A bound for the number of automorphisms of an arithmetic Riemann surface

A paper by Mikhail Belolipetsky and Gareth Jones

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Outline

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We will be considering automorphism groups of compact Riemann surfaces of genus $g \geq 2$.

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Theorem (Hurwitz, Accola, Maclachlan)

We have

$$8(g+1) \leq N(g) \leq 84(g-1),$$

the upper bound due to Hurwitz, and the lower bound due to Accola and Maclachlan.

By the uniformization theorem, each surface with $g \geq 2$ can be represented as

$$\mathcal{S} = \Gamma_{\mathcal{S}} \backslash \mathcal{H},$$

where $\mathcal{H}$ is the hyperbolic upper half plane and $\Gamma_{\mathcal{S}}$ is a cocompact torsion-free discrete subgroup of $PSL_2(\mathbb{R})$.

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We will be restricting our attention to the *arithmetic* surfaces: those coming from arithmetic subgroups $\Gamma_{\mathcal{S}}$.

Definition

Let K be a totally real number field, let $a, b \in K$, and let $A = (\frac{a,b}{K})$ be a quaternion algebra. Suppose that we have $\rho\colon (\frac{a,b}{\mathbb{R}}) \to M_2(\mathbb{R})$ an isomorphism and $(\frac{\sigma(a),\sigma(b)}{\mathbb{R}}) \cong \mathbb{H}$ for every non-identity $\sigma: K \to \mathbb{R}$. Let $\mathcal{O}$ be an order in A and let $\mathcal{O}^1$ be the elements of norm 1 in $\mathcal{O}$. We call a subgroup of $PSL(2,\mathbb{R})$ that is commensurable with the image in $PSL(2,\mathbb{R})$ of some $\rho(\mathcal{O}^1)$ an *arithmetic subgroup*.

An *arithmetic surface* is a Riemann surface $\mathcal{S}$ that can be expressed as $\mathcal{S} \cong \Gamma_{\mathcal{S}} \backslash \mathcal{H}$ with $\Gamma_{\mathcal{S}}$ arithmetic. A *non-arithmetic surface* is one that cannot be expressed in this way.

$$N_{ar}(g) = \sup\{|\mathrm{Aut}(\mathcal{S})| : \ \mathcal{S} \text{ genus } g, \text{ arithmetic}\},$$
$$N_{nar}(g) = \sup\{|\mathrm{Aut}(\mathcal{S})| : \ \mathcal{S} \text{ genus } g, \text{ non-arithmetic}\}.$$

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We will be concerned with the lower arithmetic bound.

We call a discrete subgroup of $PSL_2(\mathbb{R})$ a *Fuchsian group*. Any cocompact Fuchsian group has a presentation

$$\Gamma(g; m_1, \ldots, m_k) = \langle \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g, \gamma_1, \ldots \gamma_k | \prod_{i=1}^{g} [\alpha_i, \beta_i] \prod_{j=1}^{k} \gamma_j = 1, \gamma_j^{m_j} = 1 \rangle.$$

We call $(g; m_1, \ldots, m_k)$ the *signature*, and write $(m_1, \ldots, m_k)$ if $g = 0$.

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Theorem (Riemann-Hurwitz)

Recall that the Euler characteristic of a Riemann surface M is defined in terms of its genus g by $\chi(M) = 2 - 2g$.
If $f : M \to N$ has degree n, and if $e_f(P)$ is the ramification number at $P \in M$, then

$$\chi(N) = n\chi(M) + \sum_{P \in M} (e_f(P) - 1).$$

We define $\mu(\Gamma)$ to be the hyperbolic measure of $\Gamma\backslash\mathcal{H}$,

$$\mu(\Gamma) = \mu(g; m_1, \ldots, m_k) = 2\pi \left(2g - 2 + \sum_{j=1}^{k} (1 - \frac{1}{m_j}) \right).$$

