{G}'$ of $\mathbf{G}$. For any of the bijections $J(\theta)$, the representation corresponding to χ will be a representation of $\mathbf{G}'$.

- All of the pure inner forms of $\mathbf{G}$ have the same center $\mathbf{Z}$ over K. Every representation in Π_φ has the same central character, which we will thus denote by ω_φ. We give a construction of this central character in the case that $\mathbf{Z}$ is connected (see [22, §8.2] for the general case). When $\mathbf{Z}$ is connected, $\hat{\mathbf{Z}} = \hat{\mathbf{G}}/(\hat{\mathbf{G}}, \hat{\mathbf{G}})$ is the quotient of $\hat{\mathbf{G}}$ by its derived subgroup, and we have a natural projection $\hat{\mathbf{G}} \to \hat{\mathbf{Z}}$. $\mathcal{W}_K$ acts on $\hat{\mathbf{G}}$ via pinned automorphisms, and we get a map

$$\mathrm{H}^1(K, \hat{\mathbf{G}}) \to \mathrm{H}^1(K, \hat{\mathbf{Z}}),$$

which we can compose with the local Langlands correspondence for tori of the next section

$$\mathrm{H}^1(K, \hat{\mathbf{Z}}) \to \mathrm{Hom}(\mathbf{Z}(K), \mathbb{C}^\times).$$

The image of φ under this composition is the central character ω_φ (our formula differs from [22] since we use the arithmetic Frobenius rather than the geometric Frobenius).

2.11 The Local Langlands Correspondence for Tori

For an excellent account of the local Langlands correspondence for tori see Yu's article [55]. Another exposition is available in [36, §I.8]. In this section I merely record a statement of the main theorem for later use.

We say that a torus $\mathbf{T}$ over K is *induced* if $X^*(\mathbf{T})$ has a basis that is permuted by Γ. For example, if K' is any separable extension of K then the Weil restriction $\mathbf{T} = \mathrm{Res}_{K'/K}\,\mathbb{G}_m$ is induced. In this case $X^*(\mathbf{T})$ is a free $\mathbb{Z}$-module with basis permuted simply transitively by $\mathcal{W}_K/\mathcal{W}_{K'}$. Write $\mathcal{W}$ for $\mathcal{W}_K$ and $\mathcal{W}'$ for $\mathcal{W}_{K'}$. Then

$$\hat{\mathbf{T}} = \mathrm{Ind}_{\mathcal{W}'}^{\mathcal{W}}\,\mathbb{C}^\times.$$

Theorem 2.3.1 and Shapiro's lemma imply that the composition

$$\mathrm{Hom}(\mathbf{T}(K), \mathbb{C}^\times) = \mathrm{Hom}((K')^\times, \mathbb{C}^\times) \xrightarrow{\sim} \mathrm{Hom}(\mathcal{W}', \mathbb{C}^\times) = \mathrm{H}^1(\mathcal{W}', \mathbb{C}^\times) \xrightarrow{\sim} \mathrm{H}^1(\mathcal{W}, \hat{\mathbf{T}}) \tag{2.11.1}$$

is an isomorphism.

Theorem 2.11.1 ([55, §7.5])**.** *There is a unique family of homomorphisms*

$$\beta_{\mathbf{T}}\colon\ \mathrm{Hom}(\mathbf{T}(K), \mathbb{C}^\times) \to \mathrm{H}^1(\mathcal{W}_K, \hat{\mathbf{T}})$$

with the following properties:

(i) $\beta_{\mathbf{T}}$ is additive functorial in $\mathbf{T}$, i.e. it is a natural transformation between two additive functors from the category of tori over K to the category of abelian groups;

(ii) *for any finite separable extension K'/K and any torus of the form* $\mathbf{T} = \mathrm{Res}_{K'/K}\,\mathbb{G}_m$, $\beta_{\mathbf{T}}$ *is the isomorphism of* (2.11.1).

Moreover, $\beta_{\mathbf{T}}$ *is an isomorphism for all tori* $\mathbf{T}$ *over* K.

2.12 Néron Models

Bosch-Lütkebohmert-Reynaud [1] serves as a comprehensive reference for Néron models. In this section we merely define some terminology related to Néron models; we don't prove that such models exist or construct them.

Classical Néron models are defined for abelian varieties. Suppose that R is a Dedekind domain, K its field of fractions, and $\mathbf{X}$ is an abelian variety over K. A *Néron model* $\mathfrak{X}$ for $\mathbf{X}$ is a separated smooth scheme over R with generic fiber $\mathbf{X}$ that satisfies the following Néron mapping property [1, 1.2.1]:

- Any K-morphism $\mathfrak{Y}_K \to \mathfrak{X}_K$ from a smooth scheme $\mathfrak{Y}$ over R extends uniquely to a morphism $\mathfrak{Y} \to \mathfrak{X}$.

Such a model is clearly unique up to unique isomorphism. Néron [39] proved that Néron models for abelian varieties exist and are group schemes of finite type over R.

For a torus $\mathbf{T}$ over K, there is still a model $\mathfrak{T}$ which satisfies the Néron mapping property, but $\mathfrak{T}$ is not necessarily of finite type over R. We call such a model the *Néron-lft-model* of $\mathbf{T}$ (it is locally of finite type as a consequence of being smooth). We call the connected component of the identity of this model the *connected Néron model* of $\mathbf{T}$ and denote it $\mathfrak{T}^\circ$. If K is a p-adic field, we can also consider the maximal bounded subgroup of $\mathfrak{T}(O_K)$; this subgroup will contain $\mathfrak{T}^\circ(\mathcal{O}_K)$ with finite index. We call it the *bounded Néron model* of $\mathbf{T}$ and denote it $\mathfrak{T}^\flat$.

Now let K be a p-adic field and $R = \mathcal{O}_K$; let's consider a few examples. If $\mathbf{T} = \mathbb{G}_{m,K}$, then

$$\mathfrak{T} = \bigcup_{v \in \mathbb{Z}} \pi_K^v \cdot \mathbb{G}_{m,\mathcal{O}_K}$$

is a union of $\mathbb{Z}$ copies of $\mathcal{O}_K^\times$ [1, Ex. 10.1.5], and is thus not of finite type over $\operatorname{Spec} \mathcal{O}_K$. But the connected Néron model is just $\mathbb{G}_{m,\mathcal{O}_K}$, which is of finite type over $\mathcal{O}_K$. In this case the bounded Néron model and the Néron model agree. In both cases, the special fiber is a group scheme over k, and in fact is just $\mathbb{G}_{m,k}$.

Now suppose E is a quadratic extension of K with $\mathrm{Gal}(E/K) = \{1, \tau\}$ and $\mathbf{T}$ is the 1-dimensional unitary group associated to the pairing $\langle x, y \rangle = x\tau(y)$. If E/K is unramified, then we reduce to the previous case by the fact that Néron models are compatible with unramified base change. In this case τ is just Frobenius, and Γ_{nr} acts on the Néron model of $\mathbb{G}_m/K_{nr}$ by mapping an $\mathcal{O}_{K_{nr}}$ valued point x to $F(x)^{-1}$. We have $\mathfrak{T}(\mathcal{O}_K) = \mathbf{T}(K)$ as required, but once again the Néron-lft-model is not of finite type over $\mathcal{O}_K$, and the connected Néron model and bounded Néron model agree.

If E/K is ramified on the other hand, the Néron-lft-model is actually of finite type by [1, Thm. 10.2.1] and is equal to the bounded Néron model. Suppose for simplicity that $p \neq 2$ and π_E is a uniformizer of E with $\pi_E^2 \in K$. Suppose that $x = x_0 + x_1\pi_E \in \mathcal{O}_E$ has norm 1 in K. Then the reduction of the condition that $\mathrm{Nm}_{E/K}\, x = 1$ modulo π_E implies that $x_0^2 \equiv 1 \pmod{\pi_E}$. The two possibilities for x_0 modulo π_E correspond to the two connected components of the Néron model of $\mathbf{T}$.

Theorem 2.12.1 (c.f. [41, pp. 314-315])**.** *The component group of the special fiber of*

Néron-lft-model of $\mathbf{T}$ *is given by*

$$\pi_0(\mathfrak{T} \times_{O_K} \operatorname{Spec} k) = (X_*(\mathbf{T})_{\mathcal{I}})^F.$$

In the case of a ramified U_1, $X_*(\mathbf{T}) \cong \mathbb{Z}$ and $\mathcal{I}$ acts through its quotient $\mathrm{Gal}(E/K)$, with the nontrivial element acting by negation. Thus $X_*(\mathbf{T})_{\mathcal{I}} \cong \mathbb{Z}/2\mathbb{Z}$. Frobenius acts triviallly, and we recover the result of the previous paragraph, that the Néron model has two components.

Any maximal K-torus in a simply connected group, on the other hand, will have a torsion-free component group. We can see this directly for the quasi-split tori in $\mathrm{SU}(V)$ using Theorem 2.12.1. If V has dimension $2m+1$, then the quasi-split torus will be $\mathbf{S} \cong (\mathrm{Res}_{E/K}\,\mathbb{G}_m)^m$, and $X_*(\mathbf{S})$ will have basis $\lambda_{-m}, \ldots, \lambda_{-1}, \lambda_1, \ldots, \lambda_m$ where $\tau(\lambda_i) = -\lambda_{-i}$. Since $(\tau - 1)\lambda_i = -\lambda_{-i} - \lambda_i$, the group $X_*(\mathbf{S})_{\mathcal{I}}$ is free of rank m, and every element is F-invariant since $\mathbf{S}$ splits over E.

2.13 Moy-Prasad Filtrations

Moy and Prasad [38] defined a filtration on tori over p-adic fields. More recently, Yu's article [53, §4-5] provides a readable introduction to the theory and a slightly different approach that behaves better in the presence of wild ramification.

Let $\tilde{\mathbb{R}} = \mathbb{R} \cup \{r+ \mid r \in \mathbb{R}\} \cup \{\infty\}$ be the ordered monoid of "increasing rays in $\mathbb{R}$" (c.f. [2, §6.4.1]). If we have any decreasing filtration $\{G^r\}_{r\in\mathbb{R}}$ of a group G indexed by $\mathbb{R}$ (or by

$\mathbb{R}_{\geq 0}$), we can extend it to a filtration indexed by $\tilde{\mathbb{R}}$ (or $\tilde{\mathbb{R}}_{\geq 0}$) by setting

$$G^{r+} = \bigcup_{s>r} G^s,$$

and $G^\infty = \bigcap_{r\in\mathbb{R}} G^r$. We say that r is a *break* in the filtration if $G^r \neq G^{r+}$.

Suppose that $\mathbf{T}$ is a torus over K; we define the *Moy-Prasad filtration* on $\mathbf{T}(K)$, indexed by $\mathbb{R}_{\geq 0}$. Set $\mathbf{T}(K)^0 = \mathfrak{T}^\circ(O_K)$, the O_K-points of the connected Néron model of $\mathbf{T}$. For $r > 0$,

(i) if $\mathbf{T} = \prod_{i=1}^{j} \mathrm{Res}_{K_i/K}\,\mathbb{G}_m$ is an induced torus, then define $\mathbf{T}(K)^r$ to be the following subgroup of $\mathbf{T}(K) = \prod K_i$:

$$\mathbf{T}(K)^r = \{(x_i) \in \prod K_i^\times \mid v_K(x_i - 1) \geq r \ \ \forall i\},$$

where v_K denotes the valuation on K_i extending the one on K.

(ii) In general, we choose an induced torus $\mathbf{R}$ containing $\mathbf{T}$ and set

$$\mathbf{T}(K)^r = \mathbf{R}(K)^r \cap \mathfrak{T}^\circ(O_K).$$

This definition does not depend on the choice of $\mathbf{R}$.

Consider as an example a Hermitian space V with the maximal torus $\mathbf{S} \subset \mathrm{U}(V)$ defined in Section 2.7.2. There are subtori $\mathbf{S}_0$ and $\mathbf{S}_1$ so that

$$\mathbf{S} \cong \mathbf{S}_0 \times \mathbf{S}_1,$$

$$\mathbf{S}_0 \cong \prod_{i=1}^{j} \mathrm{Res}_{E/K}\,\mathbb{G}_m,$$

and $\mathbf{S}_1$ is anisotropic. If V is even dimensional and quasi-split then $\mathbf{S}_1$ is trivial; if V is odd dimensional then $\mathbf{S}_1 \cong \mathrm{U}_1$. If V is not quasi-split then there are many possible for $\mathbf{S}_1$, so we will consider the easiest case that $\mathbf{S}_1 \cong \mathrm{U}_1 \times \mathrm{U}_1$. This description of $\mathbf{S}$ thus reduces the computation of the Moy-Prasad filtration to the analogous computation for $\mathrm{Res}_{E/K}\,\mathbb{G}_m$ and for U_1.

The filtration on $\mathrm{Res}_{E/K}\,\mathbb{G}_m$ comes straight from the definition. To compute $\mathrm{U}_1(K)^r$, we use the natural embedding $\mathrm{U}_1 \hookrightarrow \mathrm{Res}_{E/K}\,\mathbb{G}_m$ as the elements of $E^\times$ with norm 1. In both the unramified and ramified cases, for $r > 0$

$$\mathrm{U}_1(K)^r = \{x \in E^\times \mid \mathrm{Nm}_{E/K}(x) = 1,\ v_K(x-1) \geq r\},$$

since $\mathrm{U}_1(K) \cap (\mathrm{Res}_{E/K}\,\mathbb{G}_m)^r$ is already contained in the points of the connected Néron model of U_1.

From our description of these filtrations, we see that the breaks occur at integers when $\mathbf{S}$ splits over an unramified extension, and half-integers when E/K is quadratic and tamely ramified. This phenomenon is generalized by the following proposition:

Proposition 2.13.1 (c.f. [53, Lem. 4.4.1])**.** *Suppose that* $\mathbf{T}$ *is a torus over* K *that is split by a tame extension. Then there are finitely many rational numbers* $r_1, \ldots, r_d \in [0,1)$ *such that* $\mathbf{T}(K)^r \neq \mathbf{T}(K)^{r+}$ *if and only if* $r = r_i + n$ *for some* i *and some* $n \in \mathbb{Z}_{\geq 0}$. *Moreover, if* e *is the ramification index of the splitting field of* $\mathbf{T}$, *then* $er_i \in \mathbb{Z}$ *for all* i.

We will apply the Moy-Prasad filtration by passing from characters on $\mathbf{T}(K)$ to characters on various quotients within the filtration. To this end, we define the *depth* of a character $\chi\colon \mathbf{T}(K) \to \mathbb{C}^\times$ to be the infimum of $r \in \mathbb{R}$ so that χ vanishes on $\mathbf{T}(K)^r$. This notion also makes sense for characters defined on some piece $\mathbf{T}(K)^s$ of the filtration on $\mathbf{T}$.

There is also a notion of depth for $\mathrm{H}^1(\mathcal{W}_K, \hat{\mathbf{T}})$ for tamely ramified $\mathbf{T}$. Recall the upper filtration $\mathcal{W}_K^r$ on $\mathcal{W}_K$ from Fröhlich [13, Eq. I.9.(8)]. We define the *depth* of $\varphi \in \mathrm{H}^1(\mathcal{W}_K, \hat{\mathbf{T}})$ to be the infimum infimum over all $r \geq 0$ with $\ker(\varphi) \supset \mathcal{W}_K^r$ [55, §7.9]. These two notions of depth are related by the following theorem:

Theorem 2.13.2 (c.f. [55, §7.10])**.** *The local Langlands correspondence*

$$\mathrm{Hom}(\mathbf{T}(K), \mathbb{C}^\times) \xrightarrow{\sim} \mathrm{H}^1(\mathcal{W}_K, \hat{\mathbf{T}})$$

preserves depth.

2.14 Buildings

The standard introduction to Bruhat-Tits theory is Tits' Corvallis article [50]. It is much more accessible than the series of articles by Bruhat and Tits [2, 3, 4, 6, 5] that remain the canonical reference for the subject. Yu's survey article [54] provides a complement to Tits' introduction as well as a comprehensive list of references. Garret's book [19] gives a concrete description of the buildings of the split classical groups as simplicial complexes.

Suppose that $\mathbf{G}$ is a reductive group over a non-archimedian local field K. The reduced Bruhat-Tits building is a set $\mathcal{B}_{\mathrm{red}}(\mathbf{G}/K)$ (with additional structures outlined below) associated to a connected reductive group $\mathbf{G}$ over a non-archimedian local field K. We will frequently refer to it just as $\mathcal{B}_{\mathrm{red}}(\mathbf{G})$.

- $\mathcal{B}_{\mathrm{red}}(\mathbf{G})$ is a complete metric space;

- If $\mathbf{G}$ is a simple algebraic group then $\mathcal{B}_{\mathrm{red}}(\mathbf{G})$ has the structure of a simplicial complex; in general $\mathcal{B}(\mathbf{G})$ has the structure of a poly-simplicial complex. We define a *facet* to be either the interior of a (poly)simplex or a vertex, and define an *alcove* to be a facet of maximal dimension;

- $\mathbf{G}(K)$ acts isometrically on $\mathcal{B}_{\mathrm{red}}(\mathbf{G})$ by poly-simplicial automorphisms;

- $\mathcal{B}_{\mathrm{red}}(\mathbf{G})$ is the union of a collection of distinguished subsets, known as apartments, which are indexed by the maximal K-split tori of $\mathbf{G}$. The apartment $\mathcal{A}(\mathbf{A})$ associated to $\mathbf{A}$ is an affine space for the real vector space $X_*(\mathbf{A}/\mathbf{Z}) \otimes \mathbb{R}$, where $\mathbf{Z}$ is the maximal split torus contained in the center of $\mathbf{G}$. In the case that $\mathbf{G}$ is quasi-split, $\mathbf{A}$ is determined by its centralizer $\mathbf{S}$ (it is the maximal K-split torus contained in the maximal torus $\mathbf{S}$) and we will also write $\mathcal{A}(\mathbf{S})$ for the corresponding apartment.

We will not focus on the construction of buildings, but rather on some examples as well as applications of buildings:

(i) To each point on the building we associate a collection of group schemes over $\mathcal{O}_K$. These schemes allow us to lift representations of reductive groups over finite fields to representations of $\mathbf{G}(K)$.

(ii) We can determine the structure of these schemes in terms of the geometry of the building.

2.14.1 The Building of GL_n

The building of $\mathrm{GL}_n(K)$ provides the inspiration for many of the structures we attach to buildings of more general reductive groups $\mathbf{G}$. Moreover, we can give concrete descriptions of buildings of other reductive groups using an embedding into GL_n over $\bar{K}$ and the description of $\mathcal{B}_{\mathrm{red}}(\mathrm{GL}_n/\bar{K})$. For references and more details, see Yu's survey [54, §2.1.3] .

Definition 2.14.1. Let V be a vector space over a local field K with valuation v_K. A *norm* on V is a function

$$\alpha\colon V \to \mathbb{R} \cup \{\infty\},$$

satisfying the following properties:

- $\alpha(x + y) \geq \min(\alpha(x), \alpha(y))$ for $x, y \in V$,
- $\alpha(\lambda x) = v_K(\lambda) + \alpha(x)$ for $\lambda \in K, x \in V$,
- $\alpha(x) = \infty$ if and only if $x = 0$.

We define an equivalence relation on the set of norms on V: two norms α and β are equivalent if there is some $c \in \mathbb{R}$ such that $\alpha(v) = \beta(v) + c$ for all $v \in V$. We then define $\mathcal{B}_{\mathrm{red}}(\mathrm{GL}(V)/K)$ to be the set of equivalence classes of norms on V.

Definition 2.14.2. Suppose that α is a norm on V. We say that a basis $\mathfrak{B} = \{v_1, \ldots, v_n\}$ of V is a *splitting basis* for α if there exist $c_1, \ldots, c_n \in \mathbb{R}$ so that

$$\alpha\Big(\sum \lambda_i v_i\Big) = \min\left(v_K(\lambda_i) + c_i\right)$$

for all $\sum \lambda_i v_i \in V$.

Note that if α and β are equivalent norms, then $\mathfrak{B}$ is a splitting basis for α if and only if it is a splitting basis for β. The tuple $(c_1, \ldots, c_n)$ is well defined up to multiples of $(1, \ldots, 1)$; if $\mathfrak{B}$ is a splitting basis for a norm α, write $\alpha_{\mathfrak{B}}$ for the image of $(c_1, \ldots, c_n)$ in $\mathbb{R}^n/\langle(1, \ldots, 1)\rangle$.

Proposition 2.14.3. *Suppose that V is a vector space over a local field K.*

(i) Every norm on V admits a splitting basis.

(ii) Any two norms on V admit a common splitting basis.

We can now describe apartments, the metric, the simplicial structure and the action of $\mathrm{GL}(V)$ using splitting bases.

- *Apartments.* Note that each basis of V determines a maximal K-split torus in $\mathrm{GL}(V)$ consisting of those elements of $\mathrm{GL}(V)$ that scale each basis vector. The apartment in the building of $\mathrm{GL}(V)$ associated to the basis $\mathfrak{B}$ is the set of norms for which $\mathfrak{B}$ is a splitting basis. The map $\alpha \mapsto \alpha_{\mathfrak{B}}$ defines a bijection between the apartment associated to $\mathfrak{B}$ and $\mathbb{R}^{n-1}$. Multiplying each basis vector by a scalar results in a new basis corresponding to the same K-split torus and the same apartment, but with a different identification of that apartment with $\mathbb{R}^{n-1}$. Proposition 2.14.3 tells us that every point in $\mathcal{B}_{\mathrm{red}}(\mathrm{GL}(V))$ lies in an apartment, and any two points lie in a common apartment.

- *Metric.* We can define a metric on $\mathcal{B}_{\mathrm{red}}(\mathrm{GL}(V))$ that makes the bijections $\alpha \to \alpha_{\mathfrak{B}}$ isometries between the apartments of $\mathcal{B}_{\mathrm{red}}(\mathrm{GL}(V))$ and $\mathbb{R}^{n-1}$. More explicitly, suppose α and β are norms with splitting basis $\mathfrak{B}$; we may assume that $\alpha(v_n) = \beta(v_n)$ by choosing equivalent representatives. Then we define

$$d(\alpha,\beta) = \sqrt{\sum_{i=1}^{n-1} (\alpha(v_i) - \beta(v_i))^2}.$$

  This definition is independent of the choice of common splitting basis, and makes

$\mathcal{B}_{\mathrm{red}}(\mathrm{GL}(V))$ into a complete metric space. One can show that there is a unique geodesic between any two points, consisting of the line between them in a common apartment.

- *Simplices.* The *hyperspecial norm* α associated to $\mathfrak{B}$ is the norm corresponding to the point $c = 0$:

$$\alpha\left(\sum \lambda_i v_i\right) = \min v_K(\lambda_i).$$

We say that a point $x \in \mathcal{B}_{\mathrm{red}}(\mathrm{GL}(V))$ is hyperspecial if some norm α in the given equivalence class is a hyperspecial norm. To any hyperspecial norm α we associate the lattice $L_\alpha = \mathcal{O}_K\langle v_1, \ldots, v_n\rangle$. We can define α in terms of L_α by

$$\alpha(v) = \max(m \mid v \in \pi_K^m L_\alpha).$$

The equivalence relation on norms translates to lattices L and L' being equivalent if $L' = \pi_K^c L$ for some $c \in \mathbb{Z}$. The hyperspecial points in $\mathcal{B}_{\mathrm{red}}(\mathrm{GL}(V))$ then correspond exactly to such equivalence classes of lattices in V; these will be the vertices in our simplicial decomposition of $\mathcal{B}_{\mathrm{red}}(\mathrm{GL}(V))$.

