The Local Langlands Correspondence for tamely ramified groups

David Roe

Department of Mathematics
Harvard University

Harvard Number Theory Seminar

Outline

What is the Langlands Correspondence?

- A generalization of class field theory to non-abelian extensions.
- A tool for studying L-functions.
- A correspondence between representations of Galois groups and representations of algebraic groups.

Local Class Field Theory

Irreducible 1-dimensional representations of $\mathcal{W}_{\mathbb{Q}_p}$

$$\updownarrow$$

Irreducible representations of $GL_1(\mathbb{Q}_p)$

The 1-dimensional case of local Langlands is local class field theory.

Conjecture

Irreducible n-dimensional representations of $\mathcal{W}_{\mathbb{Q}_p}$

$$\updownarrow$$

Irreducible representations of $\mathrm{GL}_n(\mathbb{Q}_p)$

In order to make this conjecture precise, we need to modify both sides a bit.

Smooth Representations

For $n > 1$, the representations of $GL_n(\mathbb{Q}_p)$ that appear are usually infinite dimensional.

Definition

A *smooth* $\mathbb{C}$*-representation* of $GL_n(\mathbb{Q}_p)$ is a pair (π, V), where

- V is a $\mathbb{C}$-vector space (possibly infinite dimensional),
- $\pi\colon GL_n(\mathbb{Q}_p) \to GL(V)$ is a homomorphism,
- The stabilizer of each $v \in V$ is open in $GL_n(\mathbb{Q}_p)$.

The only finite-dimensional irreducible smooth π are

$$g \mapsto \chi(\det(g))$$

for some character $\chi\colon \mathbb{Q}_p^\times \to \mathbb{C}^\times$.

Langlands Parameters

We also need to clarify what kinds of representations of $\mathcal{W}_{\mathbb{Q}_p}$ to focus on.

Definition

A *Langlands parameter* is a pair (φ, V) with

$$\varphi\colon \mathcal{W}_{\mathbb{Q}_p} \to \mathrm{GL}(V) \qquad \dim_{\mathbb{C}} V = n$$

such that φ is continuous and semisimple.

Parabolic Subgroups

Given a number of Langlands parameters $\varphi_i : \mathbf{W}_{\mathbb{Q}_p} \to \mathrm{GL}(V_i)$, one can form their direct sum. There should be a corresponding operation on the $\mathrm{GL}_n(\mathbb{Q}_p)$ side.

Definition

A *parabolic subgroup* of GL_n is a subgroup P conjugate to one consisting of block triangular matrices of a given pattern. For example:

$$\begin{pmatrix} * & * & * & * & * \\ 0 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \end{pmatrix}$$

Such a subgroup has a Levi decomposition $P = M \ltimes N$, where M is conjugate to the corresponding subgroup of block diagonal matrices, and N consists of the subgroup of P with identity blocks on the diagonal.

Parabolic Induction

Since each Levi subgroup M is just a direct product of GL_{n_i}, a collection of representations $\pi_i : \mathrm{GL}_{n_i}(\mathbb{Q}_p) \to \mathrm{GL}(V_i)$ yields a representation $\boxtimes_i \pi_i$ of M. We can pull this back to P and then induce to obtain

$$\pi = \mathrm{Ind}_P^{\mathrm{GL}_n(\mathbb{Q}_p)} \boxtimes_i \pi_i.$$

Definition

We say that π is the *parabolic induction* of the π_i. We say that π is *supercuspidal* if π is not parabolically induced from any proper parabolic subgroup of $\mathrm{GL}_n(\mathbb{Q}_p)$.

The Weil-Deligne Group

There is a natural bijection

Supercuspidal representations of $\mathrm{GL}_n(\mathbb{Q}_p)$	$\leftrightarrow$	n-dimensional irreducible representations of $\mathcal{W}_{\mathbb{Q}_p}$.