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Using Riemann-Hurwitz, we can show that if $\Gamma' \leq \Gamma$ is of finite index, then

$$\mu(\Gamma') = [\Gamma : \Gamma'] \cdot \mu(\Gamma).$$

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Then its automorphisms can be obtained from the automorphisms of $\mathcal{H}$:

$$\begin{aligned} \mathrm{Aut}(\mathcal{S}) &= \{\alpha \in \mathrm{PSL}(2,\mathbb{R}) : \alpha \Gamma_{\mathcal{S}} \alpha^{-1} = \Gamma_{\mathcal{S}}\} / \Gamma_{\mathcal{S}} \\ &= N(\Gamma_{\mathcal{S}}) / \Gamma_{\mathcal{S}} \end{aligned}$$

(Think: Given $\gamma \in \Gamma_{\mathcal{S}}$, we need $\alpha(\gamma(x)) = \gamma'(\alpha(x))$ for some $\gamma' \in \Gamma_{\mathcal{S}}$.)

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Then, if we determine $\mathcal{S}$ by $\Gamma_{\mathcal{S}} = K$, we have

$$\begin{tikzcd}
1 \arrow[r] & \Gamma_{\mathcal{S}} \arrow[r] & N(\Gamma_{\mathcal{S}}) \arrow[r] & \mathrm{Aut}(\mathcal{S}) \arrow[r] & 1 \\
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We call this a *surface-kernel epimorphism* or SKE.

To verify that the kernel is torsion free, we must check that every element of Γ of finite order has its order preserved by $p : \Gamma \longrightarrow G$.

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Then we know that G is a subgroup of $\mathrm{Aut}(\mathcal{S})$.

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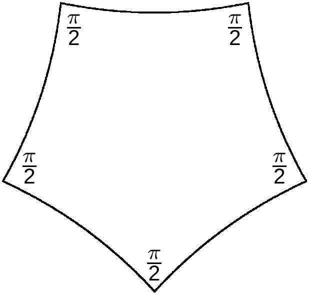
Arithmetic:

$$\begin{aligned} &(2,3,n), \quad n = 7,8,9,10,11,12,14,16,18,24,30 \\ &(2,4,n), \quad n = 5,6,7,8,9,10,12,18 \\ &(2,5,n), \quad n = 5,6,8,10,20,30 \\ &\qquad\qquad \mathit{etc}. \end{aligned}$$

K. Takeuchi. Arithmetic triangle groups. *J. Math. Soc. Japan* **29** (1977), 91-106.

Consider the right-angled hyperbolic pentagon:



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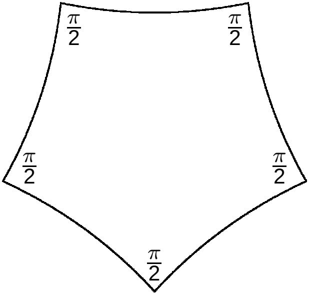
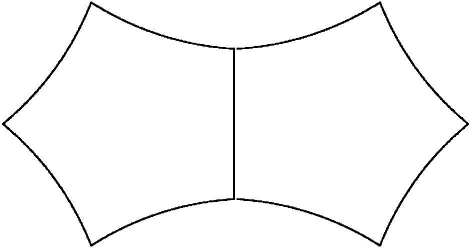


Let Γ be the orientation-preserving subgroup of the group of reflections in its sides.

The fundamental domain for Γ is two copies of the pentagon:



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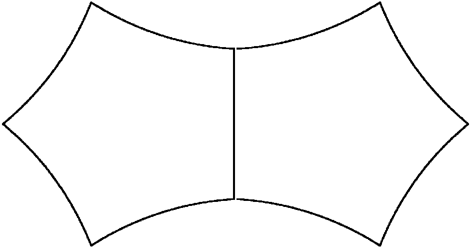

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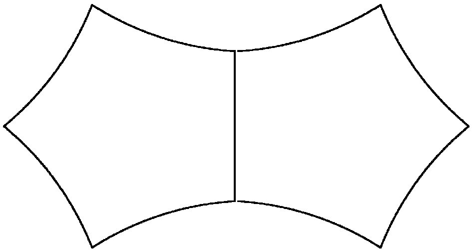