A set of $k + 1$ vertices form a simplex if there are lattices $L_0, \ldots, L_k$ representing the corresponding lattice classes such that

$$L_0 \supsetneq L_1 \supsetneq \cdots \supsetneq L_k \supsetneq \pi_K L_0.$$

- $\mathrm{GL}(V)$-*action*. For $g \in \mathrm{GL}(V)$ and α a norm on V, define

$$g.\alpha = \alpha \circ g^{-1} : v \mapsto \alpha(g^{-1} \cdot v).$$

This yields another norm on V, and preserves equivalence of norms. It thus defines an action of $\mathrm{GL}(V)$ on $\mathcal{B}_{\mathrm{red}}(\mathrm{GL}(V))$.

In order to better visualize this action, it helps to consider its relationship to the decomposition of $\mathcal{B}_{\mathrm{red}}(\mathrm{GL}(V))$ into apartments. Suppose that $\mathbf{S}$ is a maximal K-split torus in $\mathrm{GL}(V)$ with normalizer $\mathbf{N}$. Then the stabilizer of $\mathcal{A}(\mathbf{S})$ in $\mathrm{GL}(V)$ is precisely $N = \mathbf{N}(K)$. We can also describe the subgroup of N that fixes every point of $\mathcal{A}(\mathbf{S})$. Let Z be the center of $\mathrm{GL}(V)$, consisting of the scalars, and let S° be the subgroup of $\mathbf{S}(K)$ that scales each eigenvector by an element of $\mathcal{O}_K^\times$. The subgroup of N fixing $\mathcal{A}(\mathbf{S})$ is then $Z \cdot S^\circ$. The quotient $N/Z \cdot S^\circ$ decomposes into a short exact sequence

$$1 \to S/Z \cdot S^\circ \to N/Z \cdot S^\circ \to \mathbf{W}(S) \to 1,$$

which in fact splits uniquely into a semidirect product (c.f. [54, §2.2.1]). The valuation of K induces an isomorphism

$$S/Z \cdot S^\circ \to \mathbb{Z}^n/\langle(1, \ldots, 1)\rangle,$$

and $S \subset N$ acts on $\mathcal{A}(\mathbf{S}) \cong \mathbb{R}^n/\langle(1, \ldots, 1)$ as translation by its image under this map. Any element of N acts by an affine automorphism of $\mathcal{A}(\mathbf{S})$. The space of affine automorphisms of $\mathcal{A}(\mathbf{S})$ breaks up as a semidirect product of translations with the linear automorphisms:

$$\mathrm{Aut}_{\mathrm{affine}}(\mathcal{A}(\mathbf{S})) \cong \mathcal{A}(\mathbf{S}) \rtimes \mathrm{GL}(\mathcal{A}(\mathbf{S})).$$

The image in $\mathrm{GL}(\mathcal{A}(\mathbf{S}))$ of the automorphism of $\mathcal{A}(\mathbf{S})$ induced by an element $n \in N$ is precisely the linear automorphism of $\mathcal{A}(\mathbf{S})$ defined by the image of n in the Weyl group N/S of S. In order to determine it more precisely, one needs to non-canonically fix a vertex of $\mathcal{A}(\mathbf{S})$ to consider the origin.

From this description its clear that N acts on $\mathcal{A}(\mathbf{S})$ by isometries. In addition, the translations defined by $S \subset N$ map the vertices of $\mathcal{A}(\mathbf{S})$ into themselves. The description of the simplicial decomposition of $\mathcal{A}(\mathbf{S})$ in terms of lattice chains allows one to deduce that in fact every element of N defines a simplicial automorphism of $\mathcal{A}(\mathbf{S})$. This property extends to the action of $\mathrm{GL}(V)$ on the whole building $\mathcal{B}_{\mathrm{red}}(\mathrm{GL}(V))$, and generalizes to buildings of other reductive groups.

2.14.2 Affine Roots

The canonical pairing between $X_*(\mathbf{A})$ and $X^*(\mathbf{A})$ allows us to identify the roots $\Phi_{\mathbf{A}}$ with linear functions on $X_*(\mathbf{A})_{\mathbb{R}} = X_*(\mathbf{A}) \otimes \mathbb{R}$. The root hyperplanes in $X_*(\mathbf{A})_{\mathbb{R}}$ defined by the vanishing of the roots divide $X_*(\mathbf{A})_{\mathbb{R}}$ into Weyl chambers, permuted transitively by the action of the rational Weyl group $\mathbf{W}(K)$. The apartment $\mathcal{A}(\mathbf{A})$ is an affine space under $X_*(\mathbf{A})_{\mathbb{R}}$, and given an affine function $f\colon \mathcal{A}(\mathbf{A}) \to \mathbb{R}$ we can determine a linear function on $X_*(\mathbf{A})_{\mathbb{R}}$ by

$$v \mapsto f(v_0 + v) - f(v_0),$$

which does not depend on the choice of $v_0 \in \mathcal{A}(\mathbf{A})$. We will call this function the *vector part* of f. Among the affine functions $\mathcal{A}(\mathbf{A}) \to \mathbb{R}$ whose vector part is a root in $\Phi_{\mathbf{A}}$, Bruhat and Tits pick out a discrete set that they term the *affine roots* [50, §1.6]; we will denote this set by $\widetilde{\Phi}(\mathbf{G}, \mathbf{A})$, or just $\widetilde{\Phi}$ if $\mathbf{G}$ and $\mathbf{A}$ are fixed. Given an affine root α, there is a rational number c_α so that the other affine roots with the same vector part as α are precisely $\alpha + nc_\alpha$ for $n \in \mathbb{Z}$.

Just as in the non-affine case, we can define the root hyperplane associated to $\alpha \in \widetilde{\Phi}$ as the subset of $X_*(\mathbf{A})_{\mathbb{R}}$ on which α vanishes. We introduce an equivalence relation on $\mathcal{A}(\mathbf{A})$: two points x and x' are equivalent if the sign of $\alpha(x)$ and $\alpha(x')$ are identical for every affine root α. This equivalence relation gives us the polysimplicial decomposition of $\mathcal{A}(\mathbf{A})$.

2.14.3 Affine Weyl Groups

The torus $S = \mathbf{S}(K)$ acts on $\mathcal{A}(\mathbf{S})$ by translations, just as in the case of GL_n. We can extend this to an action of the normalizer $N = \mathbf{N}(K)$, and the kernel of the action is the bounded Néron model $S^\flat = \mathfrak{S}^\flat(O_K)$ of $\mathbf{S}$. The quotient

$$\widetilde{W} = N/S^\flat$$

is known as the *extended affine Weyl group* of $\mathbf{S}$.

There are two important types of decompositions of $\widetilde{W}$, each resulting from a distinguished normal subgroup. The first such subgroup is the translations within $\widetilde{W}$, isomorphic to $X_*(\mathbf{A})$. For $x \in \mathcal{A}(\mathbf{A})$, we denote by $\widetilde{\Phi}_x$ the affine roots vanishing at x, and by $\widetilde{W}_x$ the subgroup of $\widetilde{W}$ generated by reflections in the root hyperplanes passing through x. We say that a point x is *special* if $\widetilde{W}_x$ is isomorphic to the standard Weyl group $\mathbf{W}(K)$. If x is special then

$$\widetilde{W} = X_*(\mathbf{A}) \rtimes \widetilde{W}_x.$$

Special points exist for any connected reductive $\mathbf{G}$, and both $\widetilde{\Phi}_x$ and $\widetilde{W}_x$ depend only on the facet F in which x lies, so we will also write $\widetilde{\Phi}_F$ and $\widetilde{W}_F$.

Our second decomposition is associated to the subgroup of $\widetilde{W}$ fixing a particular alcove. The extended affine Weyl group acts transitively on the set of alcoves in $\mathcal{A}(\mathbf{A})$. The subgroup generated by the reflections in the root hyperplanes is known as the *affine Weyl group*: we will denote it by $\widetilde{W}^\circ$. Fix a fundamental alcove C, then the subgroup Ω of $\widetilde{W}$ fixing C forms a complement for $\widetilde{W}^\circ$:

$$\widetilde{W} = \widetilde{W}^\circ \rtimes \Omega.$$

The group $\widetilde{W}$ acts freely and transitively on the alcoves in $\mathcal{A}(\mathbf{A})$. If $x \in \mathcal{A}(\mathbf{A})$ is special, then

$$\widetilde{W}^\circ = \mathbb{Z}\Phi^\vee \rtimes \widetilde{W}_x.$$

Suppose that K_f is an unramified extension of K with $\mathbf{A}_f$ a maximal K_f-split torus of $\mathbf{G}$ containing $\mathbf{A}$. Then the affine roots of $\mathbf{A}$ over K are just the non-constant restrictions of the affine roots over K_f, and any vertex that's special over K_f will also be special over K. We say that a point x is *hyperspecial* if there is such an unramified extension so that $\mathbf{G}$ splits over K_f and x is special over K_f. Such points obviously can't exist for groups splitting over a ramified extension.

2.14.4 Extended Dynkin Diagrams

The group $\widetilde{W}^\circ$ is generated by the reflections in the walls $\{L_\alpha\}$ of the fundamental alcove C. We can associate an extended Dynkin diagram to the affine root system $\widetilde{\Phi}$ where each vertex corresponds to a reflection in one of the L_α, and the labels on the edges come in the standard way from inner products between the vector parts of the α. The underlying Coxeter graph gives a presentation for $\widetilde{W}^\circ$ with generators the reflections in the walls L_α.

We can determine the ordinary Dynkin diagram associated to the root system $\widetilde{\Phi}_x$ from the extended Dynkin diagram of $\widetilde{W}^\circ$. After applying an element of $\widetilde{W}^\circ$, we may assume that x lies in the closure of C. Let I_x denote the set of vertices in the Dynkin diagram corresponding to hyperplanes not containing x. Then the Dynkin diagram of $\widetilde{\Phi}_x$ can be obtained by removing all vertices in I_x and the incident edges.

If we identify $\Omega \cong \widetilde{W}/\widetilde{W}^\circ \cong X_*(\mathbf{A})/\Phi^\vee(\mathbf{G},\mathbf{A})$, then we get an action of the fundamental group of the root system Φ on the fundamental alcove, which induces a permutation action on the extended Dynkin diagram.

2.14.5 Filtrations on reductive groups

The foundations of the theory of filtrations of reductive groups over p-adic fields lie in Bruhat-Tits [2, 3]; see Yu [53] for a more concise exposition.

Given a point $x \in \mathscr{A}(\mathbf{A})$ and a root $\alpha \in \Phi(\mathbf{G},\mathbf{A})$, Bruhat-Tits define a filtration $\{\mathbf{U}_\alpha(K)_x^r\}_{r\in\mathbb{R}}$ of $\mathbf{U}_\alpha(K)$ (in fact, the affine roots are derived in Tits [50, §1.4] from this filtration). In order to unify notation with the Moy-Prasad filtration, we let

$$\Phi_0(\mathbf{G},\mathbf{A}) = \Phi(\mathbf{G},\mathbf{A}) \cup \{0\},$$

define $\mathbf{U}_0 = \mathbf{S}$, and write $\mathbf{U}_0(K)^r$ for the r^{th} piece of the Moy-Prasad filtration on $\mathbf{S}(K)$. We can hope to obtain subgroups of $\mathbf{G}(K)$ by taking products of various pieces of these filtrations, just as we can obtain all of $\mathbf{G}(K)$ from $\mathbf{U}_\alpha(K)$ for $\alpha \in \Phi_0(\mathbf{G}, \mathbf{A})$. In fact, even more is true: we can define group schemes over O_K whose O_K points can be expressed as appropriate products of $\mathbf{U}_\alpha(K)^r_x$.

We say that a function $f : \Phi_0 \to \tilde{\mathbb{R}}$ is *concave* if, for any set $\{\alpha_i\}_{i=1}^j \subset \Phi_0(\mathbf{G}, \mathbf{A})$ such that $\sum_i \alpha_i \in \Phi_0(\mathbf{G}, \mathbf{A})$,

$$\sum_{i=1}^{j} f(\alpha_i) \geq f\left(\sum_{i=1}^{j} \alpha_i\right).$$

The simplest examples of concave functions are just the ones that assign a constant non-negative value to every element of $\Phi_0(\mathbf{G}, \mathbf{A})$.

Given a concave function f, and a point $x \in \mathcal{B}(\mathbf{G})$, define $\mathbf{G}(K)^f_x$ to be the subgroup generated by $\mathbf{U}_\alpha(K)^{f(\alpha)}_x$ for $\alpha \in \Phi_0(\mathbf{G}, \mathbf{A})$. We will write r for the constant function r; the groups $\mathbf{G}(K)^0_x$ are called *parahoric subgroups*. They are filtered by the groups $\mathbf{G}(K)^r_x$ for $r \in \tilde{\mathbb{R}}_{\geq 0}$. A representation of $\mathbf{G}(K)^0_x$ is said to have *depth r* if it factors through the quotient $\mathbf{G}(K)^0_x/\mathbf{G}(K)^{r+}_x$. All of the representations that we construct in this thesis will have depth 0.

In order to understand the structure of the quotients $\mathbf{G}(K)^0_x/\mathbf{G}(K)^{r+}_x$, it's useful to think of $\mathbf{G}(K)^f_x$ as being the O_K-points of a smooth group scheme $\mathfrak{G}^f_x$ over O_K (c.f. [53, Thm. 8.3]). Then

$$\mathfrak{G}^0_x(k) \cong \mathfrak{G}^0_x(O_K)/\mathfrak{G}^1_x(O_K),$$

and $\mathfrak{G}^{0+}_x(O_K)/\mathfrak{G}^1_x(O_K)$ is a connected, normal, unipotent subgroup. The quotient $\mathfrak{G}^0_x/\mathfrak{G}^{0+}_x$ is the maximal reductive quotient of $\mathfrak{G}^0_x(k)$.

2.14.6 Subgroups associated to facets

Since $\mathbf{G}(K)$ acts on $\mathcal{B}(\mathbf{G}/K)$ by simplicial automorphisms, the parahoric subgroup $\mathbf{G}(K)_x^0$ associated to a point $x \in \mathcal{B}(\mathbf{G}/K)$ depends only on the facet in which x lies. So for any facet $F \subset \mathcal{B}(\mathbf{G}/K)$, we may pick some $x \in F$ and define $\mathbf{G}(K)_F^\circ = \mathbf{G}(K)_x^0$. There is an inclusion reversing bijection between facets and parahoric subgroups:

$$\mathbf{G}(K)_F^\circ \subseteq \mathbf{G}(K)_{F'}^\circ \qquad \Leftrightarrow \qquad F' \subseteq \bar{F}.$$

In addition to $\mathbf{G}(K)_F^\circ$, we will associate two other subgroups of $\mathbf{G}(K)$ to a facet F: we will denote by $\mathbf{G}(K)_F$ the subgroup that fixes every point of F, and by $\mathbf{G}(K)_F^\natural$ the subgroup that stabilizes F. Both of these have models as O_K-points of schemes over O_K, the inclusions

$$\mathbf{G}(K)_F^\circ \subseteq \mathbf{G}(K)_F \subseteq \mathbf{G}(K)_F^\natural$$

hold, and $\mathbf{G}(K)_F^\circ$ is of finite index in $\mathbf{G}(K)_F^\natural$.

A parahoric subgroup associated to an alcove is known as an *Iwahori subgroup*. For any facets F and F', if $F' = gF$ for some $g \in \mathbf{G}(K)$ then $\mathbf{G}(K)_{F'}^\circ = g \cdot \mathbf{G}(K)_F^\circ \cdot g^{-1}$. Since $\mathbf{G}(K)$ acts transitively on alcoves, all Iwahori subgroups are conjugate. Moreover, the Iwahori subgroup associated to an alcove C will be contained in $\mathbf{G}(K)_F^\circ$ for every facet F in the closure of C.

As an example, let V be a vector space over K with basis $\{v_1, \ldots, v_n\}$, and consider the alcove of $\mathrm{GL}(V)$ determined by the lattices

$$\Lambda_r = \bigoplus_{i=1}^{r} O_K v_i \oplus \bigoplus_{i=r+1}^{n} \pi O_K v_i,$$

where r ranges from 0 to n. The stabilizer of Λ_r consists of those matrices of the form

$$\left(\begin{array}{c|c} \mathcal{O}_K & \pi_K^{-1}\mathcal{O}_K \\ \hline \pi_K\mathcal{O}_K & \mathcal{O}_K \end{array}\right)$$

with unit determinant, where the top left block is $r \times r$ and the bottom right is $(n-r)\times(n-r)$ [50, §3.10]. The parahoric subgroup associated to a facet F will be the intersection of the parahorics associated to vertices in the closure of F. In considering such intersections, we may assume that the vertex associated to Λ_0 is contained in the closure of F by scaling our basis if necessary (corresponding to changing our fundamental alcove). Then the parahoric subgroup associated to F is just the preimage in $\mathrm{GL}(\Lambda_0) \cong \mathrm{GL}_n(\mathcal{O}_K)$ of an appropriate parabolic subgroup of $\mathrm{GL}(\Lambda_0/\pi_K\Lambda_0) \cong \mathrm{GL}_n(k)$. In particular, the parahoric associated to the vertex Λ_0 itself is just $\mathrm{GL}(\Lambda_0)$; the Iwahori subgroup is the preimage in $\mathrm{GL}(\Lambda_0)$ of the Borel subgroup of $\mathrm{GL}(\Lambda_0/\pi_K\Lambda_0)$ consisting of matrices that are upper triangular modulo π_K.

2.14.7 Reductions of Parahorics

We will be using reductions of these subgroups in our construction of supercuspidal representations. Each such reduction can be thought of either as the k-points of the corresponding $\mathcal{O}_K$-group scheme, or as the quotient by the first stage of the filtration described in Section 2.14.5. In general, these reductions will not be reductive, so we define $\mathcal{G}_F$ as the quotient of $\mathbf{G}(K)_F/\mathbf{G}(K)_F^{0+}$ by its unipotent radical, and define $\mathcal{G}_F^\circ$ and $\mathcal{G}_F^\flat$ analogously, which have $\mathcal{G}_F^\circ$ as their connected component.

We obtain a maximal torus $\mathcal{A}$ in $\mathcal{G}_F^\circ$ as the special fiber of the split torus subscheme of $\mathfrak{G}_F^\circ$ with generic fiber $\mathbf{A}$. In particular, this means that the k-rank of $\mathcal{G}_F^\circ$ is the same as the K-rank of $\mathbf{G}$; this equality is reflected in the fact that all of the $\mathcal{G}_F^\circ$ contain the Iwahori subgroup $\mathcal{G}_C^\circ$, where C is an alcove containing F. We get a lot of information about $\mathcal{G}_F^\circ$ from the following theorem.

Theorem 2.14.4 (c.f. [50, 3.5.1])**.** *The root system of $\mathcal{G}_F^\circ$ with respect to $\mathcal{A}$ is the system $\widetilde{\Phi}_F$ defined in Section 2.14.3, and in particular its Dynkin diagram is determined by deleting from the extended Dynkin diagram of $\mathbf{G}$ all of the vertices in I_F and the incident edges. Moreover, the coroot associated to $a \in \widetilde{\Phi}_F$ is the same for $\mathcal{G}_F^\circ$ as for $\mathbf{G}$.*

2.14.8 The Building of U_n

Our first tool for understanding the building of a unitary group is the following theorem:

Theorem 2.14.5 (c.f. [40, Thm. 1.9])**.** *Suppose $\mathbf{H}$ is a connected reductive group over a non-archimedian local field K and Ω is a finite group of K-automorphisms of $\mathbf{H}$ whose order is not divisible by p. Then $\mathbf{G} = (\mathbf{H}^\Gamma)^\circ$ is reductive and $\mathcal{B}_{\mathrm{red}}(\mathbf{G})$ can be identified with $\mathcal{B}_{\mathrm{red}}(\mathbf{H})^\Omega$.*

In particular, if $\mathbf{H} = \mathrm{Res}_{E/K}\,\mathbf{H}'$ for some $\mathbf{H}'$ defined over E, then $\Omega = \mathrm{Gal}(E/K)$ acts on $\mathcal{B}_{\mathrm{red}}(\mathbf{H}/K) = \mathcal{B}_{\mathrm{red}}(\mathbf{H}'/E)$, and for totally ramified E/K the condition that p does not divide $\#\Omega$ is just the requirement that E/K be tamely ramified. We can thus apply this theorem to the case that $\mathbf{H}'$ is a unitary group $\mathrm{U}(V)$ over K base changed to E (and thus isomorphic to GL_n/E); in this case we just need to consider the fixed points in $\mathcal{B}_{\mathrm{red}}(\mathrm{U}(V)/E)$ under the nontrivial involution of $\mathrm{Gal}(E/K)$.

Suppose that $\mathbf{A}$ is a maximal K-split torus in $\mathrm{U}(V)$, contained in a maximal torus $\mathbf{S}$ that is defined over K. Since $\mathbf{S}$ is defined over K, the apartment $\mathcal{A}(\mathbf{S}/E)$ is Ω-stable, and we can identify the Ω-fixed points with the apartment $\mathcal{A}(\mathbf{A}/K)$. If $\mathbf{G}$ is quasi-split, then each apartment of $\mathcal{B}_{\text{red}}(\mathbf{G})$ will be contained in a unique apartment of $\mathcal{B}_{\text{red}}(\mathbf{H})$; if $\mathbf{G}$ is not quasi-split then the dimension of the apartments of $\mathcal{B}_{\text{red}}(\mathbf{G})$ will be one less, and each apartment will be contained in many apartments of $\mathcal{B}_{\text{red}}(\mathbf{H})$.

Since Ω acts on $\mathcal{B}_{\text{red}}(\mathbf{H})$ as a simplicial involution, there will be two types of simplices that intersect $\mathcal{B}_{\text{red}}(\mathbf{H})^{\Omega}$:

- Simplices that are fixed by Ω. These simplices of $\mathcal{B}_{\text{red}}(\mathbf{H})$ will correspond to simplices of $\mathcal{B}_{\text{red}}(\mathbf{G})$ of the same dimension. For example, we can give a description of the vertices of $\mathrm{GL}(V)$ that are fixed by Ω. The Hermitian form ϕ on V gives an identification of V with its dual, and the dual of a lattice Λ will be the lattice

$$\Lambda^{\vee} = \{v \in V \mid \phi(v, \lambda) \in \mathcal{O}_E\}.$$

  We obtain some of the vertices of the building of $\mathrm{U}(V)$ from lattices $\Lambda \subset V$ satisfying $\Lambda = \Lambda^{\vee}$.