But the parabolic induction of irreducible representations does not always remain irreducible. To extend this bijection from supercuspidal representations of $\mathrm{GL}_n(\mathbb{Q}_p)$ to all smooth irreducible representations of $\mathrm{GL}_n(\mathbb{Q}_p)$, one enlarges the right hand side using the following group:

$$\mathrm{WD}_{\mathbb{Q}_p} := \mathcal{W}_{\mathbb{Q}_p} \times \mathrm{SL}_2(\mathbb{C}).$$

Theorem (Local Langlands for GL_n: Harris-Taylor, Henniart)

There is a unique system of bijections

$$\left\{\textit{Irreducible representations of } \mathrm{GL}_n(\mathbb{Q}_p)\right\} \xrightarrow{\mathrm{rec}_n} \left\{n\textit{-dimensional irreducible representations of } \mathrm{WD}_{\mathbb{Q}_p}\right\}$$

- rec_1 *is induced by the Artin map of local class field theory.*
- rec_n *is compatible with* 1*-dimensional characters:* $\mathrm{rec}_n(\pi \otimes \chi \circ \det) = \mathrm{rec}_n(\pi) \otimes \mathrm{rec}_1(\chi)$.
- *The central character* ω_π *of* π *corresponds to* $\det \circ \mathrm{rec}_n$*:* $\mathrm{rec}_1(\omega_\pi) = \det(\mathrm{rec}_n(\pi))$.
- $\mathrm{rec}_n(\pi^\vee) = \mathrm{rec}_n(\pi)^\vee$
- rec_n *respects natural invariants associated to each side, namely L-factors and* ϵ*-factors of pairs.*

A First Guess

Now suppose **G** is some other connected reductive group defined over $\mathbb{Q}_p$, such as SO_n, Sp_n or U_n. We'd like to use a Langlands correspondence to understand representations of $\mathbf{G}(\mathbb{Q}_p)$ in terms of Galois representations. Something like

Homomorphisms $\varphi\colon \mathrm{WD}_{\mathbb{Q}_p} \to \mathbf{G}(\mathbb{C})$	$\leftrightarrow$	Irreducible representations of $\mathbf{G}(\mathbb{Q}_p)$.

We need to modify this guess in two ways:

- change $\mathbf{G}(\mathbb{C})$ to a related group, ${}^L\mathbf{G}(\mathbb{C})$,
- and account for the fact that our correspondence is no longer a bijection.

Root Data

Reductive groups over algebraically closed fields are classified by root data

$$(X^*(\mathbf{S}), \Phi(\mathbf{G}, \mathbf{S}), X_*(\mathbf{S}), \Phi^\vee(\mathbf{G}, \mathbf{S})),$$

where

- $\mathbf{S} \subset \mathbf{G}$ is a maximal torus,
- $X^*(\mathbf{S})$ is the lattice of characters $\chi \colon \mathbf{S} \to \mathbb{G}_m$,
- $X_*(\mathbf{S})$ is the lattice of cocharacters $\lambda \colon \mathbb{G}_m \to \mathbf{S}$,
- $\Phi(\mathbf{G}, \mathbf{S})$ is the set of roots (eigenvalues of the adjoint action of $\mathbf{S}$ on $\mathfrak{g}$),
- $\Phi^\vee(\mathbf{G}, \mathbf{S})$ is the set of coroots ($\langle \alpha, \alpha^\vee \rangle = 2$).

Connected Langlands Dual

Given $\mathbf{G} \supset \mathbf{S}$, the connected Langlands dual group $\hat{\mathbf{G}}$ is defined to be the algebraic group over $\mathbb{C}$ with root datum

$$(X_*(\mathbf{S}), \Phi^\vee(\mathbf{G}, \mathbf{S}), X^*(\mathbf{S}), \Phi(\mathbf{G}, \mathbf{S})).$$

For semisimple groups, this has the effect of exchanging the long and short roots (as well as interchanging the simply connected and adjoint forms).