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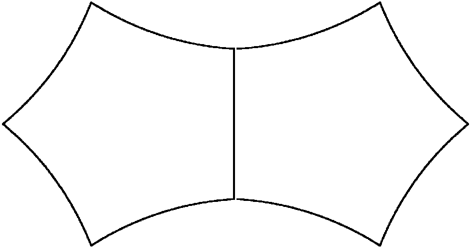

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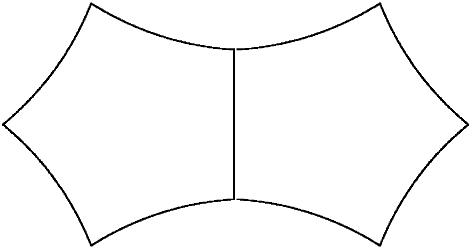

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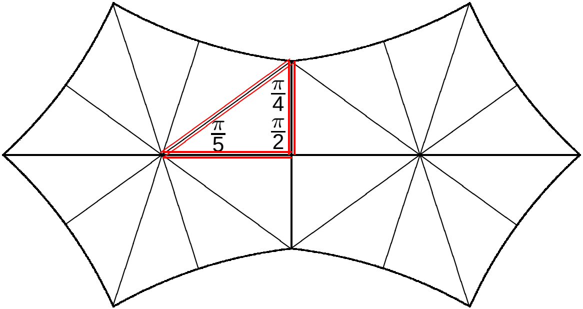
- Two reflections give rotation around an angle of π. This is order 2. There are five such elements of Γ.
- The signature of the group Γ is $(2, 2, 2, 2, 2)$.
- The Riemann surface $\mathcal{S} = \Gamma\backslash\mathcal{H}$ is of genus zero.

Subdivide the pentagon into 10 congruent triangles:



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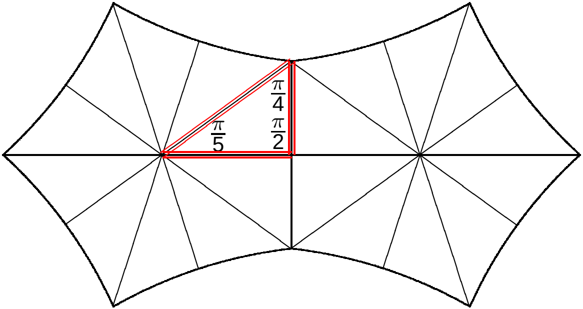

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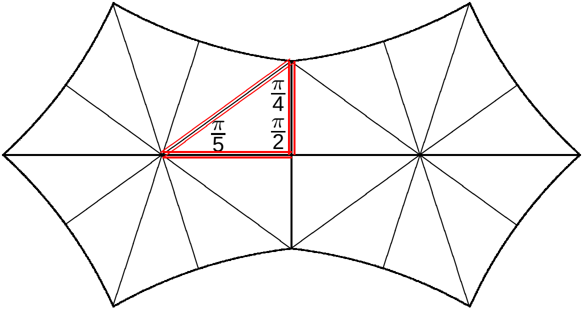

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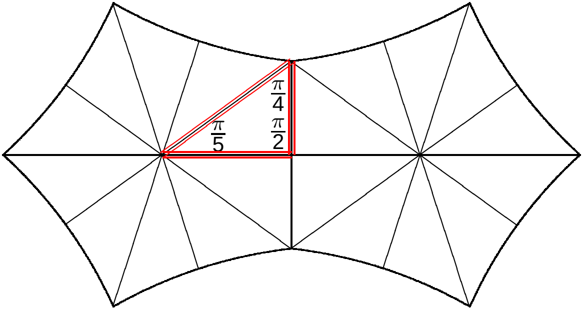

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- The triangle has angles $\pi/2$, $\pi/4$ and $\pi/5$. So Γ' is the $(2, 4, 5)$ triangle group, which is arithmetic.
- But Γ is a subgroup of Γ' of index 10. Hence the two groups are commensurable, and so Γ is arithmetic.

Outline

Lemma

Let $\{\mathcal{S}_g\}_{g\in\mathcal{G}}$ be an infinite sequence of arithmetic surfaces of different genera g, such that for each $g \in \mathcal{G}$, the group of automorphisms of $\mathcal{S}_g$ has order $a(g+b)$ for some fixed a and b. Then $b = -1$.