- Simplices that are stabilized by Ω but not fixed. Within each such simplex there will be a simplex of $\mathcal{B}_{\text{red}}(\mathbf{G})$ of one lower dimension, consisting of those points fixed by Ω. To find the vertices of this type, we need to look for the 1-simplices whose ends are exchanged by the nontrivial element of Ω. Such edges correspond to pairs of lattices Λ_0, Λ_1 with

$$\Lambda_0 \supsetneq \Lambda_1 \supsetneq \pi_E \Lambda_0$$

$$\Lambda_0^{\vee} = \Lambda_1$$

Merging these two types, we see that vertices in the simplicial decomposition of $\mathcal{B}_{\text{red}}(\mathrm{U}(V))$ correspond to lattice classes with a representative Λ satisfying

$$\Lambda \supseteq \Lambda^\vee \supsetneq \pi_E \Lambda.$$

In order to get a good description of larger simplices, we turn to a visual tool described by Tits [50, §1.11]: the local index. The local index of $\mathbf{G}$ is:

(i) the extended Dynkin diagram $\widetilde{\mathcal{D}}$ of $\mathbf{G}$ over K_{nr},

(ii) the action of Γ_{nr} on $\widetilde{\mathcal{D}}$.

This data determines the extended Dynkin diagram $\mathcal{D}$ of $\mathbf{G}$ over K according to an algorithm described by Tits. In particular, there is a bijection between vertices v of $\mathcal{D}$ and orbits $O(v)$ for the action of Γ_{nr} on $\widetilde{\mathcal{D}}$. In Figure 2.4 we give the local indices for unitary groups associated to both unramified and tamely ramified E/K. The lower diagram is $\mathcal{D}$ and the upper is $\widetilde{\mathcal{D}}$; the vertices of $O(v)$ are placed vertically above v. In the case that Γ_{nr} acts trivially on $\widetilde{\mathcal{D}}$, the two diagrams are the same and the upper is omitted. The hyperspecial vertices are denoted $\bullet$ and the other special vertices are denoted $\circ$. Thick lines are used when one simple root is a negative multiple of the other. As normal, arrows point toward the shorter root if there is a difference in length.

In addition to the diagrams, Figure 2.4 also gives the groups $\mathcal{G}_x$ for each vertex x in the closure of the fundamental alcove. Tits gives a detailed description of how these reductions are derived [50, §2.10, §3.11] in the case of odd quasi-split unitary groups. For even groups, the analogous results can be determined using Theorem 2.14.4; we also argue more directly in the proof of Theorem 5.2.1. See also Johnson [32] for a discussion of lattices in Hermitian spaces. We denote by O_n the split orthogonal group over k, and by O'_n the non-split orthogonal group.

2.15 Deligne-Lusztig Representations

Deligne and Lusztig have given a construction that produces representations of reductive groups over finite fields. We summarize some of the relevant properties of these representations in this section; see Carter [11, Ch. 7] for more details.

Suppose k is a finite field with Frobenius F, and $\mathcal{G}$ is a connected reductive group over k. Let $\mathcal{T} \subset \mathcal{G}$ be a maximal torus defined over k, and $\theta\colon \mathcal{T}(k) \to \mathbb{C}^\times$ be a character. If $\mathcal{T}$ were split then we could define a representation of $\mathcal{G}(k)$ by pulling θ back to a character an a Borel containing $\mathcal{T}$ and then parabolically inducing up to $\mathcal{G}(k)$ (see Section 2.4.1 for the definition of parabolic induction). Deligne and Lusztig generalize this process to the case that $\mathcal{T}$ is not split over k, obtaining a representation of $\mathcal{G}(k)$. We will denote this representation by

$$\pi_{\mathcal{T},\theta}\colon \mathcal{G}(k) \to \mathrm{GL}(V_{\mathcal{T},\theta}),$$

and the associated character by

$$R_{\mathcal{T},\theta}\colon \mathcal{G}(k) \to \mathbb{C}.$$

Actually, $\pi_{\mathcal{T},\theta}$ is a generalized representation: a formal integral combination of representations.

The following theorem gives a formula for the inner product of two Deligne-Lusztig characters $R_{\mathcal{T},\theta}$ and $R_{\mathcal{T}',\theta'}$, which we will use to prove that such a representation is irre-

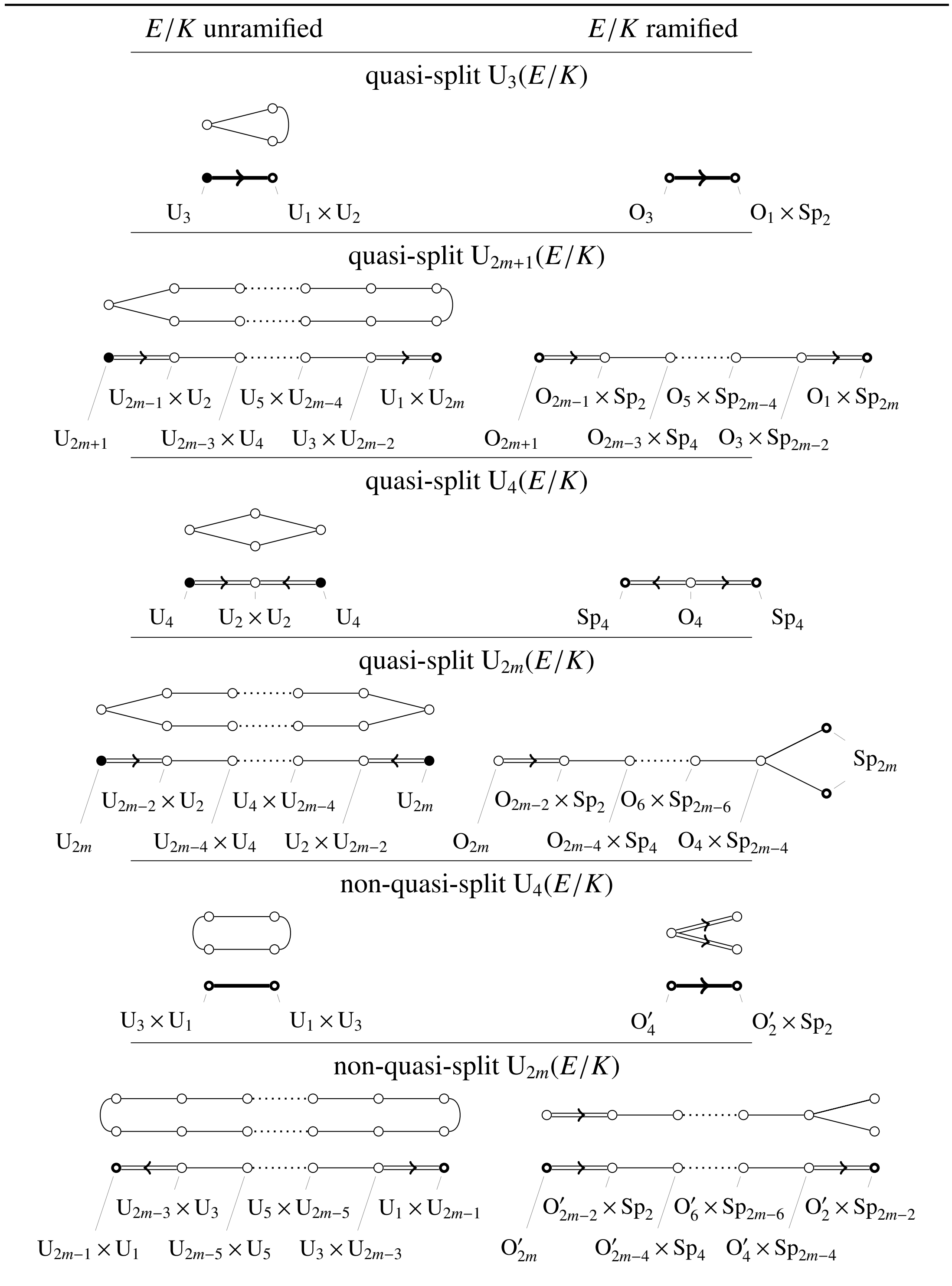

Figure 2.4: Local Indices and Reductions of Unitary Groups

ducible. Let

$$\mathrm{N}(\mathcal{T},\mathcal{T}') = \{g \in \mathcal{G} \mid \mathcal{T}g = g\mathcal{T}'\}$$

$$\mathrm{W}(\mathcal{T},\mathcal{T}') = \{\mathcal{T}g \mid g \in \mathrm{N}(\mathcal{T},\mathcal{T}')\}$$

Each $\omega \in \mathrm{W}(\mathcal{T},\mathcal{T}')$ gives a well defined map

$$\mathrm{Hom}(\mathcal{T}', \mathbb{C}^\times) \to \mathrm{Hom}(\mathcal{T}, \mathbb{C}^\times),$$

which we will denote by $\theta' \mapsto {}^\omega\theta'$. If θ' and ω are F-invariant, ${}^\omega\theta'$ will be as well.

Theorem 2.15.1 (c.f. [11, Thm. 7.3.4])**.**

$$(R_{\mathcal{T},\theta}, R_{\mathcal{T}',\theta'}) = \#\{\omega \in \mathrm{W}(\mathcal{T},\mathcal{T}')^F \mid {}^\omega\theta' = \theta\}.$$

We say that θ is in *general position* if the only F-invariant element of $\mathrm{W}(\mathcal{T})$ fixing θ is the identity. If θ is in general position then $\pm R_{\mathcal{T},\theta}$ is the character of an irreducible representation of dimension equal to the maximal factor of $[\mathcal{G}(k) : \mathcal{T}(k)]$ that is relatively prime to p [11, Thm. 7.5.1, Cor. 7.3.5]. Moreover, if $\mathcal{T}$ is obtained from a quasi-split torus $\mathcal{S}$ by twisting by $w \in \mathrm{W}(\mathcal{S})$, then the sign needed to make $R_{\mathcal{T},\theta}$ into a genuine character is $\det w$ [11, Prop. 7.5.2].

Finally, we note that every irreducible representation of $\mathcal{G}(k)$ occurs as a constituent of some $R_{\mathcal{T},\theta}$.

2.16 Representations of p-adic Groups

Casselman's notes [12] provide a useful introduction to admissible representations of p-adic groups.

Recall from Section 2.4 the notions of smooth and admissible representation as well as parabolic, smooth and compact induction.

Compact induction will provide the final step in our construction of supercuspidal representations; in order to obtain the desired properties of these representations we will need the following theorems.

A representation of a reductive group over k is said to be *cuspidal* if it is not a subquotient of the parabolic induction of a represention on any proper parabolic subgroup [11, Ch. 9]. We will be working with the following special case of an unrefined minimal K-type:

Definition 2.16.1 (c.f. [38, §3.4, 6.5])**.** (i) A *depth-zero vertex representation* for $\mathbf{G}$ is a pair (x, θ_0), where $x \in \mathcal{B}(\mathbf{G})$ is a vertex and θ_0 is the inflation to $\mathbf{G}(K)_x^\circ$ of a cuspidal representation of $\mathcal{G}_x$.

(ii) For such a pair, let $\mathcal{E}(\theta_0)$ be the set of irreducible representations of $\mathbf{G}(K)_x$ (up to equivalence) whose restriction to $\mathbf{G}(K)_x^\circ$ contains θ_0.

Note that $\mathcal{E}(\theta_0)$ consists of finite dimensional representations since $\mathbf{G}(K)_x$ is compact.

Theorem 2.16.2 (c.f. [38, §6.3, Prop. 6.6])**.** *Suppose that (x, θ_0) is a depth-zero vertex representation and $\theta \in \mathcal{E}(\theta_0)$. Then $\pi(\theta) := \mathrm{ind}_{\mathbf{G}(K)_x}^{\mathbf{G}(K)} \theta$ is irreducible and supercuspidal.*

The admissibility of $\pi(\theta)$ follows from the following two results.

Theorem 2.16.3 (c.f. [31])**.** *Every irreducible smooth representation of $\mathbf{G}(K)$ is admissible.*

Proposition 2.16.4 (c.f. [12, Thm. 2.4.1])**.** *The compact induction of a smooth representation is smooth.*

Chapter 3

Unramified Anisotropic Tori

3.1 Unramified Tori

We now begin the construction of the L-packet Π_φ. In this chapter we use φ to produce an unramified anisotropic torus $\mathbf{T}$ that will serve as an ingredient in the representations of Π_φ.

We assume from this point forward that K is a p-adic field and either $\mathbf{G} = \mathrm{U}(V)$ is a quasi-split unitary group or $\mathbf{G} = \mathrm{SU}(V)$ is a special unitary group (though many of the results of this chapter hold for more general $\mathbf{G}$: see Section 5.5.1). In either case, we assume that V is a Hermitian space with respect to the quadratic extension E/K. Moreover, in order to deal with the case not handled by DeBacker-Reeder we assume that E/K is ramified. We also fix a quasi-split maximal torus $\mathbf{S}$ in $\mathbf{G}$ and a Langlands parameter φ for $\mathbf{G}$, which we assume is tame, discrete and regular. The condition that φ is tame implies that E/K is tame and thus that $p \neq 2$.

Recall from Section 2.7.6 that stable classes of maximal tori in $\mathbf{G}$ are classified by $\mathrm{H}^1(K, \mathbf{W})$, and the torus associated to a 1-cocycle $\rho \in \mathcal{Z}^1(K, \mathbf{W})$ is denoted $\mathbf{S}_\rho$.

Definition 3.1.1. We say a torus $\mathbf{S}_\rho$ is *unramified* if it becomes isomorphic to $\mathbf{S}$ over the maximal unramified extension K_{nr} of K.

For split groups, one would say that a torus was unramified if it splits over an unramified extension. But since no torus splits in K_{nr}, we settle for $\mathbf{S}_\rho$ becoming isomorphic to the canonical stable class of $\mathbf{S}$. We can classify unramified maximal tori using the inflation restriction sequence [45, §VII.6]:

$$0 \to \mathrm{H}^1(\Gamma_{nr}, \mathbf{W}^{\mathcal{I}}) \xrightarrow{\mathrm{Inf}} \mathrm{H}^1(K, \mathbf{W}) \xrightarrow{\mathrm{Res}} \mathrm{H}^1(\mathcal{I}, \mathbf{W})^F \to \cdots$$

Since $\mathrm{H}^1(\mathcal{I}, \mathbf{W})$ classifies tori in $\mathbf{G}$ up to K_{nr}-isomorphism, a torus $\mathbf{S}_\rho$ will become isomorphic to $\mathbf{S}$ over K_{nr} if and only if the image of ρ in $\mathrm{H}^1(\mathcal{I}, \mathbf{W})$ is trivial. So we get the following proposition:

Proposition 3.1.2. *Stable classes of maximal unramified tori are classified by* $\mathrm{H}^1(\Gamma_{nr}, \mathbf{W}^{\mathcal{I}})$.

In the case that E/K is totally ramified, Γ_{nr} acts trivially on $\mathbf{W}$ and $\mathbf{W}^{\mathcal{I}}$, and thus stable classes of maximal unramified tori are in bijection with conjugacy classes in $\mathbf{W}^{\mathcal{I}}$. We will explore this bijection more fully for unitary groups in Section 3.3.

3.2 Construction From a Langlands Parameter

Before further exploring unramified tori, we explain how they arise in the context of Langlands parameters. Recall from Section 2.1 that $\tilde{\tau}$ is a topological generator for $\mathcal{I}_t$, and τ is the image in $\mathrm{Gal}(E/K)$ of $\tilde{\tau}$. Define $z \in \hat{\mathbf{G}}$ by

$$\varphi(\tilde{\tau}) = z\tau.$$

Proposition 3.2.1. *We may conjugate φ by an element of $\hat{\mathbf{G}}$ so that $z \in \hat{\mathbf{S}}^\tau$.*

Proof. An automorphism of $\hat{\mathbf{G}}$ is said to be semisimple if its action on $\hat{\mathfrak{g}}$ is diagonalizable. Since $\varphi(\tilde{\tau})$ has finite order, conjugation by it is a semisimple automorphism. We now apply [42, Lem. 3.2], the proof of which we include for clarity.

By [48, Thm. 7.5], conjugation by $\varphi(\tilde{\tau})$ preserves a Borel of $\hat{\mathbf{G}}$ and maximal torus within; we may conjugate so that it preserves our fixed $\hat{\mathbf{B}} \supset \hat{\mathbf{S}}$. Thus z normalizes $\hat{\mathbf{B}}$ and $\hat{\mathbf{S}}$; since the intersection of these two normalizers is just $\hat{\mathbf{S}}$ we have $z \in \hat{\mathbf{S}}$.

Consider the map $p\colon \hat{\mathbf{S}}^\tau \to \hat{\mathbf{S}}/(1-\tau)\hat{\mathbf{S}}$ defined as the restriction of the natural quotient $\hat{\mathbf{S}} \to \hat{\mathbf{S}}/(1-\tau)\hat{\mathbf{S}}$. If e is the order of τ, then the kernel of p is contained within the e-torsion subgroup of $\hat{\mathbf{S}}$ since the map $t \mapsto t \cdot \tau(t) \cdots \tau^{e-1}(t)$ sends $(1-\tau)\hat{\mathbf{S}}$ to 1 and $t \in \hat{\mathbf{S}}^\tau$ to t^e. Thus $\ker(p)$ is finite. Moreover, $\hat{\mathbf{S}}^\tau$ and $\hat{\mathbf{S}}/(1-\tau)\hat{\mathbf{S}}$ have the same dimension, and $\hat{\mathbf{S}}/(1-\tau)\hat{\mathbf{S}}$ is connected and thus p is surjective. So we can find $t \in \hat{\mathbf{S}}$ with

$$t^{-1} z \tau(t) \in \hat{\mathbf{S}}^\tau.$$

But this is exactly the image of $\tilde{\tau}$ under φ after conjugating by t. □

From now on, we assume that $z \in \hat{\mathbf{S}}^\tau$. To construct our unramified anisotropic torus, we want to obtain an elliptic element of $\mathbf{W}^{\mathcal{I}}$. The first step in this process is the following lemma:

Lemma 3.2.2. *Assume that φ is regular. Then the centralizer of $\varphi(\tilde{\tau})$ is given by*

$$\mathrm{Z}_{\hat{\mathbf{G}}}(\varphi(\tilde{\tau})) = \hat{\mathbf{S}}^\tau.$$

Proof. The group $\mathrm{Z}_{\hat{\mathbf{G}}}(\varphi(\tilde{\tau}))$ certainly contains $\hat{\mathbf{S}}^\tau$. In fact, $\hat{\mathbf{S}}^\tau$ is a maximal torus in $\mathrm{Z}_{\hat{\mathbf{G}}}(\varphi(\tilde{\tau}))$ using a result of Steinberg [48]. But our assumption that φ is regular implies that $\mathrm{Z}_{\hat{\mathbf{G}}}(\varphi(\tilde{\tau}))$ is a torus, and we thus obtain the desired result.

Alternatively, one can use a result of Reeder [42, Prop. 3.8] to equate the Lie algebras $\hat{\mathfrak{g}}^{\varphi(\tilde{\tau})}$ and $\hat{\mathfrak{s}}^\tau$.

□

Remark 3.2.3. If we assume only that φ is tame, discrete and trivial on $\mathrm{SL}_2(\mathbb{C})$ then we already get that

$$Z_{\hat{\mathbf{G}}}(\varphi(\tilde{\tau}))^\circ = \hat{\mathbf{S}}^\tau.$$

In order to get an unramified torus, we need to obtain an element of $\mathrm{H}^1(\Gamma_{nr}, \mathbf{W}^{\mathcal{I}})$. We do so by reducing $\varphi(F)$ modulo $\hat{\mathbf{S}}$. As long as φ maps F into $\mathrm{N}_{\hat{\mathbf{G}}}(\hat{\mathbf{S}})$, we obtain a cocycle in $\mathrm{H}^1(\Gamma_{nr}, \mathbf{W}^{\mathcal{I}})$.

Proposition 3.2.4. *After conjugating so that $z \in \hat{\mathbf{S}}^\tau$, we have $\varphi(F) \in \mathrm{N}_{\hat{\mathbf{G}}}(\hat{\mathbf{S}})$.*

Proof. We first show that it suffices to find a regular element $z_0 \in \hat{\mathbf{S}}^\tau$. The centralizer of z_0 would then would be the unique maximal torus containing z_0 [28, Prop. 2.3] and would also contain $\hat{\mathbf{S}}^\tau$:

$$Z_{\hat{\mathbf{G}}}(\hat{\mathbf{S}}^\tau) = \hat{\mathbf{S}}. \tag{3.2.1}$$

The image of F under φ must normalize $\hat{\mathbf{S}}^\tau$ since F normalizes the powers of $\tilde{\tau}$. Now (3.2.1) implies that $\varphi(F) \in \mathrm{N}_{\hat{\mathbf{G}}}(\hat{\mathbf{S}})$.

To find z_0, let $2\rho^\vee$ be the sum of the positive coroots of $\hat{\mathbf{S}}$ in $\hat{\mathbf{G}}$, which is τ-invariant since the corresponding Borel subgroup of $\hat{\mathbf{G}}$ is stable under τ. We claim that for $\epsilon \neq 0$, $z_0 = \rho^\vee(1+\epsilon)$ is an element of $\hat{\mathbf{S}}^\tau$ and a regular semisimple element of $\hat{\mathbf{G}}$. The first claim follows since $\rho^\vee$ is τ-invariant, and the second since no root of $\hat{\mathbf{S}}$ vanishes on $\rho^\vee$. □

Proposition 3.2.4 allows us to define an element $\omega \in \mathbf{W} \cong \mathrm{N}_{\hat{\mathbf{G}}}(\hat{\mathbf{S}})/\hat{\mathbf{S}}$ by projecting $\varphi(F)$. If we further assume that $q \equiv 1 \pmod{[E:K]}$, then $\omega \in \mathbf{W}^{\mathcal{I}}$, since the projection $\tau \in \mathbf{W} \rtimes \mathrm{Gal}(E/K)$ of $\varphi(\tilde{\tau})$ will satisfy $\omega\tau\omega^{-1} = \tau^q = \tau$. By Proposition 3.1.2 we get an isomorphism class of unramified tori; we will denote by $\mathbf{T}$ an abstract torus in this isomorphism class. Moreover, since we assume that the centralizer of the image of φ is finite, Frobenius acts without fixed points on $X_*(\mathbf{S}^\tau)$ and thus ω is an elliptic element of $\mathbf{W}^{\mathcal{I}}$

and $\mathbf{T}$ is anisotropic. Tracing through the bijection between $\mathrm{H}^1(\Gamma_{nr}, \mathbf{W}^{\mathcal{I}})$ and stable classes of tori (see Section 2.7.6), we can describe the Galois action on $X_*(\mathbf{T})$.