$\mathbf{G}$	GL_n	SL_n	PGL_n	Sp_{2n}	SO_{2n}	U_n
$\hat{\mathbf{G}}$	GL_n	PGL_n	SL_n	SO_{2n+1}	SO_{2n}	GL_n

Langlands Dual Group

For non-split $\mathbf{G}$, such as U_n, we need to work a little harder. Suppose that $\mathbf{G}$ is quasi-split with Borel $\mathbf{B} \supset \mathbf{S}$, splitting over a finite extension $E/\mathbb{Q}_p$. Then $\mathrm{Gal}(E/\mathbb{Q}_p)$ acts on the root datum, and we get an action on $\hat{\mathbf{G}}$ via pinned automorphisms. Define

$$^L\mathbf{G} := \hat{G} \rtimes \mathrm{Gal}(E/\mathbb{Q}_p).$$

Unitary Groups

- $E/\mathbb{Q}_p$ a quadratic extension (so for $p \neq 2$ there are three possibilities),
- $\tau \in \mathrm{Gal}(E/\mathbb{Q}_p)$ the nontrivial element,
- V an n-dimensional E-vector space,
- Non-degenerate Hermitian form $\langle , \rangle$ (so $\langle x, y \rangle = \tau \langle y, x \rangle$).

Then $\mathrm{U}(V)$ is the group of automorphisms of V preserving $\langle , \rangle$. Over $\bar{\mathbb{Q}}_p$, U becomes isomorphic to GL_n, so $\hat{\mathrm{U}}_n$ is GL_n, but ${}^L\mathbf{G}$ is non-connected: τ acts on $\mathrm{GL}_n(\mathbb{C})$ by the outer automorphism

$$g \mapsto (g^{-1})^{\mathrm{T}}.$$

Langlands Parameters

A Langlands parameter is now an equivalence class of homomorphisms

$$\varphi \colon \mathrm{WD}_{\mathbb{Q}_p} \to {}^L\mathbf{G}.$$

- We require that the composition of φ with the projection ${}^L\mathbf{G} \to \mathrm{Gal}(E/\mathbb{Q}_p)$ agrees with the standard projection $\mathcal{W}_{\mathbb{Q}_p} \to \mathrm{Gal}(E/\mathbb{Q}_p)$.
- We consider two parameters to be equivalent they are conjugate by an element of $\hat{\mathbf{G}}$. This definition of equivalence is chosen to match up with the notion of isomorphic representations on the $\mathbf{G}(\mathbb{Q}_p)$ side.

A Map

Conjecture

There is a natural map

Irreducible representations of $\mathbf{G}$ $\rightarrow$ Langlands parameters $\varphi\colon \mathrm{WD}_{\mathbb{Q}_p} \to {}^L\mathbf{G}$

It is surjective and finite-to-one; the fibers are called *L-packets*.

L-packets

Moreover, we can naturally parameterize these fibers. Given a Langlands parameter φ, let $\mathrm{Z}_{\hat{\mathbf{G}}}(\varphi)$ be the centralizer in $\hat{\mathbf{G}}$ of φ, and let ${}^LZ$ be the center of ${}^L\mathbf{G}$. Define

$$A_\varphi = \pi_0(\mathrm{Z}_{\hat{\mathbf{G}}}(\varphi)/{}^LZ).$$

The fibers should be in bijection with

$$A_\varphi^\vee = \{\text{irreducible representations of } A_\varphi\}.$$

So we get a natural bijection

Irreducible representations of $\mathbf{G}$	$\leftrightarrow$	(φ, ρ) with $\varphi\colon \mathrm{WD}_{\mathbb{Q}_p} \to {}^L\mathbf{G}$ and $\rho \in A_\varphi^\vee$

Restrictions on φ

From now on we fix a totally ramified quadratic extension $E/\mathbb{Q}_p$ and set $\mathbf{G} = \mathrm{U}(V)$ for V a quasi-split Hermitian space over E.
We say that a Langlands parameter φ is

- *discrete* if $\mathrm{Z}_{\hat{\mathbf{G}}}(\varphi)$ is finite,
- *tame* if φ factors through the maximal tame quotient (and thus $p \neq 2$).
- *regular* if $\mathrm{Z}_{\hat{\mathbf{G}}}(\varphi(\tilde{\tau}))$ is connected and minimum dimensional (here $\tilde{\tau}$ is a procyclic generator of tame inertia).