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Proof. Let $\mathcal{S}$ be a surface from the given sequence.

Then $\mathrm{Aut}(\mathcal{S}) \cong N(\Gamma_{\mathcal{S}})/\Gamma_{\mathcal{S}}$, where $\Gamma_{\mathcal{S}}$ is the surface group corresponding to $\mathcal{S}$.

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Then $\mathrm{Aut}(\mathcal{S}) \cong N(\Gamma_{\mathcal{S}})/\Gamma_{\mathcal{S}}$, where $\Gamma_{\mathcal{S}}$ is the surface group corresponding to $\mathcal{S}$.

The Riemann-Hurwitz formula yields

$$\mu(N(\Gamma_{\mathcal{S}})) = \frac{\mu(\Gamma_{\mathcal{S}})}{|\mathrm{Aut}(\mathcal{S})|} = \frac{2\pi(2g-2)}{a(g+b)},$$

so $\mu(N(\Gamma_{\mathcal{S}})) \to 4\pi/a$ as $g \to \infty$.

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So for all but finitely many $g \in \mathcal{G}$,

$$\frac{2\pi(2g-2)}{a(g+b)} = \mu(N(\Gamma_{\mathcal{S}})) = \frac{4\pi}{a} .$$

Therefore $b = -1$.

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In fact it is never attained by arithmetic surfaces, since the extremal surfaces for this bound are uniformized by surface subgroups of (2, 4, 2(g+1))-groups with $g \geq 24$ (Maclachlan), and these are not arithmetic (Takeuchi).

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- We prove the inequality by considering a family of arithmetic surfaces $\mathcal{W}_g$, one for each genus g, and show that $|\mathrm{Aut}(\mathcal{W}_g)| \geq 4(g-1)$.
- We then assume that $G := Aut(\mathcal{S})$ has order $|G| > 4(g-1)$ for some compact arithmetic surface $\mathcal{S}$ of genus $g \geq 2$.

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- Imposing specific conditions on g we get a contradiction.
- We show that infinitely many values of g satisfy these conditions. For these g, $N_{\mathrm{ar}}(g) = 4(g-1)$.

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Define $\theta_g : \Gamma \to D_{2(g-1)}$ by $\gamma_j \mapsto ab, b, a^{g-2}b, b, a^{g-1}$.

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θ_g is a SKE and thus $K_g = \ker(\theta_g)$ is a surface group.

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$$\mu(K_g) = \mu(\Gamma)|D_{2(g-1)} = 2\pi(2g-2).$$

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Then $N_{ar}(g) \geq |\mathrm{Aut}(\mathcal{W}_g)| \geq |D_{2(g-1)}| = 4(g-1)$ as required.

Outline of proof that the bound is strict

- Only finitely many signatures with $\mu(\Gamma)$ allowing $|K\backslash\Gamma| = 4(g-1)$.
- We set $p = g - 1$ prime and big enough, based on these signatures.
- Then we have a p-Sylow subgroup, which we lift to $\Delta \leq \Gamma$ and set $Q = \Delta\backslash\Gamma$.
- $\mathcal{T} := \Delta\backslash\mathcal{H}$ has genus 2 and $Q \subset \mathrm{Aut}(\mathcal{T})$.
- We have a faithful action of Q on $H_1(\mathcal{T}, \mathbb{F}_p)$.
- It decomposes into 1-dimensional submodules.
- We find $Q \subset \mathrm{GL}_1(\mathbb{F}_p)^4$, which constrains the exponent ϵ of Q.
- Thus ϵ divides $\gcd(E, p-1)$, which we can force to be 2.
- This gives a contradiction using the area formula.
- We have infinitely many p satisfying our conditions.