Construction 3.2.5. *Suppose that E/K is totally ramified and $q \equiv 1 \pmod{[E:K]}$. Then the construction described in this section produces an unramified anisotropic torus $\mathbf{T}$. The splitting field M of $\mathbf{T}$ is naturally identified with the subgroup of $\mathbf{W}^{\mathcal{I}} \times \mathrm{Gal}(E/K)$ generated by ω and $\mathrm{Gal}(E/K)$. The character and cocharacter groups $X^*(\mathbf{T})$ and $X_*(\mathbf{T})$ are identified with $X^*(\mathbf{S})$ and $X_*(\mathbf{S})$, but Frobenius now acts via ω rather than trivially; the action of $\tilde{\tau}$ remains the same.*

We can summarize the action of Γ on $\hat{\mathbf{T}}$ as follows. As a complex algebraic group, we identify $\hat{\mathbf{S}}$ with $\hat{\mathbf{T}}$. Let D_φ be the subgroup of ${}^L\mathbf{G}$ generated by $\hat{\mathbf{T}} \rtimes \mathrm{Gal}(E/K)$ and $\varphi(F)$. Then there is an exact sequence

$$1 \to \hat{\mathbf{T}} \to D_\varphi \, \mathrm{Gal}(M/K) \to 1 \tag{3.2.2}$$

so that the action of $\mathrm{Gal}(M/K)$ on $\hat{\mathbf{T}}$ is given by conjugating by a lift in D_φ.

Note that in our case, where $\mathbf{G}$ is a unitary group or a special unitary group, the condition that $q \equiv 1 \pmod{[E:K]}$ is implied by the requirement that φ be tame. We include this as an assumption because the constructions in this section work for more general $\mathbf{G}$; see Section 5.5.1 for more discussion of other $\mathbf{G}$.

3.3 Conjugacy Classes in Weyl Groups

In this section we describe the conjugacy classes of $\mathbf{W}^{\mathcal{I}}$ more concretely for unitary groups in the interest of describing the possible tori $\mathbf{T}$ that may arise in Construction 3.2.5.

This description allows us to determine which conjugacy classes yield anisotropic tori.

3.3.1 Tori in Unitary Groups

Recall from Section 2.7.4 that Γ acts on $\mathbf{W}$ through its quotient $\mathrm{Gal}(E/K)$, and the nontrivial element of $\mathrm{Gal}(E/K)$ acts as conjugation by $\eta = (-1,1)\cdots(-m,m)$. In the case that E/K is ramified, $\mathcal{I}$ will act nontrivially on $\mathbf{W}$ and $\mathbf{W}^{\mathcal{I}} = \mathrm{Z}_{\mathbf{W}}(\eta)$.

Proposition 3.3.1. $\mathbf{W}^{\mathcal{I}} = \mathrm{Z}_{\mathbf{W}}(\eta)$ *is isomorphic to* $(\mathbb{Z}/2\mathbb{Z})^m \rtimes \Sigma_m$*, where* Σ_m *denotes the symmetric group on m elements. The transpositions* $(-i,i)$ *generate the* $(\mathbb{Z}/2\mathbb{Z})$ *terms, and the* Σ_m *term is generated by* $(1,2)(-1,-2)$ *and* $(1,2,\ldots,m)(-1,-2,\ldots,-m)$.

Proof. Each of these elements commutes with η by direct computation, so it's enough to check that the cardinality of $\mathrm{Z}_{\mathbf{W}}(\eta)$ is $2^m m!$. By the orbit-stabilizer theorem, it's enough to check that the size of the conjugacy class of η is $\frac{n!}{2^m m!}$. The number of ways to obtain a conjugate element is the number of ways to choose m pairs of distinct elements of $\{1,\ldots,n\}$, which is

$$\frac{1}{m!}\prod_{j=0}^{m-1}\binom{n-2j}{2} = \frac{n!}{2^m m!}.$$

□

This group is in fact the Weyl group of the root system B_m, which we will denote by $\mathrm{W}(\mathrm{B}_m)$.

Remark 3.3.2. Reeder [43] includes a discussion of centralizers in Weyl groups; he focuses on the Weyl group of E_8, but the techniques he applies are applicable to a general Weyl group. Previously, Carter [10] described the conjugacy classes in a uniform manner for all the Weyl groups, though these results had been previously published separately. Of course, knowledge of the conjugacy classes allows us to compute orders of centralizers, but not their group structure.

We will need a description of the conjugacy classes in $\mathbf{W}^{\mathcal{I}}$; since $\mathbf{W}^{\mathcal{I}} \cong \mathrm{W}(\mathrm{B}_m)$ we can turn to Carter [10, pp. 25-26] once again. The following description of conjugacy classes in $\mathrm{W}(\mathrm{B}_m)$ is due to him.

$\mathrm{W}(\mathrm{B}_m)$ acts on the vectors $\{\pm e_i\}_{i=1}^m$, and for a given element $w \in \mathrm{W}(\mathrm{B}_m)$ one can decompose it into cycles on these vectors. Such cycles take the form

$$e_{i_1} \mapsto \pm e_{i_2} \mapsto \pm e_{i_3} \mapsto \cdots \mapsto \pm e_{i_r} \mapsto \pm e_{i_1}.$$

Definition 3.3.3. We say that such a cycle $(i_1, \ldots, i_r)$ is *positive* if $w^r(e_{i_1}) = e_{i_1}$ and *negative* if $w^r(e_{i_1}) = -e_{i_1}$. We call r the *length* of w; w has order r if w is positive and $2r$ if negative. The collection of lengths and signs of the cycles of w is called the *signed cycle type* of w.

Given such a signed cycle type, we can define a pair of partitions μ and ν by setting μ to be the collection of lengths of positive cycles and ν to be the collection of lengths of negative cycles.

Proposition 3.3.4 (c.f. [10, Prop. 24])**.**

(i) A signed cycle type occurs for some element of $\mathrm{W}(\mathrm{B}_m)$ *if and only if* $|\mu| + |\nu| = m$.

(ii) The conjugacy classes of $\mathrm{W}(\mathrm{B}_m)$ *are in bijection with the possible signed cycle types.*

For example, if $n = 6$ (so $m = 3$) we list the following conjugacy classes for $\mathbf{W}^{\mathcal{I}}$ in Figure 3.1.

Positive Cycle Lengths	**Negative Cycle Lengths**	**Conjugacy Class Rep.**
1,1,1	∅	()
2,1	∅	(1,2)(-1,-2)
3	∅	(1,2,3)(-1,-2,-3)
1,1	1	(1,-1)
2	1	(1,2)(-1,-2)(3,-3)
1	1,1	(1,-1)(2,-2)
1	2	(1,2,-1,-2)
∅	1,1,1	(1,-1)(2,-2)(3,-3)
∅	2,1	(1,2,-1,-2)(3,-3)
∅	3	(1,2,3,-1,-2,-3)

Figure 3.1: Conjugacy classes in the rational Weyl group of U_6

Let c be a conjugacy class in $\mathbf{W}^{\mathcal{I}}$ and c_0 a representative for c. Write ξ_c' for the element of $\mathrm{H}^1(\Gamma_{nr}, \mathbf{W}^{\mathcal{I}})$ given by the homomorphism ρ_c' with $\rho_c'(F) = c_0$, and let $\mathbf{T}_c$ be a torus in the associated stable class.

Proposition 3.3.5. $\mathbf{T}_c$ *is anisotropic if and only if c has no positive cycles.*

Proof. Let $\rho_c\colon \Gamma \to \mathbf{W} \rtimes \mathrm{Gal}(E/K)$ be the homomorphism corresponding to the image of ξ_c' under inflation. By the definition of inflation,

$$\rho_c(\sigma) = (\rho_c'(\sigma \bmod \mathcal{I}), \sigma \bmod \Gamma_E).$$

Suppose that the order of c_0 is f. Then the image of ρ_c will be isomorphic to $\mathbb{Z}/f\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, generated by $(c_0, 1)$ and $(1, \tau)$. In order to determine $X^*(\mathbf{T}_c)^\Gamma$, consider the elements of $X^*(\mathbf{T}_c)$ fixed by both $(c_0, 1)$ and $(1, \tau)$. The elements invariant under $(1, \tau)$ are exactly the span of

$$\{\chi_i - \chi_{-i} \mid 1 \le i \le m\}.$$

What about $(c_0, 1)$? Consider a positive cycle of length r; up to conjugation we can take this cycle to be $(1, 2, \ldots, r)(-1, -2, \ldots, -r)$. This cycle leaves $\chi_1 + \cdots \chi_r - \chi_{-1} - \cdots - \chi_{-r}$ invariant. But this element is also invariant under $(1, \tau)$. Since all the other cycles in c_0 are disjoint from this one, we get a nonzero element of $X^*(\mathbf{T}_c)^\Gamma$ if c contains any positive cycles.

Now consider a negative cycle of length r; up to conjugation we can take this cycle to be $(1, 2, \ldots, r, -1, -2, \ldots, -r)$. On the subspace of $X^*(\mathbf{T}_c)$ spanned by $\chi_1, \ldots, \chi_r, \chi_{-1}, \ldots, \chi_{-r}$, the only line left invariant by $(1, 2, \ldots, r, -1, -2, \ldots, -r)$ is the one spanned by $\chi_1 + \cdots \chi_r + \chi_{-1} + \cdots \chi_{-r}$. But this vector is negated by the action of $(1, \tau)$. Since there are no positive cycles in c_0, every pair $i, -i$ with $1 \leq i \leq m$ occurs in one of the negative cycles. This leaves only χ_0 in the case that $n = 2m + 1$, which is negated by $(1, \tau)$. Thus if every cycle is negative, $X^*(\mathbf{T}_c)^\Gamma = 0$. □

3.4 Elemental Tori

The decomposition of an acceptable $c \in \mathbf{W}^{\mathcal{I}}$ into negative cycles gives a corresponding decomposition of the torus $\mathbf{T}_c$ as a product of simpler tori. We will give an intrinsic definition of these "elemental" tori first, prove the product decomposition, and then study elemental tori in more detail.

For any r, let K_r be the unramified extension of K of degree r, and note that $E_r = E \cdot K_r$ is an unramified extension of E of degree r. We have $\mathrm{Gal}(E_r/K) \cong \mathbb{Z}/r\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, since the element τ_r of order 2 fixing K_r is central. Let σ_r be the image of $F \in \Gamma_E$ in $\mathrm{Gal}(E_r/K)$; it will be an element of order r in $\mathrm{Gal}(E_s/K)$ fixing E.

We will assume from now on that $s = 2r$ is even. In this case, define $\eta_s = \tau_s \sigma_s^r$ and let

L_r be the fixed field of η_s. The diagram of fields is:

$$
\begin{tikzcd}
 & & E_s \arrow[dl, dash, "\mathrm{Gal}(E_s/E_r)=\langle\sigma_s^r\rangle"'] \arrow[d, dash, "\langle\eta_s\rangle"] \arrow[dr, dash, "\mathrm{Gal}(E_s/K_s)=\langle\tau_s\rangle"] & \\
 & E_r \arrow[dl, dash] \arrow[dr, dash] & L_r \arrow[d, dash] & K_s \arrow[dl, dash] \\
E \arrow[dr, dash] & & K_r \arrow[dl, dash] & \\
 & K & &
\end{tikzcd}
$$

We will frequently suppress the subscript and write τ for τ_s, σ for σ_s, η for η_s and L for L_r. Note that both $\tau \in \mathrm{Gal}(E_s/K_s)$ and η induce $\tau \in \mathrm{Gal}(E/K)$ on E.

We define a torus $\mathbf{T}_s$ over K by

$$\mathbf{T}_s = \mathrm{Res}_{L/K} \mathrm{U}_1(E_s/L). \tag{3.4.1}$$

These tori will form the building blocks for all maximal anisotropic unramified tori, as we will see in Theorem 3.4.1. We first describe the character group $X^*(\mathbf{T}_s)$ more explicitly using Proposition 2.7.1:

$$X^*(\mathbf{T}_s) \cong \mathrm{Ind}_{\Gamma_L}^{\Gamma} X^*(\mathrm{U}_1(E_s/L)).$$

Since $\mathrm{U}_1(E_s/L)$ splits over E_s, Γ acts on $X^*(\mathbf{T}_s)$ through its quotient $\mathrm{Gal}(E_s/K)$. Let $\{\chi\}$ be a basis for $X^*(\mathrm{U}_1(E_s/L)) \cong \mathbb{Z}$. Each coset of $\mathrm{Gal}(E_s/L)$ in $\mathrm{Gal}(E_s/K)$ contains a unique power of σ, and we can choose a basis for the induction where each basis function evaluates to χ on one power of σ and zero on the others. It will be convenient to denote this basis by $\{\chi_{-n}, \ldots, \chi_{-1}, \chi_1, \ldots, \chi_n\}$, where σ acts by the cyclic permutation υ of $\{-n, \ldots, -1, 1, \ldots, n\}$ defined by

$$\begin{tikzcd}
(-n) \arrow[rrrr, bend right] & \cdots \arrow[l] & (-2) \arrow[l] & (-1) \arrow[l] & (1) \arrow[r] & (2) \arrow[r] & \cdots \arrow[r] & (n). \arrow[llll, bend right]
\end{tikzcd}$$

Then the action of $\mathrm{Gal}(E_s/K)$ on $X^*(\mathbf{T}_s)$ is given by

$$\begin{aligned}
\sigma(\chi_i) &= \chi_{\upsilon(i)} \qquad (3.4.2)\\
\eta(\chi_i) &= -\chi_i,
\end{aligned}$$

and thus $\tau(\chi_i) = -\chi_{-i}$. The action on $X_*(\mathbf{T}_s)$ is given similarly. We can now describe any unramified anisotropic torus in terms of the $\mathbf{T}_s$:

Theorem 3.4.1. *Suppose that $\mathbf{T} \subset \mathbf{G}$ is a maximal anisotropic unramified torus, whose stable class corresponds to $c \in \mathbf{W}^{\mathcal{I}}$. Suppose $c = c_1 \cdots c_j$ is the decomposition of c into negative cycles, and let s_i be the length of c_i.*

(i) If $\mathbf{G} = \mathrm{U}_{2m+1}$, then $\mathbf{T} \simeq \prod_{i=1}^{j} \mathbf{T}_{s_i} \times \mathrm{U}_1$.

(ii) If $\mathbf{G} = \mathrm{U}_{2m}$, then $\mathbf{T} \simeq \prod_{i=1}^{j} \mathbf{T}_{s_i}$.

(iii) If $\mathbf{G} = \mathrm{SU}_{2m+1}$, then $\mathbf{T} \simeq \prod_{i=1}^{j} \mathbf{T}_{s_i}$.

(iv) If $\mathbf{G} = \mathrm{SU}_{2m}$, then $\mathbf{T}$ is isomorphic to the subtorus of $\mathbf{T} \simeq \prod_{i=1}^{j} \mathbf{T}_{s_i}$ of dimension $2m - 1$ whose character group is the quotient by the sum of all χ_i.

Proof. In order to define each of these isomorphisms of tori, we may give a Γ-equivariant isomorphism between $X^*(\mathbf{T})$ and the character group of each right hand side.

If c breaks up as the product of disjoint negative cycles of lengths $s_1, s_2, \ldots, s_k$ and $\mathbf{G} = \mathrm{U}_2\, m$, then $X^*(\mathbf{T})$ will decompose as a representation of Γ into a direct sum of submodules of dimensions $2s_1, 2s_2, \ldots, 2s_k$, each spanned by the χ_i for i occurring in a single negative cycle. The action of Γ is precisely the one on $\mathbf{T}_s$ given in (3.4.2)

The case that $\mathbf{G} = \mathrm{U}_{2m+1}$ is similar, but there will be an additional 1-dimensional summand on which Γ acts through $\mathrm{Gal}(E/K)$, with τ negating χ_0.

Finally, the results for special unitary groups can be obtained from the first two cases by noting how the determinant one condition translates to the character group, as in Section 2.7.2. □

3.4.1 Néron model

For any positive even s, consider the torus $\mathbf{T}_s$ defined in (3.4.1). We saw in Section 2.12 that $\mathbf{T}_s$ has a locally finite type Néron model $\mathfrak{T}_s$, and the component group $\mathfrak{T}_s/\mathfrak{T}_s^0$ is given by $(X_*(\mathbf{T}_s)_{\mathcal{I}})^F$. In fact, these Néron models are all connected:

Proposition 3.4.2. $(X_*(\mathbf{T}_s)_{\mathcal{I}})^F = 0$

Proof. Since $(\tau - 1)\lambda_i = -\lambda_{-i} - \lambda_i = (\tau - 1)\lambda_{-i}$, the quotient $X_*(\mathbf{T}_s)/(\tau - 1)X_*(\mathbf{T}_s)$ is free of rank r, generated by the images of $\lambda_1, \ldots, \lambda_r$:

$$(\tau - 1)X_*(\mathbf{T}_s) = \bigoplus_{i=1}^{r} \mathbb{Z}(\lambda_i + \lambda_{-i}).$$

Since σ is the image of F in $\mathrm{Gal}(E_s/K)$, and it acts on the quotient by

$$\bar{\lambda}_1 \mapsto \bar{\lambda}_2 \mapsto \cdots \mapsto \bar{\lambda}_r \mapsto \bar{\lambda}_{-1} = -\bar{\lambda}_1 \mapsto \cdots \mapsto -\bar{\lambda}_r \mapsto \bar{\lambda}_1,$$

a fixed vector $\sum_{i=1}^{r} \alpha_i \lambda_i$ would satisfy

$$\alpha_1 = \alpha_2 = \cdots = \alpha_r = -\alpha_1$$

and thus $\alpha_i = 0$ for all i. $\square$

Next, we compute the Moy-Prasad filtration on $\mathfrak{T}_s$.

3.4.2 Moy-Prasad filtration

Recall from Section 2.13 that the Moy-Prasad filtration is defined by embedding $\mathbf{T}_s$ into an induced torus $\mathbf{R}$ and then setting $\mathbf{T}_s(K)^\alpha = \mathbf{R}(K)^\alpha \cap \mathfrak{T}_s^\circ(\mathcal{O}_K)$ for $\alpha > 0$. In our case, we can take $\mathbf{R} = \mathrm{Res}_{E_s/K}\,\mathbb{G}_m$ and $\mathfrak{T}_s^\circ = \mathfrak{T}_s$, so

$$\mathbf{T}_s(K)^\alpha = \mathbf{R}(K)^\alpha \cap \mathbf{T}_s(K).$$

For fixed α, let d be the floor of 2α. Then

$$\mathbf{T}_s(K)^\alpha = \{1 + \pi_E^d \alpha \ \mid \alpha \in \mathcal{O}_{E_s} \text{ and } \alpha + (-1)^d \eta(\alpha) = -\pi_E^d \alpha \eta(\alpha)\}.$$

We see that the breaks in the filtration occur at half integers, in accordance with Proposition 2.13.1. The successive quotients give another picture of the filtration. Denote by $\mathrm{Tr}_0(k_s/k_r)$ the elements of k_s with trace 0 in k_r.

Proposition 3.4.3. *The quotients of the Moy Prasad filtration on $\mathbf{T}_s(K)$ are given by*

$$\mathbf{T}_s(K)^{d/2}/\mathbf{T}_s(K)^{(d/2)+} = \begin{cases} \mathrm{U}_1(k_s/k_r) & \textit{if } d = 0 \\ k_r & \textit{if } d \textit{ odd} \\ \mathrm{Tr}_0(k_s/k_r) & \textit{if } d \textit{ even} \end{cases}$$

Proof. We can identify the quotient $\mathbf{T}_s(K)^{d/2}/\mathbf{T}_s(K)^{(d/2)+}$ with a subgroup of

$$\mathbf{R}(K)^{d/2}/\mathbf{R}(K)^{(d/2)+} = \begin{cases} k_s^\times & \text{if } d = 0, \\ \pi^d \mathcal{O}_{E_s}/\pi^{d+1}\mathcal{O}_{E_s} \cong k_s & \text{if } d > 0, \end{cases}$$

using Serre [45, Prop. IV.2.5]. Note that E_s/L is unramified, so the subgroup of $k_s^\times$ defined by the condition that $\mathrm{Nm}_{E_s/L}(1+\alpha) = 1$ is precisely the elements of k_s with norm 1 in k_r. For even positive d, the condition that $\alpha + \eta(\alpha) \equiv 0 \pmod{\pi_E}$ translates to a quotient of $\mathrm{Tr}_0(k_s/k_r)$. For odd d, the analogous condition that $\alpha \equiv \eta(\alpha) \pmod{\pi_E}$ translates to a quotient of k_r. □

Starting from the torus $\mathbf{T}$ constructed in Section 3.2, we define a torus $\mathcal{T}$ over k as the special fiber of the connected Néron model of $\mathbf{T}$. We can describe the k-points of $\mathcal{T}$ using the Moy-Prasad filtration:

$$\mathcal{T}(k) = \mathbf{T}(K)^0/\mathbf{T}(K)^{0+}.$$

We define $\mathcal{T}_s$ from $\mathbf{T}_s$ similarly, though now the Néron model is already connected. As a corollary to Proposition 3.4.3 we have:

Corollary 3.4.4. $|\mathcal{T}_s(k)| = q^r + 1$

Proof. This follows from the fact that the norm map is surjective for finite fields and the resulting factorization

$$|k_s^\times| = |\mathrm{U}_1(k_s/k_r)| \cdot |k_r^\times|.$$

$\square$

Theorem 3.4.1 translates to an analogous decomposition of $\mathcal{T}$ as a product of $\mathcal{T}_{s_i}$, with an additional term coming from the U_1 in the odd case.

Chapter 4

A Regular Character

Having associated an anisotropic unramified torus $\mathbf{T}$ to the Langlands parameter φ in Section 3.2, our next objective is to define a character on $\mathbf{T}(K)$ that we can use to construct representations of pure inner forms of $\mathbf{G}$. We don't quite succeed in this goal, but we do define a character χ_φ on $\mathbf{T}(K)^0 \subseteq \mathbf{T}(K)$. Note that for many of the tori constructed in the previous chapter $\mathbf{T}(K) = \mathbf{T}(K)^0$. Only in the case of odd unitary groups does a U_1 term contribute factor of 2 to the index of $\mathbf{T}(K)^0$ in $\mathbf{T}(K)$. This chapter is devoted to the construction of the character χ_φ.

The construction ultimately relies on the Langlands correspondence for tori (see Section 2.11). After conjugating φ so that $\varphi(\mathcal{I}) \subset \hat{\mathbf{S}} \rtimes \mathrm{Gal}(E/K)$, the whole image of φ then lies within $\hat{\mathbf{N}} \rtimes \mathrm{Gal}(E/K)$ by Proposition 3.2.4. If we could fit the image into a semidirect product of $\hat{\mathbf{S}}$ with some finite group H projecting onto $\mathrm{Gal}(E/K)$, then we could use the local Langlands correspondence directly to obtain a character of $\mathbf{T}(K)$ for some twist $\mathbf{T}$ of $\mathbf{S}$ splitting over a field M with $\mathrm{Gal}(M/K) \cong H$. Unfortunately, while we can fit the image into some extension of $\mathrm{Gal}(M/K)$ by $\hat{\mathbf{S}}$, this extension is not split in general. So, we have to work harder in order to determine a character, and the result is not generally defined on all of $\mathbf{T}(K)$.