We will construct an L-packet of supercuspidal representations of pure inner forms of $\mathbf{G}(\mathbb{Q}_p)$ given a tame, discrete regular parameter.

Filtrations

$\mathbf{G}(\mathbb{Q}_p)$ acts on the Bruhat-Tits building $\mathcal{B}(\mathbf{G})$, and we can classify the compact subgroups of $\mathbf{G}(\mathbb{Q}_p)$ as stabilizers of convex subsets of $\mathcal{B}(\mathbf{G})$

- Each such compact $\mathbf{H}$ has the structure of a $\mathbb{Z}_p$-scheme.
- There is a decreasing filtration on each $\mathbf{H}$.
- $\mathbf{H}^0$ is just the connected component of the identity (as a $\mathbb{Z}_p$-scheme) and is of finite index in $\mathbf{H}$.
- The special fiber $\mathbf{H}(\mathbb{F}_p)$ is given by $\mathbf{H}/\mathbf{H}^{0+}$.
- The filtration on $\mathbf{T}$ is the one given by Moy and Prasad, coming from the filtration on $\mathbb{Q}_p^\times$.

We can thus obtain representations of compact subgroups of $\mathbf{G}$ by pulling back representations of reductive groups over finite fields.

Outline

Our plan for constructing an L-packet from φ is as follows. We construct:

- A maximal unramified anisotropic torus **T**, which embeds into **G** in various ways,
- A character χ_φ on $\mathbf{T}^0$ that vanishes on $\mathbf{T}^{0+}$,
- For each $\rho \in A_\varphi^\vee$, an embedding of **T** into a maximal compact subgroup $\mathbf{H} \subset \mathbf{G}$.
- We get a Deligne-Lusztig representation of $\mathbf{H}^0(\mathbb{F}_p) = \mathbf{H}^0/\mathbf{H}^{0+}$ associated to the torus $\mathbf{T}^0(\mathbb{F}_p) = \mathbf{T}^0/\mathbf{T}^{0+}$ and the character χ_φ.
- We induce this representation up to a representation of **G**.

An Unramified Anisotropic Torus

We have a specified torus $\hat{\mathbf{S}} \subset \hat{\mathbf{G}}$, dual to the centralizer $\mathbf{S}$ of a maximal $\mathbb{Q}_p$-split torus.

- We can conjugate φ so that $\varphi(\tilde{\tau}) \in \hat{\mathbf{S}} \rtimes \mathrm{Gal}(E/\mathbb{Q}_p)$.
- $\varphi(\mathrm{F})$ then lies in the normalizer of $\hat{\mathbf{S}}$. It's projection onto the Weyl group gives an element of

$$\mathrm{H}^1(\langle \mathrm{F} \rangle, W^{\mathcal{I}}) \hookrightarrow \mathrm{H}^1(\mathbb{Q}_p, W),$$

  which is exactly the data we need to define a maximal unramified torus as a twist of $\mathbf{S}$.
- Discreteness of φ implies that

$$(\hat{\mathfrak{g}}^{\mathcal{I}})^{\mathrm{F}} = 0 \Rightarrow \hat{\mathfrak{g}}^{\mathcal{I}} = \hat{\mathfrak{s}}^{\mathcal{I}} \text{ and } X_*(\mathbf{S})^{\mathrm{F}} = 0$$

  and thus that $\mathbf{T}$ is anisotropic.

A Character

- The image of φ lands inside some group containing $\hat{\mathbf{S}}$ with finite index. If it were a semidirect product, we could just use the local Langlands correspondence for tori:

$$\mathrm{H}^1(\mathbb{Q}_p, \hat{\mathbf{T}}) \cong \mathrm{Hom}(\mathbf{T}(\mathbb{Q}_p), \mathbb{C}^\times).$$

- In general this extension is not a semidirect product. But we can use a modification of the Langlands correspondence for tori to obtain a character χ_φ of $\mathbf{T}^0(\mathbb{Q}_p)$, where $\mathbf{T}^0$ is the connected component in the Néron model of $\mathbf{T}$.
- The depth preservation feature of the Langlands correspondence for tori, together with the fact that φ is tamely ramified, implies that χ_φ vanishes on $\mathbf{T}^{0+}$.