We now assume that $G \cong K \backslash \Gamma$ for some co-compact arithmetic group Γ and normal surface subgroup $K = \Gamma_{\mathcal{S}} \leq \Gamma$, with

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For each $\sigma \in \Sigma$, the number

$$\frac{\mu(\Gamma)}{4\pi} = \frac{g-1}{|G|} =: q = \frac{r_\sigma}{s_\sigma}$$

is rational and depends only on the signature $\sigma \in \Sigma$. We have $|G| = (g-1)/q = (g-1)s/r$.

Let $R = \text{lcm}\{r_\sigma | \sigma \in \Sigma\}$, and $S = max\{s_\sigma | \sigma \in \Sigma, r_\sigma = 1\}$.

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Since $|Q|$ is coprime to p, the natural epimorphism $G \to Q$ preserves the orders of the images of all elliptic generators of Γ.

The inclusions $K \trianglelefteq \Delta \trianglelefteq \Gamma$ induce an étale $\mathbb{Z}/p\mathbb{Z}$-covering of Riemann surfaces

$$\begin{tikzcd} \mathcal{S} \cong K\backslash\mathcal{H} \arrow[d, "P\cong\mathbb{Z}/p\mathbb{Z}"'] \arrow[dd, bend left, "G"] \\ \mathcal{T} \cong \Delta\backslash\mathcal{H} \arrow[d, "Q"'] \\ \Gamma\backslash\mathcal{H} \end{tikzcd}$$

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In particular we have that $Q \leq Aut(\mathcal{T})$, and $\mathcal{T}$ has genus $1 + (g-1)/p = 2$.

Then Q is contained in a group of automorphisms of a Riemann surface $\mathcal{T}$ of genus 2.

Notice that $|Aut(\mathcal{T})| \leq 84$, thus there are just finitely many possibilities for $Aut(\mathcal{T})$ and hence for Q.

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In particular $E \equiv 0 \pmod 2$.

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with each V_i a 1-dimensional Q-invariant subspace of V.

Therefore we have a map $Q \to \mathrm{GL}_1(\mathbb{F}_p)^4$.

Lemma

Lemma (Farkas & Kra, V.3.4, due to Serre) If $A \in \mathrm{SL}_k(\mathbb{Z})$ *has finite order* $m > 1$ *and* $A \equiv I \pmod{n}$ *then* $m = n = 2$.

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ϵ thus divides $\gcd(E, p - 1)$.

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Recall our area formula:

$$\mu(\Gamma) = 2\pi(2g - 2 + \sum_{i=1}^{k}(1 - \frac{1}{m_i})).$$

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This contradicts $0 < \mu(\Gamma) < \pi$.

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Therefore we have an infinitely many g that lead to a contradiction.

So we've proven:

Theorem

$N_{ar}(g)\geq 4(g-1)$ for all $g\geq 2$, and this bound is attained for infinitely many values of g.

Outline

1. Definitions, Geometric Preliminaries and an Example
2. A Sharp Lower Bound on $N_{ar}(g)$
3. An Effective Version of the Main Theorem

Main Theorem

- **Main Theorem:** Let Σ be the set of all signatures of cocompact arithmetic Fuchsian groups with volume strictly less than π.

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  Assume that $g - 1 =: p$ is a prime such that $\gcd(p, R) = 1$, $p \notin \Pi$, $p > S$ and such that $\gcd(p - 1, E) = 2$, where E is the least common multiple of the exponents of all automorphism groups of Riemann surfaces of genus 2.

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  Then the size of the automorphism group of any surface of genus g cannot be greater than $4(g - 1)$, so we have to have equality.

Explicit Sequence Theorem

Goal

Construct a specific sequence of genera g such that N_{ar} attains the lower bound.

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Theorem (Explicit Theorem)

For all primes $p \equiv 23, 47, 59 \pmod{60}$, *we have* $N_{ar}(g) = 4(g-1)$. *The least genus g for which the lower bound* $N_{ar}(g) = 4(g-1)$ *is attained is* $g = 24$.

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Idea

Construct primes p satisfying the hypotheses of the Main Theorem. Then $g = p + 1$ will be such that:

$$N_{ar}(g) = 4(g-1).$$

Strategy

1. Listing all Arithmetic Fuchsian Signatures
2. The Conditions on Sufficiently Large Primes p
3. Smaller Primes

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- Then by the proof of the Main Theorem, for any prime p not dividing R, not contained in Π and greater than S, we cannot have

$$|G| > 4(g-1)$$

  if we impose the additional condition that $\gcd(p-1, E) = 2$.