Recall from Section 3.2 that $\varphi(F) \in \hat{\mathbf{N}}$ projects onto an elliptic element $w \in \mathbf{W}$, and the torus $\mathbf{T}$ is defined by giving a twisted Galois action on $X_*(\mathbf{S})$ factoring through $\mathrm{Gal}(M/K) \cong \langle w \rangle \times \mathrm{Gal}(E/K)$ that makes the sequence

$$1 \to \hat{\mathbf{T}} \to D_\varphi \to \mathrm{Gal}(M/K) \to 1$$

exact. In Section 4.1 we study such sequences, and define an analogue of the restriction map from group cohomology. In Section 4.2 we construct the character χ_φ using this restriction map. Finally, in Section 4.3 we prove that this character has depth zero, and in Section 4.4 that it satisfies a regularity property used to prove the irreducibility of the Deligne-Lusztig representation we will construct in Section 5.3.

4.1 Groups of type L

We begin by defining a generalization of the L-group of $\mathbf{T}$.

Definition 4.1.1. A *group of type L* associated to a torus $\mathbf{T}$ that splits over M/K is an extension D of the form

$$1 \to \hat{\mathbf{T}} \to D \to \mathrm{Gal}(M/K) \to 1.$$

Such extensions are classified up to isomorphism by $\mathrm{H}^2(\mathrm{Gal}(M/K), \hat{\mathbf{T}})$.

A *split group of type L* is a group of type L together with a chosen section of $D \to \mathrm{Gal}(M/K)$ that yields an isomorphism $D \cong \hat{\mathbf{T}} \rtimes \mathrm{Gal}(M/K)$.

The notion of a group of type L is similar to that of Vogan's weak extended group for $\mathbf{G}$ [51, Def. 2.3], but with a torus $\mathbf{T}$ in place of a more general reductive group $\mathbf{G}$, and with $\mathrm{Gal}(M/K)$ in place of Γ.

For any group D of type L, let $P_K(D, \mathbf{T})$ denote the set of homomorphisms from $\mathcal{W}_K$ to D that yield the standard projection $\mathcal{W}_K \to \mathrm{Gal}(M/K)$ when composed with $D \to \mathrm{Gal}(M/K)$, where we consider two such homomorphisms equivalent if one can be obtained from the other via conjugation by an element of $\hat{\mathbf{T}}$. One can consider $P_K(D, \mathbf{T})$ to be a generalization of $\mathrm{H}^1(K, \hat{\mathbf{T}})$, since in the case that D is split

$$P_K(D, \mathbf{T}) = \mathrm{H}^1(K, \hat{\mathbf{T}})$$

(see Section 2.2). For D not split, $P_K(D, \mathbf{T})$ is no longer a group, since it's precisely the splitting that allows one to define a group operation on each fiber of the projection $D \to \mathrm{Gal}(M/K)$.

The group D_φ mentioned above is an example of a group of type L associated to $\mathbf{T}$. Since the image of φ is contained within D_φ, we can consider $\varphi \in P_K(D_\varphi, \mathbf{T})$.

4.1.1 Restriction

For any extension N of K, we can consider the extension of scalars $\mathbf{T}_N$ of $\mathbf{T}$ to N. The splitting field of $\mathbf{T}_N$ is just given by NM, and we have $\mathrm{Gal}(NM/N) \cong \mathrm{Gal}(M/N \cap M)$. If we have a group D of type L associated to $\mathbf{T}$, then we can obtain a group D_N associated to $\mathbf{T}_N$ as follows. The dual group $\widehat{\mathbf{T}_N}$ is just $\hat{\mathbf{T}}$ with the subgroup $\Gamma_N \subset \Gamma$ acting, and thus we will denote it as $\hat{\mathbf{T}}$ as well. We can thus define $D_N \subseteq D$ as the inverse image of $\mathrm{Gal}(M/N \cap M) \subseteq \mathrm{Gal}(M/K)$. The canonical isomorphism between $\mathrm{Gal}(NM/N)$ and $\mathrm{Gal}(M/N \cap M)$ then gives us the exact sequence

$$1 \to \hat{\mathbf{T}} \to D_N \to \mathrm{Gal}(NM/N) \to 1.$$

We can now define a restriction map

$$\mathrm{res}_{N/K} \colon P_K(D, \mathbf{T}) \to P_N(D_N, \mathbf{T}_N)$$

by just restricting to Γ_N. If D is split then D_N will be split by the restriction of the splitting map $\mathrm{Gal}(M/K) \to D$ to $\mathrm{Gal}(M/M \cap N)$; in this case $\mathrm{res}_{N/K}$ is just the normal restriction map of group cohomology from $\mathrm{H}^1(K, \hat{\mathbf{T}}) \to \mathrm{H}^1(N, \hat{\mathbf{T}})$.

Theorem 4.1.2. *Suppose that K_f/K is the maximal unramified subextension of the splitting field M/K of $\mathbf{T}$. Then each fiber of*

$$\mathrm{res}_{K_f/K}\colon P_K(D,\mathbf{T}) \to P_{K_f}(D_{K_f},\mathbf{T}_{K_f})$$

is either empty or a principal homogeneous space for $\mathrm{H}^1(K,\hat{\mathbf{T}}^{\mathcal{I}})$.

Proof. For a fixed $\varphi \in P_K(D,\mathbf{T})$, let $\varphi_f = \mathrm{res}_{K_f/K}(\varphi)$. We want to describe the set of all φ' with $\mathrm{res}_{K_f/K}(\varphi') = \varphi_f$. Since K_f/K is unramified, in order to extend φ_f to all of Γ, we need only specify the image of some Frobenius element $F \in \Gamma$. By multiplying F by an element of Γ_{nr} if necessary, we may assume that F^f acts trivially on M. Since $F^f \in \Gamma_M$, we must therefore have $\varphi_f(F^f) \in \hat{\mathbf{T}}$. Whatever value $\varphi'(F)$ takes, it must satisfy

$$\varphi'(F)^f = \varphi_f(F^f). \tag{4.1.1}$$

Write $x = \varphi(F)$, $x' = \varphi'(F)$ and $x' = xy$. Since x and x' have the same image in $\mathrm{Gal}(M/K)$, we must have $y \in \hat{\mathbf{T}}$.

The image of Frobenius must satisfy an additional condition, arising from the conjugation of Frobenius on inertia. For any $\alpha \in \mathcal{I}_t$, $F\alpha F^{-1} = \alpha^q$. Thus

$$x\varphi_f(\alpha)x^{-1} = \varphi_f(\alpha)^q$$

and

$$x(y\varphi_f(\alpha)y^{-1})x^{-1} = \varphi_f(\alpha)^q.$$

Equating the left hand sides of these two equations we see that y must commute with $\varphi_f(\alpha)$

for every $\alpha \in \mathcal{I}_t$. So in fact $y \in \hat{\mathbf{T}}^{\mathcal{I}}$.

Now we return to (4.1.1). Since $y \in \hat{\mathbf{T}}^{\mathcal{I}}$, for any integer j we have

$$x^j y x^{-j} = F^j(y). \tag{4.1.2}$$

Thus

$$\begin{aligned}
\varphi_f(F^f) &= (xy)^f \\
&= xyx^{-1}x^2yx^{-2}x^3 \cdots x^f yx^{-f}x^f \\
&= \left(\prod_{j=1}^{f} F^j(y)\right) x^f \\
&= \left(\prod_{j=1}^{f} F^j(y)\right) \varphi_f(F^f).
\end{aligned}$$

So if we define a norm map $\mathrm{Nm}_{\hat{\mathbf{T}}^{\mathcal{I}}} : \hat{\mathbf{T}}^{\mathcal{I}} \to \hat{\mathbf{T}}^{\mathcal{I}}$ by $t \mapsto \prod_{j=1}^{f} F^j(t)$, then we must have $y \in \ker(\mathrm{Nm}_{\hat{\mathbf{T}}^{\mathcal{I}}})$.

Conversely, suppose that $y \in \ker(\mathrm{Nm}_{\hat{\mathbf{T}}^{\mathcal{I}}})$. Then setting $x' = xy$ and working backward through the same steps we find that x' satisfies all the identities required for the image of F, and thus defines an element $\varphi' \in P_K(D, \mathbf{T})$ with the same restriction to $P_{K_f}(D_{K_f}, \mathbf{T}_{K_f})$.

Since elements of $P_K(D, \mathbf{T})$ are only defined up to $\hat{\mathbf{T}}$ conjugacy, different values of x' may yield the same element. In fact, x and x' will yield the same element of $P_K(D, \mathbf{T})$ if and only if $y \in (F-1)\hat{\mathbf{T}}^{\mathcal{I}}$. Suppose first that $y = F(z)z^{-1}$ for some $z \in \hat{\mathbf{T}}^{\mathcal{I}}$. Since the image of φ_f projects onto $\mathrm{Gal}(M/K_f)$, each element commutes with $F(z) \in \hat{\mathbf{T}}^{\mathcal{I}}$ and thus conjugating by $F(z)$ leaves the restriction φ_f fixed. Therefore, xy yields the same element of $P_K(D, \mathbf{T})$

as

$$F(z)xyF(z)^{-1} = F(z)xz^{-1} = x.$$

Conversely, suppose x and xy are identified after conjugating by some element $z \in \hat{\mathbf{T}}$. We've already fixed φ_f, so z must commute with every element of the image of φ_f; since the image projects surjectively onto $\mathrm{Gal}(M/K_f)$ we must in fact have $z \in \hat{\mathbf{T}}^{\mathcal{I}}$. We now have

$$\begin{aligned} xy &= zxz^{-1} \\ &= xF^{-1}(z)z^{-1} \\ &= x \cdot F(F^{-1}(z)^{-1}) \cdot F^{-1}(z), \end{aligned}$$

and thus $y = (F-1)(F^{-1}(z)^{-1}) \in (F-1)\hat{\mathbf{T}}^{\mathcal{I}}$.

We now finish with the observation that $\mathrm{H}^1(K, \hat{\mathbf{T}}^{\mathcal{I}})$ can be identified with the quotient $\ker(\mathrm{Nm}_{\hat{\mathbf{T}}^{\mathcal{I}}})/(F-1)\hat{\mathbf{T}}^{\mathcal{I}}$. □

In the case that D is split, theorem 4.1.2 reduces to the inflation-restriction sequence [45, Prop. VII.6.4]:

$$1 \to \mathrm{H}^1(\mathrm{Gal}(K_f/K), \hat{\mathbf{T}}^{\mathcal{I}}) \xrightarrow{\mathrm{inf}} \mathrm{H}^1(K, \hat{\mathbf{T}}) \xrightarrow{\mathrm{res}_{K_f/K}} \mathrm{H}^1(K_f, \hat{\mathbf{T}}).$$

By the remark at the end of that section, this sequence extends to

$$\begin{aligned} 1 \longrightarrow \mathrm{H}^1(\mathrm{Gal}(K_f/K), \hat{\mathbf{T}}^{\mathcal{I}}) &\xrightarrow{\mathrm{inf}} \mathrm{H}^1(K, \hat{\mathbf{T}}) \xrightarrow{\mathrm{res}} \mathrm{H}^1(K_f, \hat{\mathbf{T}})^{\mathrm{Gal}(K_f/K)} \\ &\longrightarrow \mathrm{H}^2(\mathrm{Gal}(K_f/K), \hat{\mathbf{T}}^{\mathcal{I}}) \longrightarrow \mathrm{H}^2(K, \hat{\mathbf{T}}). \quad (4.1.3) \end{aligned}$$

Our next objective is to generalize the later part of this sequence to the case that D is not split. We run into problems defining a $\mathrm{Gal}(K_f/K)$ action on $P_{K_f}(D_{K_f}, \mathbf{T}_{K_f})$: since Γ acts only on $\hat{\mathbf{T}}$ and not on all of D_{K_f}, we can't merely follow Serre [45, Prop. VII.6.3]. In the next section we consider a class of groups of type L that allow us to proceed.

4.1.2 Unramified Groups of type L

We can't define an action of $\mathrm{Gal}(K_f/K)$ on $P_{K_f}(D_{K_f}, \mathbf{T}_{K_f})$ in general, but we can if D_{K_f} is split: in this case $P_{K_f}(D_{K_f}, \mathbf{T}_{K_f}) \cong \mathrm{H}^1(K_f, \hat{\mathbf{T}})$ and we have a standard action of $\mathrm{Gal}(K_f/K)$. This feature motivates the following definition:

Definition 4.1.3. We say that D is *unramified* if D_N is split for some unramified extension N/K.

Proposition 4.1.4. *Set* $I = \mathrm{Gal}(M/K_f)$. *The following conditions are equivalent:*

(i) D is unramified,

(ii) there is a function $\iota \colon I \to D$ *splitting the map* $D \to \mathrm{Gal}(M/K)$*,*

(iii) there is an exact sequence

$$1 \to \hat{\mathbf{T}} \rtimes I \to D \to \mathrm{Gal}(K_f/K) \to 1$$

compatible with the one defining D.

Proof.

- (*i*) $\Rightarrow$ (*ii*): If D is unramified, then there is some $N \subset K_f$ together with a homomorphism $\iota' \colon \mathrm{Gal}(M/N) \to D_N$ splitting the sequence

$$1 \to \hat{\mathbf{T}} \to D_N \to \mathrm{Gal}(M/N) \to 1.$$

  The restriction of ι' to I yields the desired splitting of $D \to \mathrm{Gal}(M/K)$.

- (*ii*) $\Rightarrow$ (*iii*): If we identify I with its image under ι, we get a subgroup $\hat{\mathbf{T}} \rtimes I \subset D$. The quotient is just $\mathrm{Gal}(M/K)/I \cong \mathrm{Gal}(K_f/K)$.

- (*iii*) $\Rightarrow$ (*i*): The restriction of $\hat{\mathbf{T}} \rtimes I \to D$ to I provides the desired splitting for $N = K_f$.

$\square$

We can now return to the exact sequence (4.1.3).

Proposition 4.1.5. *Suppose that D is unramified, and that* $\mathrm{Gal}(M/K_f)$ *is abelian. Then the image of* $\mathrm{res}_{K_f/K} \colon P_K(D, \mathbf{T}) \to P_{K_f}(D_{K_f}, \mathbf{T}_{K_f})$ *is fixed by* $\mathrm{Gal}(K_f/K)$.

Proof. Since D is unramified, $P_{K_f}(D_{K_f}, \mathbf{T}_{K_f}) \cong \mathrm{H}^1(K_f, \hat{\mathbf{T}})$, and we want to use the standard action of $\mathrm{Gal}(K_f/K)$ on $\mathrm{H}^1(K_f, \hat{\mathbf{T}})$ to give an action of Γ on $P_{K_f}(D_{K_f}, \mathbf{T}_{K_f})$. We'd like to define, for $\varphi \in P_{K_f}(D_{K_f}, \mathbf{T}_{K_f})$, $\sigma \in \Gamma$ and $\epsilon \in \Gamma_{K_f}$

$$(\sigma.\varphi)(\epsilon) = \sigma.\varphi(\sigma^{-1}\epsilon\sigma).$$

Here the action of Γ on $\hat{\mathbf{T}} \rtimes I$ should come from conjugation within D, using the exact sequence from Proposition 4.1.4: to determine how σ acts we first project it to $\mathrm{Gal}(K_f/K)$, then lift it arbitrarily to D and conjugate. This doesn't actually yield an action on $\hat{\mathbf{T}} \rtimes I$, since the action would depend on our choice of lift. Suppose that x and x' are two different lifts, and thus $x' = (t,i)x$ for some $(t,i) \in \hat{\mathbf{T}} \rtimes I$. Then conjugation by x and by x' differs by conjugation by (t,i). Since I is assumed to be abelian, a simple computation shows that conjugating by (t,i) is the same as conjugating by $t \in \hat{\mathbf{T}}$. Thus the ambiguity in the definition of the action of $\mathrm{Gal}(K_f/K)$ on $\hat{\mathbf{T}} \rtimes I$ disappears once we note that elements of $P_{K_f}(D_{K_f}, \mathbf{T}_{K_f})$ are defined up to conjugation by an element of $\hat{\mathbf{T}}$. Similarly, modifying σ by an element of Γ_{K_f} has the effect of conjugating $\varphi(\sigma^{-1}\epsilon\sigma)$ by an element of $\hat{\mathbf{T}} \rtimes I$, and thus by an element of $\hat{\mathbf{T}}$ by the same reasoning. So we get a genuine action of $\mathrm{Gal}(K_f/K)$ on $P_{K_f}(D_{K_f}, \mathbf{T}_{K_f})$, and one can check that in fact this is the same action as the one on $\mathrm{H}^1(K_f, \hat{\mathbf{T}})$ described in [45, Prop. VII.6.3].

If $\varphi \in P_{K_f}(D_{K_f}, \mathbf{T}_{K_f})$ is in the image of restriction, write $\tilde{\varphi}$ for a homomorphism on Γ with restriction φ. Then for $\sigma \in \Gamma$,

$$\begin{aligned}(\sigma.\varphi)(\epsilon) &= \sigma.\varphi(\sigma^{-1}\epsilon\sigma)\\ &= \tilde{\varphi}(\sigma)\tilde{\varphi}(\sigma^{-1})\varphi(\epsilon)\tilde{\varphi}(\sigma)\tilde{\varphi}(\sigma)^{-1}\\ &= \varphi(\epsilon),\end{aligned}$$

where all equalities are defined up to conjugation by an element of $\hat{\mathbf{T}}$ that depends on σ but not ϵ. □

Note that it is not necessary for the group $\mathbf{G}$ to be unramified in order for groups of type L associated to it to be unramified. In fact, every group of type L associated to a tame Langlands parameter will be unramified:

Proposition 4.1.6. *Suppose that φ is a tame, discrete, regular Langlands parameter. Then*

D_φ is unramified.

Proof. The inertia subgroup of $\mathrm{Gal}(M/K) \subset \mathbf{W}^\tau \times \mathrm{Gal}(E/K)$ is just the $\mathrm{Gal}(E/K)$ factor. The obvious homomorphism $\mathrm{Gal}(E/K) \subset \mathbf{W}^\tau \times \mathrm{Gal}(E/K) \to \hat{\mathbf{T}} \cdot \mathrm{N}_{\hat{\mathbf{G}}}(\hat{\mathbf{T}})^\tau \rtimes \mathrm{Gal}(E/K)$ provides a partial splitting for the sequence

$$1 \to \hat{\mathbf{T}} \to \hat{\mathbf{T}} \cdot \mathrm{N}_{\hat{\mathbf{G}}}(\hat{\mathbf{T}})^\tau \rtimes \mathrm{Gal}(E/K) \to \mathbf{W}^\tau \times \mathrm{Gal}(E/K) \to 1,$$

and since its image lies within D_φ, this same map splits $D_\varphi \to \mathrm{Gal}(M/K)$. ◻

4.2 Constructing a Character

Suppose that D is an unramified group of type L associated to a torus $\mathbf{T}$ that splits over a tame extension M of K. In this section we define a map

$$\psi_D\colon P_K(D, \mathbf{T}) \to \mathrm{Hom}(\mathbf{T}(K)^0, \mathbb{C}^\times)$$

that will allow us to associate a character χ_φ to each $\varphi \in P_K(D, \mathbf{T})$.

We begin with a lemma. Recall from Section 2.13 that $\mathbf{T}(N)^0$ are the O_N points of the connected Néron model of $\mathbf{T}$.

Lemma 4.2.1. *For any unramified extension N/K, the norm map induces an isomorphism from $\mathbf{T}(N)^0_{\mathrm{Gal}(N/K)}$ to $\mathbf{T}(K)^0$.*

Proof. Consider the Tate cohomology sequence for the $\mathrm{Gal}(N/K)$-module $\mathbf{T}(N)^0$:

$$0 \to \hat{\mathrm{H}}^{-1}(\mathrm{Gal}(N/K), \mathbf{T}(N)^0) \to \mathbf{T}(N)^0_{\mathrm{Gal}(N/K)} \to \mathbf{T}(K)^0 \to \hat{\mathrm{H}}^0(\mathrm{Gal}(N/K), \mathbf{T}(N)^0) \to 0.$$

Since the central map is precisely that induced by the norm, it suffices to prove that the outside two groups are trivial.

Note that $\mathrm{Gal}(N/K)$ is cyclic, and thus

$$\hat{\mathrm{H}}^{-1}(\mathrm{Gal}(N/K), \mathbf{T}(N)^0) \cong \mathrm{H}^1(\mathrm{Gal}(N/K), \mathbf{T}(N)^0)$$

and

$$\hat{\mathrm{H}}^{0}(\mathrm{Gal}(N/K), \mathbf{T}(N)^0) \cong \mathrm{H}^2(\mathrm{Gal}(N/K), \mathbf{T}(N)^0);$$

if we can prove that $\mathbf{T}(N)^0$ is cohomologically trivial, then both Tate cohomology groups will vanish.

To do so, consider the filtration on $\mathbf{T}(N)^0$:

$$\mathbf{T}(N)^0 \supset \mathbf{T}(N)^{a_1} \supset \mathbf{T}(N)^{a_2} \supset \cdots,$$

and for convenience, write $a_0 = 0$ and $A^i = \mathbf{T}(N)^{a_i}$ for the duration of this proof. Each quotient A^i/A^{i+1} is the k_f points of an algebraic group: a torus for $i = 0$ and an additive group for $i > 0$. By Lang's theorem [34, Thm. 1], these quotients are cohomologically trivial. The long exact sequence associated to

$$1 \to A^i/A^{i+1} \to A^0/A^{i+1} \to A^0/A^i \to 1$$

inductively implies that A^0/A^i has trivial cohomology for each i. If $f \in \mathcal{Z}^m(\mathrm{Gal}(N/K), A^0)$

is some cocycle, we can approximate f as

$$f \equiv \partial g_i \pmod{A^i},$$

where $g_i \in C^{m-1}(\mathrm{Gal}(N/K), A^0)$ for each positive integer i; this approximation is possible since $\mathrm{H}^m(\mathrm{Gal}(N/K), A^0/A^i) = 0$. Moreover, we may choose the g_i so that

$$g_{i+1} \equiv g_i \pmod{A^i}.$$

Since K is complete, there is some $g \in C^{m-1}(\mathrm{Gal}(N/K), A^0)$ with

$$g \equiv g_i \pmod{A^i}$$

for each i. Then $f \equiv \partial g \pmod{A^i}$ for all i, and thus $f = \partial g$. Since every cocycle is a coboundary, we have

$$\mathrm{H}^m(\mathrm{Gal}(N/K), \mathbf{T}(N)^0) = 0$$

for all $m > 0$, as desired. □

We can now construct ψ_D. Since M/K is tame, $\mathrm{Gal}(M/K_f)$ will be abelian, where K_f is the maximal unramified subextension of M/K. By Proposition 4.1.5, we have a map

$$P_K(D, \mathbf{T}) \to \mathrm{H}^1(K_f, \hat{\mathbf{T}})^{\mathrm{Gal}(K_f/K)}.$$

Theorem 2.11.1 defines an isomorphism

$$\mathrm{Hom}(\mathbf{T}(K_f), \mathbb{C}^\times) \to \mathrm{H}^1(K_f, \hat{\mathbf{T}}),$$

and Lemma 4.2.1 gives an isomorphism

$$\mathrm{Hom}(\mathbf{T}(K)^0, \mathbb{C}^\times) \to \mathrm{Hom}(\mathbf{T}(K_f)^0_{\mathrm{Gal}(K_f/K)}, \mathbb{C}^\times).$$

Finally, restriction induces a homomorphism

$$\mathrm{Hom}(\mathbf{T}(K_f), \mathbb{C}^\times) \to \mathrm{Hom}(\mathbf{T}(K_f)^0, \mathbb{C}^\times),$$

and those characters fixed by $\mathrm{Gal}(K_f/K)$ are precisely those descending to a well defined homomorphism from the co-invariants $\mathbf{T}(K_f)^0_{\mathrm{Gal}(K_f/K)}$. Putting all of these together, we define ψ_D as the composition

$$\begin{aligned} P_K(D, \mathbf{T}) \to \mathrm{H}^1(K_f, \hat{\mathbf{T}})^{\mathrm{Gal}(K_f/K)} &\xrightarrow{\sim} \mathrm{Hom}(\mathbf{T}(K_f), \mathbb{C}^\times)^{\mathrm{Gal}(K_f/K)} \\ &\to \mathrm{Hom}(\mathbf{T}(K_f)^0_{\mathrm{Gal}(K_f/K)}, \mathbb{C}^\times) \xrightarrow{\sim} \mathrm{Hom}(\mathbf{T}(K)^0, \mathbb{C}^\times). \quad (4.2.1) \end{aligned}$$

If φ is a tame, discrete, regular Langlands parameter then by Proposition 4.1.6 D_φ is unramified. Moreover, $\mathbf{T}$ splits over a tame extension of K, so ψ_{D_φ} exists. For such a φ we will denote the element $\psi_{D_\varphi}(\varphi) \in \mathrm{Hom}(\mathbf{T}(K)^0, \mathbb{C}^\times)$ by χ_φ.