Construction of χ_φ

Let $D \subset {}^L\mathbf{G}$ be the group generated by $\hat{\mathbf{S}} \rtimes \mathrm{Gal}(E/\mathbb{Q}_p)$ and $\varphi(\mathrm{F})$. Denote by $M = \mathbb{Q}_{p^s} \cdot E$ the splitting field of $\mathbf{T}$.

$$1 \to \hat{\mathbf{T}} \to D \to \mathrm{Gal}(M/\mathbb{Q}_p) \to 1.$$

Let $P_K(D, \mathbf{T})$ be the set of homomorphisms from $\mathrm{Gal}(\bar{K}/K)$ to D that project correctly onto $\mathrm{Gal}(M/\mathbb{Q}_p)$, modulo conjugacy by $\hat{\mathbf{T}}$. Set D_s as the preimage in D of $\mathrm{Gal}(M/\mathbb{Q}_{p^s})$ and let $\Gamma = \mathrm{Gal}(\mathbb{Q}_{p^s}/\mathbb{Q}_p)$. Then χ_φ is the image of φ under

$$\begin{aligned} P_{\mathbb{Q}_p}(D, \mathbf{T}) \xrightarrow{\text{res}} P_{\mathbb{Q}_{p^s}}(D_s, \mathbf{T})^\Gamma &\cong \mathrm{H}^1(\mathbb{Q}_{p^s}, \hat{\mathbf{T}})^\Gamma \\ &\cong \mathrm{Hom}(\mathbf{T}(\mathbb{Q}_{p^s})_\Gamma, \mathbb{C}^\times) \to \mathrm{Hom}(\mathbf{T}^0(\mathbb{Q}_{p^s})_\Gamma, \mathbb{C}^\times) \\ &\cong \mathrm{Hom}(\mathbf{T}^0(\mathbb{Q}_p), \mathbb{C}^\times). \end{aligned}$$

Basic Tori

We classify unramified anisotropic twists of the "quasi-split" torus $\mathbf{S}$. Essentially, they are products of basic tori. For each $s = 2r$, define $\mathbf{T}_s = \{x \in E_s : \mathrm{Nm}_{E_s/L_r} x = 1\}$,

$$\begin{tikzcd}
 & & E_s \arrow[dl, dash, "\mathrm{Gal}(E_s/E_r)=\langle\sigma_s^r\rangle"'] \arrow[d, dash, "\langle\eta_s\rangle"] \arrow[dr, dash, "\mathrm{Gal}(E_s/\mathbb{Q}_{p^s})=\langle\tau_s\rangle"] & \\
 & E_r \arrow[dl, dash] \arrow[dr, dash] & L_r \arrow[d, dash] & \mathbb{Q}_{p^s} \arrow[dl, dash] \\
E \arrow[dr, dash] & & \mathbb{Q}_{p^r} \arrow[dl, dash] & \\
 & \mathbb{Q}_p & &
\end{tikzcd}$$

Embeddings of Basic Tori

In order to get Deligne-Lustig representations, we need to embed **T** into maximal compacts of **G**. We do so by building a Hermitian space around each basic torus in the product decomposition of **T**.

For each $\kappa \in L_r^\times$, we define a Hermitian product on E_s

$$\phi_\kappa(x, y) = \mathrm{Tr}_{E_s/E}\left(\frac{\kappa}{\pi_L} x \cdot \eta_s(y)\right)$$

This Hermitian space is quasi-split if and only if $v_L(\kappa)$ is even.

Embeddings of General Tori

In general, we choose a κ_i for each basic torus in the decomposition of **T**. This choice corresponds to a choice of $\rho \in A_\varphi^\vee$ as long as the sum of the valuations of the κ_i is even.