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- Let $(g; m_1; \ldots; m_r)$ be the signature of a Fuchsian group Γ. Then

$$\frac{1}{\pi}\mu(\Gamma) = 4(g-1) + 2\sum_{k=1}^{r}\left(1 - \frac{1}{m_k}\right) < 1 \qquad (2)$$

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  has no solution unless $g = 0$.
- If $g = 0$, then since $m_k \geq 2$, we must have $r < 5$, so all signatures have length 3 or 4.
- Takeuchi gave a complete list of cocompact arithmetic triangle groups; almost all of these have volume less than π.

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- It can be shown that there are only 12 signatures for which $(2,2,2,n)$ is arithmetic.

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- Further examining the list of possible signatures, and putting $\frac{\mu(\Gamma)}{4\pi}$ into lowest terms, we find that $R = 4 \cdot 3 \cdot 5 \cdot 7$ is the least common multiple of the numerators of all $\frac{\mu(\Gamma_\sigma)}{4\pi}$ and $s = 84$ is the largest occurring denominator.

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Lemma

If $\mathcal{S}$ is a Riemann surface of genus $\gamma \geq 2$, then it has no automorphisms of prime order greater than $2\gamma + 1$.

Sufficiently Large Primes

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Sufficiently Large Primes

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If f is an automorphism of $\mathcal{S}$ of order p, let T be the Riemann surface corresponding to $\mathcal{S}$ modulo $< f >$, and γ' its genus. Then $f : \mathcal{S} \longrightarrow T$ is a smooth p-sheeted covering of T, so the Riemann-Hurwitz formula reads:

$$2(\gamma - 1) = 2p(\gamma' - 1) + m(p - 1)$$

where m is the number of fixed points of f. Assume that $p \geq 2\gamma$, then

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- for $\gamma' = 1$, $2(\gamma - 1) = m(p - 1) \geq p - 1 \geq 2\gamma - 1$, a contradiction.

$\square$

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- For $\gamma' = 0$, $2(\gamma - 1) = -2pg + m(p-1)$, we have $m = \frac{2\gamma}{p-1} + 2 \leq \frac{p}{p-1} + 2 \leq 3$, so $m = 3$.

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- **Conclusion:** No prime other than $\{2, 3, 5\}$ divides E, the least common multiple of the exponents of automorphism groups of surfaces of genus 2. Thus the condition that $\gcd(p-1, E) = 2$ is satisfied by all p such that $p - 1$ is not divisible by $3, 4, 5$.

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- **Conclusion:** No prime other than $\{2, 3, 5\}$ divides E, the least common multiple of the exponents of automorphism groups of surfaces of genus 2. Thus the condition that $\gcd(p-1, E) = 2$ is satisfied by all p such that $p - 1$ is not divisible by $3, 4, 5$.
- Since we also require that $p \not\equiv 0 \mod q$ for $q =$ 2,3,5, this leaves the possibilities that $p \equiv 2 \pmod 3$, $p \equiv 3 \mod 4$ and $p \equiv 2, 3, 4 \mod 5$. The first two lift to the congruence $p \equiv 11 \pmod{12}$; combining with the last one gives $p \equiv 23, 47, 59 \pmod{60}$ as the equivalent congruence.

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- We have shown that any prime $p > 84$ congruent to one of 23,47,59 modulo 60 satisfies the conditions of the Main Theorem.

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- Thus, surfaces of genus $p+1$ for any such p satisfy the lower bound: $N_g = 4(g-1)$.
- What about $p = 23, 47, 59$ or 83?

Smaller Primes: p=59

- $p = 59$, $\mathcal{S}$ of genus $g = 60$:

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- 59 is coprime to R, so $|\mathrm{Aut}(\mathcal{S})| = |G| = (g-1)s = 59s$ for some s.