4.3 Depth of character

Just as we defined the depth of an element of $\mathrm{H}^1(\mathcal{W}_K, \hat{\mathbf{T}})$ in section 2.13, we can define the depth of an element of $P_K(D, \mathbf{T})$.

Definition 4.3.1. Suppose $\varphi \in P_K(D, \mathbf{T})$. Then the *depth* of φ is the infimum over $r \geq 0$ with

$$\ker(\varphi) \supset \mathcal{W}_K^r.$$

Note that $\ker(\varphi)$ is well defined even though $\varphi \in P_K(D, \mathbf{T})$ is only defined up to conjugation by an element of $\hat{\mathbf{T}} \subseteq D$.

We can now generalize the depth preservation of the local Langlands correspondence for tori.

Theorem 4.3.2. *Suppose that* $\mathbf{T}$ *splits over a tame extension M of K, and D is an unramified group of type L. Then ψ_D preserves depth.*

Proof. We prove that each map going into the definition of ψ_D preserves depth. Let K_f be the maximal unramified subextension of M/K as in the construction of ψ_D.

(i) Restricting to $\mathcal{W}_{K_f} \subset \mathcal{W}_K^0$ has no effect on depth since $\mathcal{W}_{K_f}^r = \mathcal{W}_K^r \cap \mathcal{W}_{K_f}$.

(ii) The local Langlands correspondence for tori preserves depth: see theorem 2.13.2.

(iii) Restricting characters to $\mathbf{T}(K_f)^0$ has no effect on the depth since we intersect with $\mathbf{T}(K_f)^0$ in the definition of the Moy-Prasad filtration already.

(iv) Finally, we need to show that the norm map $\mathbf{T}(K_f)^0 \to \mathbf{T}(K)^0$ preserves the Moy-Prasad filtration.

Suppose first that $\mathbf{T} = \operatorname{Res}_{M/K} \mathbb{G}_m$. Then

$$\mathbf{T}(K_f) = (M^\times)^f,$$

and $\operatorname{Gal}(K_f/K)$ acts by permuting the coordinates. The norm map thus just multiplies all coordinates together, which sends

$$\mathbf{T}(K_f)^r = (1 + \pi_M^r \mathcal{O}_M)^f$$

surjectively onto $\mathbf{T}(K)^r = (1 + \pi_M^r \mathcal{O}_M)$ for any positive integer r. Since K_f/K is unramified, we get that the Moy-Prasad filtration is preserved by the norm map on $\mathbf{T}$.

For a more general $\mathbf{T}$, we embed $\mathbf{T}$ into a product $\mathbf{R}$ of restrictions of the above form. Since the Moy-Prasad filtration is defined as the intersection of the filtration on $\mathbf{R}$ with the connected Néron model of $\mathbf{T}$, our result follows from the above case and the behavior of Néron models under unramified base change [1, Prop. 10.1.3].

□

Corollary 4.3.3. *If φ is a tame, discrete, regular Langlands parameter then χ_φ has depth zero. In particular, it induces a character on*

$$\mathcal{T}(k) = \mathbf{T}(K)^0/\mathbf{T}(K)^{0+}.$$

Proof. The tameness of φ is equivalent to φ having depth zero. □

4.4 Regularity

In order to prove the irreducibility of the Deligne-Lusztig representations we construct, we need to compute the stabilizer of χ_φ in the Weyl group of $\mathcal{T}$. Currently, we've defined $\mathbf{T}$ abstractly as a twist of $\mathbf{S}$, which is not enough data to define the Weyl group for $\mathbf{T}$. In Chapter 5 we will embed $\mathbf{T}$ into pure inner forms $\mathbf{G}'$ in various ways.

Lemma 4.4.1. *Let $\mathbf{G}'$ be pure inner form of $\mathbf{G}$.*

(i) There is an isomorphism $\beta\colon \mathbf{G} \to \mathbf{G}'$ defined over K_{nr}.

(ii) There is an element $g \in \mathbf{G}'(K_{nr})$ with $\mathbf{T}(K_{nr}) = g\beta(\mathbf{S}(K_{nr}))g^{-1}$. Moreover, the identification of $X^*(\mathbf{S})$ with $X^*(\mathbf{T})$ and $X_*(\mathbf{S})$ with $X_*(\mathbf{T})$ defined by β and conjugation by g is precisely that obtained by the construction of $\mathbf{T}$ as a twist of $\mathbf{S}$.

Proof. By Steinberg's theorem [46, Ch. II §3.3 and III §2.3], $\mathrm{H}^1(K_{nr}, \mathbf{G}) = 0$, and thus all inner forms of $\mathbf{G}$ become isomorphic (and quasi-split) over K_{nr}.

Since $\mathbf{G}$ is already quasi-split over K with totally ramified splitting field, the K-rank and K_{nr}-rank of $\mathbf{G}$ are identical. Since $\mathbf{S}$ contains a maximal K-split torus, and because $\mathbf{S}$ and $\mathbf{T}$ become isomorphic over K_{nr}, they both contain a K_{nr}-split torus of dimension equal to the K_{nr}-rank of $\mathbf{G}$, which is the same as the K_{nr}-rank of $\mathbf{G}'$. Now we note that $\mathbf{G}'$ has a unique conjugacy class of such maximal tori over K_{nr} since it's quasi-split over K_{nr}.

Finally, note that $\mathbf{T}$ is constructed from $\mathbf{S}$ as a twist by an element of
$\mathrm{H}^1(\mathrm{Gal}(K_{nr}/K), \mathbf{W}^{\mathcal{I}})$. □

Conjugation by this g also takes the normalizer of $\beta(\mathbf{S})$ to the normalizer of $\mathbf{T}$, and thus defines an isomorphism of the Weyl group $\mathbf{W_S}$ of $\mathbf{S}$ with the Weyl group $\mathbf{W_T}$ of $\mathbf{T}$, as finite group schemes over K_{nr}. Moreover, since $\mathbf{T}$ splits over M, we can choose both the element g and the isomorphism of Weyl groups to be defined over K_f, the maximal unramified extension of M.

Since $\mathcal{T}$ is defined as the special fiber of the Néron model of $\mathbf{T}$, the Weyl group of $\mathcal{T}$ is naturally identified with the sub-group scheme $\mathbf{W}_{\mathbf{T}}^{\mathcal{I}} \subset \mathbf{W}_{\mathbf{T}}$. As our isomorphism $\mathbf{W}_{\mathbf{S}} \xrightarrow{\sim} \mathbf{W}_{\mathbf{T}}$ is defined over K_{nr}, we may identify $\mathbf{W}_{\mathbf{S}}^{\mathcal{I}}$ and $\mathbf{W}_{\mathbf{T}}^{\mathcal{I}}$. We define a character $\chi'_\varphi \colon \mathbf{S}(K_f) \to \mathbb{C}^\times$ by pulling χ_φ back to $\mathbf{T}(K_f)$ using the norm (essentially moving back one step in the application of ψ_{D_φ}) and then conjugating $\mathbf{T}(K_f)$ to $\mathbf{S}(K_f)$ with g. Since our identification of $\mathbf{W}_{\mathbf{S}}$ with $\mathbf{W}_{\mathbf{T}}$ is done via conjugation by g as well, an element of $\mathbf{W}_{\mathbf{S}}^{\mathcal{I}}$ will fix χ'_φ if and only if the corresponding element of $\mathbf{W}_{\mathbf{T}}^{\mathcal{I}}$ fixes χ_φ. For the rest of this chapter we will work with $\mathbf{W}_{\mathbf{S}}$, so write $\mathbf{W}$ for $\mathbf{W}_{\mathbf{S}}$.

In order to state the result on the stabilizer of χ'_φ in $\mathbf{W}_{\mathbf{S}}^{\mathcal{I}}$, we need to recall some notation from Reeder [42]. Set $Y = X^*(\mathbf{S})$ and $Y_{\mathbb{R}} = Y \otimes \mathbb{R}$. Any element $\vartheta \in \mathrm{Gal}(E/K)$ acts via a pinned automorphism on $\hat{\mathbf{G}}$. Suppose that ϑ has order m, and let

$$P_\vartheta = m^{-1}(1 + \vartheta + \cdots + \vartheta^{m-1}) \in \mathrm{End}(Y_{\mathbb{R}}).$$

Set

$$Y_\vartheta = P_\vartheta Y,$$

the projection of Y onto $Y_{\mathbb{R}}^{\vartheta}$. We then define

$$\widetilde{\mathbf{W}}_\vartheta = \mathbf{W}^\vartheta \ltimes Y_\vartheta.$$

We have the exact sequence

$$1 \to Y \to Y \otimes \mathbb{C} \xrightarrow{\exp} \hat{\mathbf{S}} \to 1,$$

and the subspace $Y_{\mathbb{R}}^{\vartheta}$ maps under exp into $\hat{\mathbf{S}}^{\vartheta}$. By [42, Lem. 3.4], if $x, x' \in Y_{\mathbb{R}}^{\vartheta}$, then the elements $\exp(x)\vartheta$ and $\exp(x')\vartheta$ of ${}^{L}\mathbf{G}$ are $\hat{\mathbf{S}}$-conjugate if and only if $x - x' \in Y_{\vartheta}$. We will apply this result to the case that $\vartheta = \tau$, and note that $\mathbf{W}_{\mathbf{S}}^{\mathcal{I}} = \mathbf{W}_{\mathbf{S}}^{\tau}$. Our next goal is to define an alcove C_τ in $Y_{\mathbb{R}}^{\tau}$.

Reeder denotes by Φ/ϑ the set of ϑ-equivalence classes of roots in $\Phi(\hat{\mathbf{G}}, \hat{\mathbf{S}})$ and for each $a \in \Phi/\vartheta$, he sets

$$\gamma_a = \sum_{\alpha \in a} \bar{\alpha} \qquad \text{and} \qquad \Phi_\vartheta = \{\gamma_a \mid a \in \Phi/\vartheta\},$$

where $\bar{\alpha}$ is the restriction of the root α to W^{ϑ}.

He defines I_ϑ as the set of orbits in $\{1, \ldots, l\}$ under the permutation induced by the action of ϑ on the set $\{\alpha_1, \ldots, \alpha_l\}$ of simple roots in $\Delta(\hat{\mathbf{G}}, \hat{\mathbf{B}})$, for $\iota \in I_\vartheta$ sets $a_\iota \in \Phi/\vartheta$ as the equivalence class containing $\{a_i \mid i \in \iota\}$, and defines $\gamma_\iota = \gamma_{a_\iota}$. The set $\Delta_\vartheta = \{\gamma_\iota \mid \iota \in I_\vartheta\}$ is a base for the reduced root system Φ_ϑ, and he can thus define $\tilde{\gamma}_0$ as the highest root of Φ_ϑ with respect to the base Δ_ϑ. He then sets

$$\tilde{I}_\vartheta = \{0\} \cup I_\vartheta, \qquad \text{and} \qquad \gamma_0 = 1 - \tilde{\gamma}_0.$$

We can now define an alcove C_ϑ in W^{ϑ} by

$$C_\vartheta = \{x \in Y_{\mathbb{R}}^{\vartheta} \mid \gamma_\iota > 0 \ \ \forall \iota \in \tilde{I}_\vartheta\}.$$

There is a unique element y in the closure of C_τ satisfying

$$\varphi(\tilde{\tau}) = \exp(y)\tau.$$

Finally, we let

$$\mathbf{W}_{\varphi(\tilde{\tau})} = \mathrm{N}_{\hat{\mathbf{G}}}(\hat{\mathbf{S}})^{\varphi(\tilde{\tau})}/\hat{\mathbf{S}}^{\tau}$$

be the subgroup of elements of $\mathbf{W}^\tau$ representable by a $\varphi(\tilde{\tau})$ fixed element of $\mathrm{N}_{\hat{\mathbf{G}}}(\hat{\mathbf{S}})$.

From the proof of [42, Lem. 3.9], the projection $\widetilde{\mathbf{W}}_\tau \to \mathbf{W}^\tau$ maps the stabilizer $\widetilde{\mathbf{W}}_{\tau,y}$ of y in $\widetilde{\mathbf{W}}_\tau$ isomorphically onto $\mathbf{W}_{\varphi(\tilde{\tau})}$.

Proposition 4.4.2.

$$\{w \in \mathbf{W}^\tau \mid w \cdot \chi'_\varphi = \chi'_\varphi\} \subseteq \mathbf{W}_{\varphi(\tilde{\tau})}.$$

Proof. The local Langlands correspondence for tori is given by the following series of isomorphisms [55, §7.7]:

$$\begin{aligned}
\mathrm{Hom}(\mathbf{T}(K_f), \mathbb{C}^\times) &\xrightarrow{\sim} \mathrm{Hom}(\mathbf{T}(M), \mathbb{C}^\times)_{\mathrm{Gal}(M/K_f)} \\
&\xrightarrow{\sim} \mathrm{Hom}(M^\times \otimes_{\mathbb{Z}} X_*(\mathbf{T}), \mathbb{C}^\times)_{\mathrm{Gal}(M/K_f)} \\
&\xrightarrow{\sim} \mathrm{Hom}(M^\times, X^*(\mathbf{T}) \otimes_{\mathbb{Z}} \mathbb{C}^\times)_{\mathrm{Gal}(M/K_f)} \\
&\xrightarrow{\sim} \mathrm{H}^1(\mathcal{W}_M, \hat{\mathbf{T}})_{\mathrm{Gal}(M/K_f)} \\
&\xrightarrow{\sim} \mathrm{H}^1(\mathcal{W}_{K_f}, \hat{\mathbf{T}}).
\end{aligned}$$

We can translate to $\mathbf{S}$ by conjugating by g, and then trace through the action of $\mathbf{W}^\tau$. The action of $\mathbf{W}^\tau$ on $\mathbf{S}(K_f)$ comes from its action on $X_*(\mathbf{S})$, and this corresponds to the standard action of $\mathbf{W}^\tau$ on $\hat{\mathbf{S}}$. Since χ_φ maps to φ, an element $w \in \mathbf{W}^\tau$ will fix χ_φ if and only if it fixes the restriction of φ to $\mathcal{W}_{K_f}$. Note that $\varphi \in \mathrm{H}^1(\mathcal{W}_{K_f}, \hat{\mathbf{T}})$ is determined by $\varphi(\tilde{\tau})$ and $\varphi(F^f)$. The condition that φ is fixed by w translates to the requirement that $w \cdot \varphi(\tilde{\tau})$ is conjugate to $\varphi(\tilde{\tau})$ by some $t \in \hat{\mathbf{T}}$, and that $w \cdot \varphi(F^f)$ is also conjugate to $\varphi(F^f)$ by the same t.

We now invoke [42, Lem. 3.4] to replace the condition that $w \cdot \varphi(\tilde{\tau}) = \exp(w \cdot y)\tau$ be conjugate to $\varphi(\tilde{\tau}) = \exp(y)\tau$ with the requirement that

$$y - w \cdot y \in Y_\tau.$$

Since $\widetilde{\mathbf{W}}_\tau = \mathbf{W}^\tau \ltimes Y_\tau$, the statement that we can translate from y to $w \cdot y$ by an element of Y_τ is equivalent to the statement that y is be fixed by some element of $\widetilde{\mathbf{W}}_\tau$. The image of this element under the projection $\widetilde{\mathbf{W}}_\tau \to \mathbf{W}^\tau$ gives us a $w \in \mathbf{W}^\tau$ so that $w \cdot \varphi(\tilde{\tau})$ is conjugate to φ.

Therefore any w fixing χ'_φ must be in the image of $\widetilde{W}_{\tau,y}$ under the projection $\widetilde{W}_\tau \to \mathbf{W}^\tau$, which is precisely $\mathbf{W}_{\varphi(\tilde{\tau})}$. The proof is complete, but we note that the impediment having an equality in the statement of the Proposition is that the same t must conjugate $w \cdot \varphi(F^f)$ to $\varphi(F^f)$. In particular, if $\varphi(F^f) = 1$ then equality holds. But the inclusion is enough for our purposes. □

Chapter 5

Induction to the Full Group

In Chapters 3 and 4, for each Langlands parameter φ, we constructed an anisotropic unramified torus $\mathbf{T}$ and a depth zero character χ_φ of $\mathbf{T}(K)^0$. Both $\mathbf{T}$ and χ_φ depend only on φ, but the L-packet Π_φ is supposed to be parameterized by $\mathrm{Irr}(A_\varphi)$. We need to construct representations of the K-rational points of pure inner forms of $\mathbf{G}$, and the dependence on the choice of character in $\mathrm{Irr}(A_\varphi)$ enters next.

We will see in Proposition 5.1.1 that isomorphism classes of embeddings of $\mathbf{T}$ into pure inner forms of $\mathbf{G}$ are parameterized by $\mathrm{Irr}(A_\varphi)$. Suppose that $\mathbf{G}'$ is a pure inner form of $\mathbf{G}$, and $\rho : \mathbf{T} \to \mathbf{G}'$ an embedding. Since $\mathbf{G}'(K)$ acts on $\mathscr{B}(\mathbf{G}')$, ρ gives an action of $\mathbf{T}(K)$ on $\mathscr{B}(\mathbf{G}')$: we show in Corollary 5.1.9 that there is a unique fixed point, corresponding to a parahoric subgroup $\mathfrak{G}_\rho(O_K) \subseteq G'$. We get an embedding $\mathbf{T}(K)^0 = \mathfrak{T}^\circ(O_K) \to \mathfrak{G}_\rho(O_K)$ and a corresponding embedding $\mathfrak{T}^\circ(k) \to \mathfrak{G}_\rho(k)$. Moreover, the fact that χ_φ is depth zero means that it descends to a character of $\mathfrak{T}^\circ(k)$.

We can now apply the Deligne-Lusztig construction of Section 2.15 to obtain a representation of $\mathfrak{G}_\rho(k)$ and thus of $\mathfrak{G}_\rho(O_K)$. Finally, we argue that the compact induction of this representation to $\mathbf{G}'(K)$ is irreducible.

5.1 Embeddings of T into Pure Inner Forms

Having constructed an anisotropic unramified torus $\mathbf{T}$ from φ, we now consider the different embeddings of $\mathbf{T}$ into $\mathbf{G}$ and its pure inner forms. These embeddings will parameterize the L-packet Π_φ:

Proposition 5.1.1. *Suppose that* $\mathbf{T}$ *is the anisotropic torus constructed from* φ *in Chapter 3. Each choice of embedding* $\iota_0 \colon \mathbf{T} \hookrightarrow \mathbf{G}$ *determines a bijection between*

- *the set of irreducible representations of A_φ and*
- *equivalence classes of embeddings of $\mathbf{T}(K)$ into $\mathbf{G}'(K)$, where $\mathbf{G}'$ ranges over K-isomorphism classes of pure inner forms of $\mathbf{G}$, and we consider two such embeddings equivalent if they differ by conjugation within $\mathbf{G}'(K)$.*

Two choices ι_0 will determine the same bijection if and only if they induce the same map $\mathrm{H}^1(K, \mathbf{T}) \to \mathrm{H}^1(K, \mathbf{G})$.

Proof. By Proposition 2.9.4, we need to identify $\mathrm{H}^1(K, \mathbf{T})$ with the set $\mathrm{Irr}(A_\varphi)$. In our case,

$$A_\varphi = \mathrm{Z}_{\hat{\mathbf{G}}}(\varphi).$$

Since $\mathcal{W}_t$ is generated by $\tilde{\tau}$ and F, the elements of A_φ are precisely those that commute with both $\varphi(\tilde{\tau})$ and $\varphi(F)$. By Lemma 3.2.2 together with the assumption that $\varphi(\tilde{\tau})$ is regular,

$$A_\varphi = \mathrm{Z}_{\hat{\mathbf{S}}^\tau}(\varphi(F)).$$

Now note that we defined $\mathbf{T}$ to be the twist of $\mathbf{S}$ such that F acts on $\hat{\mathbf{T}} = \hat{\mathbf{S}}$ by conjugation by $\varphi(F)$. Thus

$$A_\varphi = \hat{\mathbf{T}}^\Gamma.$$

We can fit $\hat{\mathbf{T}}^\Gamma$ into the long exact sequence determined by the sequence of Γ-modules

$$1 \to X^*(\mathbf{T}) \to X^*(\mathbf{T}) \otimes \mathbb{C} \xrightarrow{\exp} \hat{\mathbf{T}} \to 1.$$

Since $\mathbf{T}$ is anisotropic, $\mathrm{H}^0(K, X^*(\mathbf{T}) \otimes \mathbb{C}) = 0$, and since $X^*(\mathbf{T}) \otimes \mathbb{C}$ is a $\mathbb{C}$-vector space

$\mathrm{H}^1(K, X^*(\mathbf{T}) \otimes \mathbb{C}) = 0$. We thus get an isomorphism

$$A_\varphi \cong \mathrm{H}^1(K, X^*(\mathbf{T})).$$

We finish with the Tate duality isomorphism of Proposition 2.9.1, together with the observation that since A_φ is abelian in our case the irreducible representations are precisely the characters. □

5.1.1 Elemental Embeddings

For each elementary torus $\mathbf{T}_s$, we will define a family of hermitian spaces $\{V_{s,\kappa}\}_{\kappa \in L^\times}$, together with an embedding of $\mathbf{T}_s$ into each unitary group $\mathrm{U}(V_{s,\kappa})$. These unitary groups are not all quasi-split. Instead, we get embeddings into both pure inner forms of $\mathbf{G}$, which will eventually yield representations of the different pure inner forms.