We prove **T** fixes a unique point on the building $\mathcal{B}(\mathbf{G})$ and thus embeds in a unique maximal compact $\mathbf{H} \subset \mathbf{G}$.
The reduction of **H** is

$$\mathrm{O}(m) \times \mathrm{Sp}(m'),$$

where m is the sum of the dimensions of basic tori whose κ_i has even valuation and m' is the sum of those with $v(\kappa_i)$ odd.

Constructing a representation of $\mathbf{G}(\mathbb{Q}_p)$

Modulo p, we have a maximal torus $\mathbf{T}^0(\mathbb{F}_p)$ sitting in a connected reductive group $\mathbf{H}^0(\mathbb{F}_p)$ and a character χ_φ of $\mathbf{T}^0(\mathbb{F}_p)$. This situation was studied by Deligne and Lusztig, and they produce a representation from étale cohomology. The irreducibility of this representation follows from the regularity condition on φ.

We pull back to $\mathbf{H}^0$ and the only wrinkle in the induction process occurs between $\mathbf{H}^0$ and $\mathbf{H}$. Once we have a representation of $\mathbf{H}$, we define a representation on all of $\mathbf{G}(\mathbb{Q}_p)$ by compact induction.

A Finite Induction

There are three cases for the induction from $\mathbf{H}^0$ to $\mathbf{H}$.

- n even, $\mathbf{H}(\mathbb{F}_p) = \mathrm{Sp}(n)$. Here $\mathbf{H} = \mathbf{H}^0$ and there is no induction.
- n even, otherwise. The fact that the normalizer of $\mathbf{T}^0(\mathbb{F}_p)$ in $\mathbf{H}(\mathbb{F}_p)$ contains the normalizer in $\mathbf{H}^0(\mathbb{F}_p)$ with index 2 implies that the induction remains irreducible.
- n odd. Now the induction from $\mathbf{H}^0$ to $\mathbf{H}$ splits into two irreducible components. We can pick one using a recipe for the central character, together with the fact that in the case that n is odd the center of $\mathrm{O}(m)$ is not contained in $\mathrm{SO}(m)$.

Summary

$$
\begin{tikzcd}
\mathbf{T}(\mathbb{Q}_p) \arrow[r] & \mathbf{H}(\mathbb{Q}_p) \arrow[r] \arrow[ddr, dashed, "\mathrm{Ind}(\mathrm{DL}(\mathbf{T}^0, \chi_\varphi))"] & \mathbf{G}'(\mathbb{Q}_p) \arrow[ddr, dashed, "c\,\mathrm{Ind}(\mathrm{DL}(\mathbf{T}^0, \chi_\varphi))"] & \\
\mathbf{T}^0(\mathbb{Q}_p) \arrow[r] \arrow[u] \arrow[d] & \mathbf{H}^0(\mathbb{Q}_p) \arrow[u] \arrow[d] \arrow[ddr, dashed] & & \\
\mathbf{T}^0(\mathbb{F}_p) \arrow[r] \arrow[dr, dashed, "\chi_\varphi"'] & \mathbf{H}^0(\mathbb{F}_p) \arrow[dr, dashed, "\mathrm{DL}(\mathbf{T}^0, \chi_\varphi)"'] & \mathrm{GL}(\mathrm{Ind}\, V) & \mathrm{GL}(c\,\mathrm{Ind}\, V) \\
& \mathbb{C}^\times & \mathrm{GL}(V) &
\end{tikzcd}
$$

Further Work

- There are only a few instances where I depend on the fact that $\mathbf{G} = U(V)$. I'd like to come up with general arguments that work for any reductive group splitting over a tame extension.
- There are various properties we expect the correspondence to satisfy. All of the properties that Reeder and DeBacker prove about their L-packets (including stability and an analysis of which representations are generic) we should be able to do for L-packets in the tame case.
- I want to build a computational framework within Sage to experiment with these L-packets.