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- 59 is coprime to R, so $|\mathrm{Aut}(\mathcal{S})| = |G| = (g-1)s = 59s$ for some s.
- By inspection, s is coprime to 59, so a 59-Sylow subgroup is of order 59. Letting n_{59} be the number of 59-Sylow subgroups, we must have $n_{59}|s$ and $n_{59} \equiv 1 \pmod{59} \Rightarrow n_{59} = 1$. So the 59-Sylow subgroup P_{59} is unique.

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- $p \notin \Pi = \{2, 3, 5, 7\}$, the set of primes dividing an element of order in some Γ_σ.

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- Conclusion: $g = 60$ attains the lower bound.

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- 83 is coprime to R, so $|\mathrm{Aut}(\mathcal{S})| = |G| = (g-1)s = 83s$ for some s. By inspection, s is coprime to 83, so if P_{83} is a 83-Sylow subgroup, then $|P_{83}| = 83$. Letting n_{83} be the number of 83-Sylow subgroups, we must have $n_{83}|s$ and $n_{59} \equiv 1 \pmod{59}$.

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- Claim: P_{83} is normal in G.

Proof:

- The only possibility for the 83-Sylow subgroup P_{83} not being unique is if $n_{83} = s = 84$.
- Then the normaliser of P_{83} is just P, so G acts faithfully and transitively on P_{83} (Frobenius action).
  $\Rightarrow$ There exists a normal subgroup N of G such that G is the semidirect product of N and P_{83}.

Smaller Primes: p=83,47,23

- In particular, there exists an epimorphism $G \to \mathbb{Z}_{83}$.

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- Conclusion: $g = 84$ attains the lower bound.

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- Conclusion: $g = 84$ attains the lower bound.
- Similarly, one can show that for $p = g - 1 = 47$, there exists a unique normal subgroup of order 47, and satisfies the other conditions of the Main Theorem as well.

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- Similarly, one can show that for $p = g - 1 = 47$, there exists a unique normal subgroup of order 47, and satisfies the other conditions of the Main Theorem as well.
- Using more results from group theory, one can show that $p = 23$ attains the lower bound as well.

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- In particular, there exists an epimorphism $G \to \mathbb{Z}_{83}$.
- But since $s = 84$, $\Gamma = \Gamma(2, 3, 7)$ is a triangle group, this is impossible. Thus P_{83} must be normal as required.
- Also, $p - 1 = 82 = 2 \cdot 41$, so $\gcd(p - 1, E) = 2$. Therefore, $p = 83$ satisfies all required conditions to exclude that $|G| > 4(g - 1) = 4 \cdot 83$.
- Conclusion: $g = 84$ attains the lower bound.
- Similarly, one can show that for $p = g - 1 = 47$, there exists a unique normal subgroup of order 47, and satisfies the other conditions of the Main Theorem as well.
- Using more results from group theory, one can show that $p = 23$ attains the lower bound as well.
- In fact, one can show that $g = 24$ is the smallest genus such that $N_{ar}(g) = 4(g - 1)$.

Explicit Sequence Theorem

Theorem (Explicit Theorem)

For all primes $p \equiv 23, 47, 59 \pmod{6}0$, *we have* $N_{ar}(g) = 4(g-1)$. *The least genus* g *for which the the lower bound* $N_{ar}(g) = 4(g-1)$ *is attained is* $g = 24$.

References I

- Farkas, H.M., Kra, I.
  Riemann Surfaces.
  New York: Springer, 1980.
- Jones, G., Singerman, D.
  Complex Functions.
  Cambridge: Cambridge UP, 1987.
- Katok, S.
  Fuchsian Groups.
  Chicago: U Chicago Press, 1992.

References II

Belolipetsky, M., Jones, G.
A bound for the number of automorphisms of an arithmetic Riemann surface.
Math. Proc. Cambridge Philos. Soc., **138** (2005), no. 2, 289-299.

Sah, C.H.
Groups related to compact Riemann surfaces.
Acta Math. **123** (1969), 13-42.

Takeuchi, K.
Arithmetic Triangle Groups
J. Math. Soc. Japan **29** (1977), 91-106.