As an E-vector space, $V_{s,\kappa}$ is simply E_s. Following Euler (see [45, p. 56]), for any $\kappa \in E_s$, define a bilinear form ϕ_κ on $V_{s,\kappa}$ by

$$\phi_\kappa(x, y) = \mathrm{Tr}_{E_s/E}\left(\frac{\kappa}{\pi_L} \cdot x \cdot \eta(y)\right).$$

We divide by π_L in the definition of ϕ_κ so that Proposition 5.1.3 holds.

Proposition 5.1.2. *ϕ_κ is Hermitian if and only if $\kappa \in L$.*

Proof. Since the trace pairing is bilinear and nondegenerate (see Section 2.1), and since η induces τ on E, the only property of a Hermitian form that remains is the condition that

$\phi_\kappa(x, y) = \tau\phi_\kappa(y, x)$. Suppose first that $\kappa \in L$. Then

$$\begin{aligned}\tau\phi_\kappa(y, x) &= \eta\,\mathrm{Tr}_{E_s/E}\left(\frac{\kappa}{\pi_L} \cdot y \cdot \eta(x)\right)\\ &= \mathrm{Tr}_{E_s/E}\left(\frac{\eta(\kappa)}{\pi_L} \cdot \eta(y) \cdot x\right)\\ &= \phi_\kappa(x, y)\end{aligned}$$

Conversely, suppose that ϕ_κ is Hermitian. Then

$$\phi_\kappa(x, y) - \tau\phi_\kappa(y, x) = \mathrm{Tr}_{E_s/E}\left(\frac{\kappa - \eta(\kappa)}{\pi_L} \cdot x \cdot \eta(y)\right)$$

is 0 for every $x, y \in E_s$. Thus the nondegeneracy of the trace pairing implies that $\eta(\kappa) = \kappa$. $\square$

From now on we will assume that $\kappa \in L^\times$, in which case $V_{s,\kappa}$ is a Hermitian space. Since $\mathbf{T}_s(K) = \{\alpha \in E_s \mid \mathrm{Nm}_{E_s/L}\,\alpha = 1\}$, we have an embedding

$$\begin{aligned}\mathbf{T}_s(K) &\to \mathrm{U}(V_{s,\kappa})\\ \alpha &\mapsto \text{ multiplication by } \alpha\end{aligned}$$

Proposition 5.1.3. $\mathrm{U}(V_{s,\kappa})$ *is quasi-split if and only if* $\kappa \in \mathrm{Nm}_{E_s/L}(E_s^\times)$.

Proof. We first reduce to the case $s = 2$. Let V'_κ be the two dimensional E_r-vector space E_s with Hermitian pairing ϕ'_κ (relative to the quadratic extension E_r/K_r) defined by

$$\phi'_\kappa(x, y) = \mathrm{Tr}_{E_s/E_r}\left(\frac{\kappa}{\pi_L} \cdot x \cdot \eta(y)\right).$$

We can reconstruct ϕ_κ from ϕ'_κ via the identity $\phi_\kappa = \mathrm{Tr}_{E_r/E} \circ \phi'_\kappa$.

Lemma 5.1.4. $\mathrm{U}(V_{s,\kappa})$ *is quasi-split if and only if* $\mathrm{U}(V'_\kappa)$ *is quasi-split.*

Proof. Note that $\mathrm{U}(V'_\kappa)$ is quasi-split if and only if there is a nonzero isotropic vector $v \in V'_\kappa$, and $\mathrm{U}(V_{s,\kappa})$ is quasi-split if and only if there is an r-dimensional isotropic subspace of $V_{s,\kappa}$.

Suppose that $v \in V'_\kappa = E_s$ is a nonzero isotropic vector. Then I claim that $E_r \cdot v$ is an r-dimensional isotropic subspace of $V_{s,\kappa}$. Suppose that $\alpha, \beta \in E_r$. Then

$$\begin{aligned}
\phi_\kappa(\alpha v, \beta v) &= \mathrm{Tr}_{E_s/E}\left(\frac{\kappa}{\pi_L} \cdot \alpha v \cdot \eta(\beta v)\right) \\
&= \sum_{i=0}^{s-1} \sigma^i \left(\frac{\kappa}{\pi_L} \cdot \alpha v \cdot \eta(\beta v)\right) \\
&= \sum_{i=0}^{r-1} \sigma^i \left(\alpha\tau(\beta)\left(\frac{\kappa}{\pi_L} \cdot v \cdot \eta(v) + \sigma^r\left(\frac{\kappa}{\pi_L} \cdot v \cdot \eta(v)\right)\right)\right) \\
&= \sum_{i=0}^{r-1} \sigma^i \left(\alpha\tau(\beta)\phi'_\kappa(v, v)\right) \\
&= 0
\end{aligned}$$

Conversely, suppose that $X \subset V_{s,\kappa}$ is an r-dimensional isotropic subspace. Since $\mathrm{Tr}_{E_r/E}$ is E-linear, the set

$$Y = \{y \in E_r \mid \mathrm{Tr}_{E_r/E}(y) = 0\}$$

is an $r - 1$ dimensional E-subspace of E_r. The composition

$$X \hookrightarrow E_s \xrightarrow{\Delta} E_s \times E_s \xrightarrow{\phi'_\kappa} E_r$$

is K-linear, and has image contained in Y. But $\dim_K X = 2r$ while $\dim_K Y = 2r - 2$, so the composition has nontrivial kernel. This yields a nonzero isotropic vector in V'_κ and completes the proof of the lemma. □

Suppose that κ_1 and κ_2 are in the same equivalence class modulo $\mathrm{Nm}_{E_s/L} E_s^\times$, ie $\kappa_1 = \mathrm{Nm}_{E_s/L}(\alpha)\kappa_2$. I claim that $x \in E_s$ is isotropic for ϕ'_{κ_1} if and only if αx is isotropic for ϕ'_{κ_2}:

$$\begin{aligned}\phi'_{\kappa_1}(x,x) &= \mathrm{Tr}_{E_s/E_r}\left(\frac{\kappa_1}{\pi_L}\cdot x\cdot\eta(x)\right)\\ &= \mathrm{Tr}_{E_s/E_r}\left(\frac{\kappa_2}{\pi_L}\cdot(\alpha x)\cdot\eta(\alpha x)\right)\\ &= \phi'_{\kappa_2}(\alpha x,\alpha x).\end{aligned}$$

Thus $\mathrm{U}(V'_{\kappa_1})$ is quasi-split if and only if $\mathrm{U}(V'_{\kappa_2})$ is quasi-split. So it suffices to consider representatives for each of the two cosets in $L^\times/\,\mathrm{Nm}_{E_s/L} E_s^\times$, which we can take to be 1 and π_L.

If $\kappa = 1$ then

$$\phi'_1(\pi_L,\pi_L) = \mathrm{Tr}_{E_s/E_r}\left(\frac{1}{\pi_L}\cdot\pi_L\cdot\pi_L\right) = 0,$$

and thus $\mathrm{U}(V'_1)$ is quasi-split.

Finally, the two different equivalence classes have different Hermitian discriminants (see Section 2.6.1):

$$\det\begin{pmatrix}\mathrm{Tr}_{E_s/E_r}\left(\frac{1}{\pi_L}\cdot 1\right) & \mathrm{Tr}_{E_s/E_r}\left(\frac{1}{\pi_L}\cdot\pi_L\right)\\ \mathrm{Tr}_{E_s/E_r}\left(\frac{1}{\pi_L}\cdot\pi_L\right) & \mathrm{Tr}_{E_s/E_r}\left(\frac{1}{\pi_L}\cdot\pi_L^2\right)\end{pmatrix} = -4,$$

while

$$\det\begin{pmatrix} \mathrm{Tr}_{E_s/E_r}\left(\frac{\pi_L}{\pi_L}\cdot 1\right) & \mathrm{Tr}_{E_s/E_r}\left(\frac{\pi_L}{\pi_L}\cdot \pi_L\right) \\ \mathrm{Tr}_{E_s/E_r}\left(\frac{\pi_L}{\pi_L}\cdot \pi_L\right) & \mathrm{Tr}_{E_s/E_r}\left(\frac{\pi_L}{\pi_L}\cdot \pi_L^2\right) \end{pmatrix} = 4\pi_L^2.$$

Since the quotient of the two discriminants, $-\pi_L^2$, has valuation 1 and is a norm from $L^\times$ to $K_r^\times$, it is not a norm from $E_r^\times$, the other tamely ramified quadratic extension of K_r. With different discriminants, $\mathrm{U}(V_1')$ and $\mathrm{U}(V_{\pi_L}')$ are non-isomorphic, and therefore $\mathrm{U}(V_{\pi_L}')$ is not quasi-split. □

Let $u \in K^\times$ be a non-square unit (and thus $u \notin \mathrm{Nm}_{E/K} E^\times$).

Corollary 5.1.5. $\mathrm{disc}(V_{s,\kappa}) \equiv u^{v_L(\kappa)+r(q-1)/2} \pmod{\mathrm{Nm}_{E/K} E^\times}$.

Proof. Since E_s/L is unramified, $\kappa \in \mathrm{Nm}_{E_s/L} E_s^\times$ if and only if $v_L(\kappa) \equiv 0 \pmod 2$. We have from section 2.6.3 that the discriminant of the quasi-split unitary group of dimension s is congruent to $(-1)^r$ modulo $\mathrm{Nm}_{E/K} E^\times$. Since -1 is a unit, it's a norm from E if and only if it's a square, which occurs if and only if $q \equiv 1 \pmod 4$. □

5.1.2 Embeddings of Products

We first consider even dimensional unitary groups. By Theorem 3.4.1 we may write $\mathbf{T} \simeq \prod_{i=1}^{j} \mathbf{T}_{s_i}$; let $\underline{s} = (s_1, \ldots, s_j) = (2r_1, \ldots, 2r_j)$ be the tuple of dimensions and set $L_i = L_{r_i}$.

For odd dimensional unitary groups, Theorem 3.4.1 implies that $\mathbf{T} \simeq \prod_{i=1}^{j-1} \mathbf{T}_{s_i} \times \mathrm{U}_1$; let $s_j = 1$ and $\underline{s} = (s_1, \ldots, s_j) = (2r_1, \ldots, 2r_{j-1}, 1)$ be the tuple of dimensions. Set $L_i = L_{r_i}$ as above, and $L_j = K$; for $\kappa_j \in L_j^\times$ we can define a one dimensional Hermitian space $V_{1,\kappa_j} \cong E$ by setting $\phi_{\kappa_j}(1,1) = \kappa_j/\pi_K$. We will write $\mathbf{T}_1$ for U_1 to simplify notation: $\mathbf{T}_1(K)$ acts on

V_{1,κ_j} by multiplication just as the other T_{s_i} act on V_{s_i,κ_i}. Note that the Hermitian condition on ϕ_{κ_j} forces $\kappa_j \in K^\times$ and thus $v_E(\kappa_j)$ to be even.

In both cases we set $n = \sum_i s_i$. For every j-tuple $\underline{\kappa} = (\kappa_1, \dots, \kappa_j)$ with $\kappa_i \in L_i^\times$, we get a Hermitian space $V_{\underline{s},\underline{\kappa}} = \prod_{i=1}^j V_{s_i,\kappa_i}$ and a product embedding $\mathbf{T} \hookrightarrow \mathrm{U}(V_{\underline{s},\underline{\kappa}})$. Write $\phi_{\underline{\kappa}}$ for the Hermitian pairing on $V_{\underline{s},\underline{\kappa}}$, $\mathbf{G}_{\underline{s},\underline{\kappa}}$ for $\mathrm{U}(V_{\underline{s},\underline{\kappa}})$ and $\mathbf{T}_{\underline{s},\underline{\kappa}}$ for the image of $\mathbf{T}$ in $\mathbf{G}_{\underline{s},\underline{\kappa}}$.

Proposition 5.1.6. *For n odd, $\mathbf{G}_{\underline{s},\underline{\kappa}}$ is always quasi-split; for n even $\mathbf{G}_{\underline{s},\underline{\kappa}}$ is quasi-split if and only if*

$$\sum_{i=1}^{j} v_L(\kappa_i) \equiv 0 \pmod{2}.$$

Proof. The odd case follows from our discussion of unitary groups in section 2.7.5, so we assume that n is even. The discriminant of $\mathbb{H}$ is -1, which is a norm from E if and only if it's a square in $K^\times$, i.e. if $q \equiv 1 \pmod 4$. Since the discriminant is multiplicative for products of Hermitian spaces, by Corollary 5.1.5 $\mathbf{G}_{\underline{s},\underline{\kappa}}$ will be quasi-split if and only if

$$\begin{aligned} u^{(q-1)\sum r_i} &\equiv \mathrm{disc}(V_{\underline{s},\underline{\kappa}}) \\ &\equiv u^{\sum((q-1)r_i + v_L(\kappa_i))} \pmod{\mathrm{Nm}_{E/K} E^\times}, \end{aligned}$$

which will hold if and only if $\sum_{i=1}^j v_L(\kappa_i) \equiv 0 \pmod 2$. □

Since $\mathbf{T}$ is anisotropic, $\mathbf{T}_{\underline{s},\underline{\kappa}}(K)$ is compact and thus contained in at least one maximal compact subgroup of $\mathbf{G}_{\underline{s},\underline{\kappa}}$. In fact, it is contained in a unique maximal compact subgroup. To see this, we consider the action of $\mathbf{T}_{\underline{s},\underline{\kappa}}$ on the building $\mathcal{B}(\mathbf{G}_{\underline{s},\underline{\kappa}})$ (see Section 2.14 for background on buildings).

Theorem 5.1.7. *The action of the torus $\mathbf{T}_{\underline{s},\underline{\kappa}}(K)$ fixes a unique vertex x in $\mathcal{B}(\mathbf{G}_{\underline{s},\underline{\kappa}})$.*

Proof. Recall from Section 2.14.8 that the vertices of $\mathcal{B}(\mathrm{U}(V))$ are in bijection with lattices $\Lambda \subset V$ satisfying

$$\Lambda \supseteq \Lambda^\vee \supsetneq \pi_E \Lambda.$$

For the purpose of reducing subscripts, write E_i for E_{s_i}, L_i for L_{r_i} and $\mathcal{O}_i$ for the ring of integers of E_i for the duration of this proof. For each tuple $\underline{b} = (b_1, \ldots, b_j)$ of integers, we define a lattice

$$\Lambda_{\underline{s},\underline{b}} = \prod_{i=1}^{j} \pi_E^{b_i} \mathcal{O}_i \subset V_{\underline{s},\underline{\kappa}}.$$

Since $\mathrm{Nm}_{E_i/L_i} \alpha = 1$ implies $\alpha \in \mathcal{O}_i^\times$, the action of $\mathbf{T}_{\underline{s},\underline{\kappa}}$ on $V_{\underline{s},\underline{\kappa}}$ preserves $\Lambda_{\underline{s},\underline{b}}$. Each extension E_i/E is unramified and thus has trivial different, so the dual of $\mathcal{O}_i$ under the trace pairing is just $\mathcal{O}_i$, and the dual of $\pi_E^{b_i}\mathcal{O}_i$ under ϕ_{κ_i} is $\pi_E^{-b_i - v_L(\kappa_i)}\mathcal{O}_i$. Therefore, if we write $v_L(\underline{\kappa})$ for $(v_L(\kappa_1), \ldots, v_L(\kappa_j))$,

$$\Lambda_{\underline{s},\underline{b}}^\vee = \Lambda_{\underline{s},-v_L(\underline{\kappa})-\underline{b}}$$

In order for

$$\Lambda_{\underline{s},\underline{b}} \supseteq \Lambda_{\underline{s},\underline{b}}^\vee \supseteq \pi_E \Lambda_{\underline{s},\underline{b}},$$

every entry of $-v_L(\underline{\kappa}) - 2\underline{b}$ must be either 0 or 1. There is a unique such $\underline{b}$ for each $\underline{\kappa}$, and for this choice of $\underline{b}$, the corresponding vertex of $\mathcal{B}(\mathbf{G}_{\underline{s},\underline{\kappa}})$ will be fixed by $\mathbf{T}_{\underline{s},\underline{\kappa}}(K)$. In order to check that $\mathbf{T}_{\underline{s},\underline{\kappa}}(K)$ fixes a unique vertex, it suffices to check that any lattice fixed by $\mathbf{T}_{\underline{s},\underline{\kappa}}(K)$ must be one of the $\Lambda_{\underline{s},\underline{b}}$.

Suppose that Λ is an $\mathcal{O}_E$-lattice in $V_{\underline{s},\underline{\kappa}}$ fixed by $\mathbf{T}_{\underline{s},\underline{\kappa}}(K)$. For each i between 1 and j, let $\lambda_i = (\lambda_{i,1}, \ldots, \lambda_{i,j}) \in \Lambda$ be any element with $v_{E_i}(\lambda_{i,i})$ minimal among the valuations of i^{th} coordinates of elements of Λ; let b_i be this minimal valuation. I claim that $\Lambda = \Lambda_{\underline{s},\underline{b}}$.

First we reduce to working one coordinate at a time. Since $\mathrm{Nm}_{E_i/L_i}(-1) = 1$ for every

i, we have an element $\alpha_i = (-1, -1, \ldots, -1, 1, -1, \ldots, -1) \in \mathbf{T}_{\underline{s},\underline{\kappa}}(K)$, where the 1 occurs in position i. Therefore we may replace λ_i by $\lambda_i/2 + \alpha_i\lambda_i/2 = (0, \ldots, 0, \lambda_{i,i}, 0, \ldots, 0) \in \Lambda$, which also has minimal valuation in the i^{th} coordinate.

By our definition of b_i, we have $\Lambda \subseteq \Lambda_{\underline{s},\underline{b}}$. To show the reverse containment, it suffices to show that

$$\mathcal{O}_E \cdot \mathbf{T}_s(K) := \left\{ \sum_k x_k\alpha_k \;\mid x_k \in \mathcal{O}_E, \alpha_k \in \mathbf{T}_s(K) \right\} = \mathcal{O}_{E_s}. \tag{5.1.1}$$

When $s = 1$ this equation clearly holds, and we achieve our goal for $s = 2r$ in the following lemma.

Lemma 5.1.8. *For any field M, let $\mu_n(M)$ denote the group of n^{th} roots of unity in M.*

(i) $\mu_{q^r+1}(E_s) \subset \mathbf{T}_s(K)$

(ii) $\mu_{q^r+1}(E_s)$ *generates* $\mathcal{O}_{E_s}$ *as an* $\mathcal{O}_E$*-module.*

Proof. The nontrivial element of $\mathrm{Gal}(E_s/L)$ is F^r, which acts on elements $\alpha \in \mu_{q^r+1}$ by $\alpha \mapsto \alpha^{q^r}$. Thus $\mathrm{Nm}_{E_s/L}(\alpha) = \alpha^{q^r+1} = 1$, so $\alpha \in \mathbf{T}_s(K)$.

Now let $\bar{\alpha}$ be a generator for the cyclic group $\mu_{q^r+1}(k_s)$. Since the multiplicative order of $\bar{\alpha}$ is $q^r + 1$, $\bar{\alpha}$ is not contained in any subfield of k_s, and thus the set $\{1, \bar{\alpha}, \ldots, \bar{\alpha}^{s-1}\}$ is a basis for k_s over k. Since E_s/E is unramified we can approximate any element of $\mathcal{O}_{E_s}$ arbitrarily well with elements of $\mathcal{O}_E \cdot \mathbf{T}_s(K)$; completeness of $\mathcal{O}_{E_s}$ now finishes the proof. □

So any element with valuations in each coordinate at least the minimum given by $\underline{b}$ must lie within Λ since we can obtain it from the λ_i by a combination of addition and the action of $\mathbf{T}_s(K)$. This shows that any lattice fixed by $\mathbf{T}_{\underline{s},\underline{\kappa}}(K)$ must be one of the $\Lambda_{\underline{s},\underline{b}}$ and thus there is a unique fixed vertex. □

Corollary 5.1.9. *$\mathbf{T}_{\underline{s},\underline{\kappa}}(K)$ fixes no other point in $\mathcal{B}(\mathbf{G}_{\underline{s},\underline{\kappa}})$.*

Proof. Suppose that $\mathbf{T}_{\underline{s},\underline{\kappa}}(K)$ fixes an additional point $y \in \mathcal{B}(\mathbf{G}_{\underline{s},\underline{\kappa}})$, which we may take to lie in a common apartment $\mathcal{A}$. Since $\mathbf{T}_{\underline{s},\underline{\kappa}}(K)$ acts isometrically, it must fix the whole line between x and y. This line will pass through the interior of some facet in $\mathcal{A}$ that is not a vertex, and since $\mathbf{T}_{\underline{s},\underline{\kappa}}(K)$ acts by simplicial automorphisms, it must fix the whole facet, and thus the vertices in the closure of the facet. This contradicts Theorem 5.1.7. □

Corollary 5.1.10.

(i) *$\mathbf{T}_{\underline{s},\underline{\kappa}}(K)$ is contained in a unique maximal compact subgroup $\mathfrak{G}_{\underline{s},\underline{\kappa}}(\mathcal{O}_K) \subset \mathbf{G}_{\underline{s},\underline{\kappa}}(K)$.*

(ii) *$\mathfrak{T}^{\circ}_{\underline{s},\underline{\kappa}}(\mathcal{O}_K)$ is contained in a unique maximal parahoric subgroup $\mathfrak{G}^{\circ}_{\underline{s},\underline{\kappa}}(\mathcal{O}_K)$.*

Proof. Every maximal compact subgroup fixes a point of $\mathcal{B}(\mathbf{G})$, and every maximal parahoric subgroup fixes a vertex. □

At this point we fix $\underline{s}$ and $\underline{\kappa}$ in order to simplify the notation. Note that $\underline{s}$ is determined by $\mathbf{T}$, and the choice of $\underline{\kappa}$ is equivalent to a choice of embedding $\mathbf{T} \hookrightarrow \mathbf{G}'$ for some inner form $\mathbf{G}'$ of $\mathbf{G}$. We set

$$\begin{aligned}
G &= \mathbf{G}_{\underline{s},\underline{\kappa}}(K), \\
G^{\flat} &= \mathfrak{G}^{\flat}_{\underline{s},\underline{\kappa}}(\mathcal{O}_K) = \mathfrak{G}_{\underline{s},\underline{\kappa}}(\mathcal{O}_K), \\
G^{\circ} &= \mathfrak{G}^{\circ}_{\underline{s},\underline{\kappa}}(\mathcal{O}_K), \\
\mathcal{G}^{*} &= \mathfrak{G}^{\flat}_{\underline{s},\underline{\kappa}}(k) = \mathfrak{G}_{\underline{s},\underline{\kappa}}(k).
\end{aligned}$$

Finally, let $\mathcal{G}$ be the maximal reductive quotient of $\mathcal{G}^*$, and let $\mathcal{G}^\circ$ the connected component of the identity of $\mathcal{G}$.

5.2 Reductions of Parahorics and Maximal Compacts

Our construction of representations of G has as intermediate steps the construction of representations of G° and then $G^\flat$. We need to understand the reductions of G° and $G^\flat$ in order to pass from a representation of the first to a representation of the second.

We may assume that $\underline{\kappa}$ is sorted so that all of the κ_i with odd valuation appear at the beginning and those of even valuation at the end. If n is odd this convention aligns with our previous choice of putting the U_1 last, since $v_E(\kappa_j)$ will always be even. Let d be the cutoff so that κ_d has odd valuation and κ_{d+1} even, let $l = \sum_{i=1}^{d} s_i$ and $m = \sum_{i=d+1}^{j} s_i$.

The group $G^\flat = \mathfrak{G}_{\underline{s},\underline{\kappa}}(\mathcal{O}_K)$ has a filtration as in Section2.14.5, and the quotient gives the special fiber

$$\mathcal{G} = \mathfrak{G}_{\underline{s},\underline{\kappa}}(\mathcal{O}_K)/\mathfrak{G}_{\underline{s},\underline{\kappa}}(\mathcal{O}_K)^{0+}.$$

Theorem 5.2.1. *Suppose that* $\mathbf{G} = \mathrm{U}_n\,/K$ *is a unitary group.*

(i) The reduction $\mathcal{G}$ *is given by*

$$\mathcal{G} \cong \mathrm{Sp}_l(k) \times \mathrm{O}_m(k).$$

(ii) The connected component of the identity is given by

$$\mathcal{G}^\circ \cong \mathrm{Sp}_l(k) \times \mathrm{SO}_m(k).$$

Proof. Let

$$\Lambda = \prod_{i=1}^{j} \pi_E^{b_i} O_i$$

be the lattice corresponding to the vertex fixed by $\mathbf{G}^\flat$ as in the proof of Theorem 5.1.7. By our definitions of l and m, the first d entries of $\nu_E(\underline{\kappa}) + 2\underline{b}$ are -1 and the last $j - d$ are 0. Since G stabilizes the lattice Λ we get an action of G on

$$\bar{\Lambda} := \Lambda/\pi_K\Lambda.$$

Note that $\mathfrak{G}_{\underline{s},\underline{\kappa}}(O_K)^{0+}$ acts trivially on $\bar{\Lambda}$, and thus we get an action of $\mathcal{G}$ on $\bar{\Lambda}$.

Following Tits [50, §3.11], we consider the endomorphism ν of $\bar{\Lambda}$ induced by multiplication by π_E within Λ; ν is clearly centralized by the action of $\mathcal{G}^*$, and has kernel equal to its image. Set $\bar{\Lambda}_0 = \bar{\Lambda}/\nu(\bar{\Lambda}) \cong \Lambda/\pi_E\Lambda$. Since $\mathcal{G}^*$ centralizes ν, we get a homomorphism $\mathcal{G}^* \to \mathrm{GL}(\bar{\Lambda}_0)$ with unipotent kernel.

The skew Hermitian form $\pi_E\phi_\kappa$ takes integral values on Λ since $\Lambda^\vee \supseteq \pi_E\Lambda$, and thus induces an alternating form $\bar{\phi}_0$ on $\bar{\Lambda}_0$. This form is degenerate, with kernel $\bar{\Lambda}_1 \subset \bar{\Lambda}_0$ equal to the image of $\Lambda^\vee$ in $\bar{\Lambda}_0$. The dimension of $\bar{\Lambda}_1$ is the sum of the dimensions of the components of Λ corresponding to κ_i with even valuation, namely $\dim_k(\bar{\Lambda}_1) = m$. Our alternating form induces a nondegenerate alternating form on the quotient $\bar{\Lambda}_0/\bar{\Lambda}_1$, a k-vector space of dimension l.

The Hermitian form ϕ_κ takes integral values on $\Lambda^\vee$ since $\Lambda^\vee \subseteq \Lambda$, and thus induces a symmetric form ϕ_1 on $\bar{\Lambda}_1$. The image of $\mathcal{G}^*$ in $\mathrm{GL}(\bar{\Lambda}_0)$ preserves these two forms, and the maximal reductive quotient $\mathcal{G}$ is just the product

$$\mathcal{G} = \mathrm{Sp}(\phi_0) \times \mathrm{O}(\phi_1).$$

The second half of the theorem now follows easily. □

Corollary 5.2.2.

$$|G^\flat / G^\circ| = \begin{cases} 1 & \textit{if } n \textit{ is even and all } \kappa_i \textit{ have odd valuation} \\ 2 & \textit{otherwise} \end{cases}$$

In the case that G° sits inside $G^\flat$ with index 2, we will need to determine whether the induction of a Deligne-Lustig representation remains irreducible after inducing. To this end, we have the following proposition.

Proposition 5.2.3. *The center* $\mathrm{Z}(\mathcal{G})$ *lies within* $\mathcal{G}^\circ$ *if and only if* n *is even.*

Proof. Since $\mathcal{G} = \mathrm{Sp}_l(k) \times \mathrm{O}_m(k)$, n has the same parity as m and $\mathrm{Sp}_l(k)$ is connected, it suffices to prove the statement for $\mathcal{G} = \mathrm{O}_n(k)$.

In order for a diagonal matrix α to be orthogonal, we must have $\alpha^2 = 1$; for scalar α this reduces to $\alpha = \pm 1$.

If n is odd, the -1 matrix does not lie in $\mathrm{SO}_n(k)$ but does lie in the center of $\mathrm{O}_n(k)$. For n even, $-1 \in \mathrm{SO}_n(k)$ and thus $\mathrm{Z}(\mathcal{G}) \subset \mathcal{G}^\circ$. □

Note that the different reductions line up correctly with the reductions given in Figure 2.4. In particular, if $n = 2m$ and G is quasi-split, then there must be either no odd $v_{E_i}(\kappa_i)$ or at least two; this explains why there are no reductions of the form $\mathrm{O}_2 \times \mathrm{Sp}_{2m-2}$ for the quasi-split G. Conversely, if G is not quasi-split then there must be at least one odd $v_{E_i}(\kappa_i)$, corresponding to the lack of any reduction of the form Sp_{2m}.

In the other direction, Figure 2.4 gives us information about the orthogonal form ϕ_1 in the proof of Theorem 5.2.1: it will be split if G is quasi-split and non-split otherwise.

5.3 A representation

We may now define a complex admissible representation $\pi = \pi_{\varphi,\underline{\kappa}}$ of G in a sequence of steps.

(i) Since the character χ_φ has depth zero it descends to a character on $\mathcal{T}$. Together with the torus $\mathcal{T} \subseteq \mathcal{G}^\circ$ this provides the defining data for a Deligne-Lusztig representation π° of $\mathcal{G}^\circ$.

(ii) We obtain a representation of the parahoric subgroup G° via the natural map $G^\circ \to \mathcal{G}^\circ$; we will also call this representation π°

(iii) Define a representation $\pi^\flat$ on the maximal compact subgroup $G^\flat$ by a finite induction from G°.

(iv) Finally, define a representation π on all of G by compact induction from $G^\flat$.

In this section we elaborate on the different steps in this process and give conditions under which the representation at each step is irreducible.

(i) By Theorem 2.15.1, π° will be irreducible if and only if the only F-invariant of the Weyl group of $\mathcal{T}$ fixing χ_φ is the identity, namely that χ_φ is in general position.

Proposition 5.3.1. *Suppose that* $Z_{\hat{\mathbf{G}}}(\varphi(\tilde{\tau})) = \hat{\mathbf{S}}^\tau$. *Then* χ_φ *is in general position.*

Proof. Recall from Lemma 3.2.2 that we already know $Z_{\hat{\mathbf{G}}}(\varphi(\tilde{\tau}))^\circ = \hat{\mathbf{S}}^\tau$, so we're just assuming that this centralizer is connected. We use the notations of Section 4.4 and apply Proposition 4.4.2. If $\mathbf{W}_{\varphi\tilde{\tau}}$ is trivial, we can ignore the question of whether elements of $\mathbf{W}^\tau$ are fixed by Frobenius. But a nontrivial element of $\mathbf{W}_{\varphi\tilde{\tau}}$ provides an element of $\hat{\mathbf{G}}$ centralizing $\varphi(\tilde{\tau})$ but lying outside $\hat{\mathbf{S}}^\tau$, contradicting our assumption on the centralizer of $\varphi\tilde{\tau}$. □

(ii) The inflation of π° to G° will clearly be irreducible if and only if π° is already irreducible.

(iii) The induction from G° to $G^\flat$ is the most delicate stage. In fact, it does not always remain irreducible, but we are able to pick out one of the irreducible factors by a separate method. We begin with a lemma.

We have two methods for obtaining a character on $Z = \mathbf{Z}(K)$. Since Z is compact and central, $Z \subset G^\flat$. We can thus restrict π° to get a character ϵ from Z° to the center of a general linear group, which is isomorphic to $\mathbb{C}^\times$. On the other hand, we've seen in Section 2.10 a general construction of the central character $\omega_\varphi \colon Z \to \mathbb{C}^\times$.

Lemma 5.3.2. *The two characters* ϵ *and* ω_φ *agree on* Z°.

Proof. From the description of the Deligne-Lusztig representation given in Carter [11, §7.2] we see that central elements $z \in \mathcal{G}$ scale by $\chi_\varphi(z)$. Recall the description of ω_φ as the image of φ under the composition

$$\mathrm{H}^1(K, \hat{\mathbf{G}}) \to \mathrm{H}^1(K, \hat{\mathbf{Z}}) \to \mathrm{Hom}(Z, \mathbb{C}^\times).$$

If we instead restrict to $\mathrm{H}^1(K_f, \hat{\mathbf{G}})$ first, project onto $\mathrm{H}^1(K_f, \hat{\mathbf{Z}})$, map to $\mathrm{Hom}(Z, \mathbb{C}^\times)$ and then descend to $\mathrm{Hom}(Z^\circ, \mathbb{C}^\times)$ we'll get the same character as long as the following diagrams commute:

(a)

$$\begin{tikzcd} \mathrm{H}^1(K_f, \hat{\mathbf{T}}) \arrow[r] \arrow[d] & \mathrm{Hom}(\mathbf{T}(K_f), \mathbb{C}^\times) \arrow[d] \\ \mathrm{H}^1(K_f, \hat{\mathbf{Z}}) \arrow[r] & \mathrm{Hom}(\mathbf{Z}(K_f), \mathbb{C}^\times) \end{tikzcd}$$

where the horizontal maps are the local Langlands correspondence, the left map is induced by the quotient map $\hat{\mathbf{G}} \to \hat{\mathbf{Z}}$ and the right map is restriction.

(b)

$$\begin{tikzcd} \mathrm{H}^1(K, \hat{\mathbf{Z}}) \arrow[r] \arrow[d] & \mathrm{Hom}(\mathbf{Z}(K), \mathbb{C}^\times) \arrow[d] \\ \mathrm{H}^1(K_f, \hat{\mathbf{Z}}) \arrow[r] & \mathrm{Hom}(\mathbf{Z}(K_f), \mathbb{C}^\times) \end{tikzcd}$$

where here the left map is restriction and the right is induced by the norm $\mathbf{Z}(K_f) \to \mathbf{Z}(K)$.

□

With this lemma in hand, we can define the representation $\pi^\flat$ of $G^\flat$.

Proposition 5.3.3. *There is a unique irreducible representation $\pi^\flat$ of $G^\flat$ satisfying:*

(a) $\pi^\flat$ is a sub-representation of the induction $\mathrm{Ind}_{G^\circ}^{G^\flat} \pi^\circ$,

(b) the restriction of $\pi^\flat$ to Z agrees with $\omega_{\mathbf{Z}}$.

Proof. There are two cases. Assume first that $\mathbf{G}$ is an even unitary group. Then the induction $\mathrm{Ind}_{G^\circ}^{G^\flat} \pi^\circ$ remains irreducible [29, Thm. 6.11]. By Lemma 5.3.2 and Proposition 5.2.3, the induction satisfies the second property. So we may set

$$\pi^\flat = \mathrm{Ind}_{G^\circ}^{G^\flat} \pi^\circ.$$

Now suppose that $\mathbf{G}$ is an odd unitary group. Then the induction has two irreducible sub-representations. By Proposition 5.2.3, there is a central element z lying in the nontrivial coset of $G^\circ \subset G^\flat$. The two irreducible representations in the induction will take different values on z, and thus at most one of these will satisfy the second requirement. On the other hand, Lemma 5.3.2 guarantees that both pieces of the induction will agree with $\omega_{\mathbf{Z}}$ on Z°, so our chosen sub-representation will satisfy both desired properties. □

(iv) The representation $\pi^\flat$ acts on a finite dimensional $\mathbb{C}$-vector space. We now define

$$\pi = \mathrm{ind}_{G^\flat}^{G} \pi^\flat.$$

By Theorem 2.16.2, π is irreducible.

5.4 L-packets

Let $\mathbf{G} = \mathrm{U}(V)$ as normal. The L-packet Π_φ associated to a tame, discrete, regular Langlands parameter φ for $\mathbf{G}$ consists of the representations π constructed in the previous section, parameterized by the embeddings $\rho\colon \mathbf{T} \hookrightarrow \mathbf{G}'$ as $\mathbf{G}'$ ranges over the pure inner forms of **G**. In particular, we can give the size of Π_φ in terms of the parameter φ.

Proposition 5.4.1. *Let j be the number of cycles in the permutation obtained by projecting $\varphi(F)$ onto the Weyl group of the unique maximal torus containing $\varphi(\tilde{\tau})$. Then there are 2^j representations in Π_φ.*

Proof. I claim first that j is the number of tori $\mathbf{T}_{s_i}$ in the decomposition of Theorem 3.4.1, though in the odd case we let the U_1 factor contribute 1 to j. To see this observe that the action of Frobenius on $\mathbf{T}$ is defined by $\varphi(F)$, and the decomposition in the proof of Theorem 3.4.1 is precisely determined by the cycles in that action on $X^*(\mathbf{T})$.

Now note that for each $\mathbf{T}_{s_i}$, the representation π associated to the embedding determined by the tuple $\underline{\kappa}$ depends only on the choice of κ_i modulo $\mathrm{Nm}_{E_i/L_i} E_i^\times$, and $L_i^\times / \mathrm{Nm}_{E_i/L_i} E_i^\times$ has order 2. $\square$

For U_{2m}, the smallest L-packets, of cardinality 2, occur when the image of $\varphi(F)$ is a Coxeter element in $\mathbf{W}^{\mathcal{I}}$. The largest, of size 2^m, occur when $\varphi(F)$ is a product of m commuting transpositions. For any L-packet, each embedding determines a vertex of $\mathcal{B}_{\mathrm{red}}(\mathbf{G})$ stabilized by the image of $\mathbf{T}(K)$ in $\mathbf{G}'(K)$. Up to conjugacy within $\mathbf{G}'(K)$ each embedding is determined by the choice of even or odd valuation for each κ_i, and one can pick out the type of the stabilized vertex in the tables of Figure 2.4 using Theorem 5.2.1 and Proposition 5.1.6.

Similar results hold for U_{2m+1}: the smallest L-packets have cardinality 4 and occur when the image of $\varphi(F)$ is a Coxeer element in $\mathbf{W}^{\mathcal{I}}$. The largest have size 2^m.

In order to understand the L-packets for SU_n, we use the isomorphism

$$\mathrm{H}^1(K, \mathbf{T}) \cong \hat{\mathbf{T}}^\Gamma.$$

The group $\hat{\mathbf{T}}^\Gamma$ will be contained within $\hat{\mathbf{T}}^\tau$; we may take the torus $\hat{\mathbf{T}}$ to consist of diagonal matrices, and the action of τ to be reflecting over the anti-diagonal and inverting. So $\hat{\mathbf{T}}^\Gamma$ will be a group of diagonal matrices with 1s and -1s on the diagonal. When we change $\mathbf{G}$ from U_n to SU_n, we cut the size of $\hat{\mathbf{T}}^\Gamma$ in half since the connected dual group of SU_n is PGL_n rather than GL_n.

So the L-packets for SU_{2m} range in size from 1 for a Coxeter element to 2^{m-1} for a product of m commuting transpositions. Similarly, the L-packets for SU_{2m+1} range in size from 2 to 2^m.

5.5 Closing Remarks

The story doesn't end with the construction of Π_φ. Without a full correspondence, there is no guarantee that the representations we've constructed are the right ones to make up the L-packet associated to φ. DeBacker-Reeder give various pieces of evidence that their construction produces the correct L-packet:

(i) The formal degrees of the representations in each of their L-packets behave as expected [16, Ch. 5].

(ii) They compute which representations in each of their L-packets are generic [16, Ch. 6].

(iii) They prove that each of their L-packets is stable [16, Ch. 7-12].

Proofs of the analogous results for the L-packets constructed in this thesis are not yet complete.

5.5.1 Other Tamely Ramified Groups

Though we expect that the constructions of Π_φ carried out in the last three chapters should carry over to groups other than $\mathbf{G} = \mathrm{U}(V)$, not all of the results have yet been established for tame, discrete, regular parameters of arbitrary $\mathbf{G}$. Most of our results are phrased for a general $\mathbf{G}$; we highlight here the obstructions to moving beyond $\mathbf{G} = \mathrm{U}(V)$.

(i) In Construction 3.2.5 we need to assume that $q \equiv 1 \pmod{[E : K]}$, which holds automatically for the unitary case. This condition is needed to ensure that the image of Frobenius in the Weyl group of $\hat{\mathbf{S}}$ is $\mathcal{I}$-invariant.

(ii) Proposition 5.1.1 holds for general $\mathbf{G}$. But the proof of Theorem 5.1.7 depends on the structure of unramified anisotropic tori in unitary groups. To generalize the construction to arbitrary $\mathbf{G}$ one would need a more general proof that for each pure inner form $\mathbf{G}'$ of $\mathbf{G}$ and each embedding $\mathbf{T}(K) \hookrightarrow \mathbf{G}'(K)$, $\mathbf{T}(K)$ fixes a unique point in the building of $\mathbf{G}'$. Note that $\mathbf{T}(K)$ will always fix the image of $\mathcal{B}(\mathbf{T}/K)$ in $\mathcal{B}(\mathbf{G}/K)$, which is a single point since $\mathbf{T}$ is anisotropic. For unramified $\mathbf{T}$, this image should be a vertex, and in fact the unique point of $\mathcal{B}(\mathbf{G}/K)$ fixed by $\mathbf{T}(K) \hookrightarrow \mathbf{G}(K)$ (c.f. [16, Lem. 4.4.1]).

(iii) In section 5.3, the proof that χ_φ is in general position works for arbitrary $\mathbf{G}$. The representation π° is thus irreducible for any $\mathbf{G}$. But the construction of the representation $\pi^\flat$ works differently for even and odd unitary groups. A more general construction needs a better description of the component group of $G^\flat$ and its interaction with the center $\mathbf{Z}(K)$ of $\mathbf{G}(K)$.

5.5.2 Other Local Fields

We assume that K is a finite extension of $\mathbb{Q}_p$, but many of the stages of the construction work just as well for finite extensions of $\mathbb{F}_p((T))$. We expect the results of this thesis to carry over to the characteristic p case, though some of the arguments may need flat cohomology rather than Galois cohomology. We assume that K has characteristic zero mainly because many of our references restrict themselves to this case.

5.5.3 A Non-regular Example

In this section we examine the role regularity plays by considering a simple example of a non-regular parameter.

Note first that regularity is automatic when the derived group of $\hat{\mathbf{G}}$ is simply connected. Even when $\hat{\mathbf{G}}$ is adjoint, the computation of Ω_ϑ in [42, Table 1] implies that all parameters are regular for $\mathbf{G} = \mathrm{SU}_{2n+1}$. So suppose that $\mathbf{G} = \mathrm{SU}_4$ with Hermitian form given by a matrix with ones on the anti-diagonal. Then $\hat{\mathbf{G}} = \mathrm{PGL}_4(\mathbb{C})$, with $\tau \in \mathrm{Gal}(E/K)$ acting as reflection across the anti-diagonal and then inversion. We can take for our maximal K-split torus the set of diagonal matrices with entries (a, b, b^{-1}, a^{-1}) $(a, b \in K^\times)$, and its centralizer $\mathbf{S}$ will consist of diagonal matrices of the form $(a, b, \tau(b^{-1}), \tau(a^{-1}))$ $(a, b \in E^\times$ with $ab \in K^\times)$.

The dual torus $\hat{\mathbf{S}}$ is just the diagonal torus in $\mathrm{PGL}_4(\mathbb{C})$, and $\hat{\mathbf{S}}^\tau$ is the set of diagonal matrices of equivalent to $[a, b, b^{-1}, a^{-1}]$ for some $a, b \in \mathbb{C}^\times$. The Weyl group $\mathbf{W}$ of $\hat{\mathbf{S}}$ is represented by the permutation matrices, and $\mathbf{W}^\tau$ is generated by

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \text{and} \qquad D = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Let k be a positive integer with $q \equiv -1 \pmod{2k+4}$; for each odd $q > 3$ there exists such a k. Set $\zeta = e^{2\pi i/(2k+4)}$. We will define a tame discrete Langlands parameter with

$$\varphi(\tilde{\tau}) = \begin{bmatrix} \zeta^{k+1} & 0 & 0 & 0 \\ 0 & \zeta^{k} & 0 & 0 \\ 0 & 0 & \zeta^{-k} & 0 \\ 0 & 0 & 0 & \zeta^{-k-1} \end{bmatrix} \tau \in \hat{\mathbf{G}} \rtimes \mathrm{Gal}(E/K)$$

Note that $\varphi(\tilde{\tau})$ is centralized by D, and we can show using [42, Prop. 3.8] that in fact the centralizer of $\varphi(\tilde{\tau})$ in $\hat{\mathbf{G}}$ will contain $\hat{\mathbf{S}}^{\tau}$ with index 2: the nontrivial coset will be represented by D. It remains to define $\varphi(F)$. In order to get a discrete parameter, we need to choose $\varphi(F)$ from the nontrivial powers of C: C and C^3 would yield the unramified anisotropic torus $\mathbf{T} = \mathbf{T}_4 \cap \mathrm{SU}_4$, and C^2 would yield the unramified anisotropic torus $\mathbf{T} = \mathbf{T}_2 \times \mathbf{T}_2 \cap \mathrm{SU}_4$ (see Section 3.4). If we set

$$\varphi(F) = C^2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

then our condition on q implies that $\varphi(F)\varphi(\tilde{\tau})\varphi(F) = \varphi(\tilde{\tau})^q$, whereas C and C^3 don't normalize $\varphi(\tilde{\tau})$ correctly.

The subgroup of $\hat{\mathbf{S}}^{\tau}$ that commutes with C^2 is of order 2, generated by

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and the finite group $A_\varphi = \mathrm{Z}_{\hat{\mathbf{G}}}(\varphi)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and generated by M and D. Note that $\mathrm{H}^1(K, \mathbf{T}) \cong \hat{\mathbf{T}}^\Gamma = \langle M \rangle$ has order 2, and thus we would expect only two representations in Π_φ if φ were regular. But in this case each Deligne-Lusztig representation defined in Section 5.3 breaks up as a sum of two irreducible representations. Each of these pieces then yield a supercuspidal representation of SU_4, and the L-packet Π_φ has 4 representations, rather than the 2 one would expect from the decomposition of $\mathbf{T}$. This cardinality agrees with our computation of A_φ.

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