A bound for the number of automorphisms of an arithmetic Riemann surfaces
Exposition of a paper by Mikhail Belolipetsky and Gareth Jones

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Outline

1. Terminology and Riemann-Hurwitz (Ying Zong)
2. Surface Kernel Epimorphisms and an Example (Kate Stange)
3. The Lower Bound on $N_{ar}(g)$ (Dermot McCarthy)
4. Sharpness of Bound, part 1 (Guillermo Mantilla)
5. Sharpness of Bound, part 2 (David Roe)
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Consider a Riemann surface as a quotient of $\mathcal{H}$ by its surface group.

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Then its automorphisms can be obtained from the automorphisms of $\mathcal{H}$:

$$\text{Aut}(S) = \{ \alpha \in \text{PSL}(2, \mathbb{R}) : \alpha \Gamma_S \alpha^{-1} = \Gamma_S \}/\Gamma_S$$

$$= N(\Gamma_S)/\Gamma_S$$

(Think: Given $\gamma \in \Gamma_S$, we need $\alpha(\gamma(x)) = \gamma'(\alpha(x))$ for some $\gamma' \in \Gamma_S$.)
Given arithmetic $\Gamma$, we will build an arithmetic Riemann surface $S$ with surface group $\Gamma_S$, such that $\Gamma \leq N(\Gamma_S)$. 
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Find a torsion-free normal subgroup $K$ finite index in $\Gamma$:

\[ 1 \to K \to \Gamma \xrightarrow{p} G \to 1 \]
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Then, if we determine $S$ by $\Gamma_S = K$, we have

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We call this a *surface-kernel epimorphism* or SKE.
To verify that the kernel is torsion free, we must check that every element of $\Gamma$ of finite order has its order preserved by $\rho : \Gamma \rightarrow G$. 
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For Fuchsian groups, it suffices to check this for the elements $\gamma_1, \ldots, \gamma_k$ in the canonical presentation.
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- epimorphism $p : \Gamma \rightarrow G$ to finite group
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Then we know that $G$ is a subgroup of $\text{Aut}(S)$.
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Arithmetic:

$$(2, 3, n), \quad n = 7, 8, 9, 10, 11, 12, 14, 16, 18, 24, 30$$
$$(2, 4, n), \quad n = 5, 6, 7, 8, 9, 10, 12, 18$$
$$(2, 5, n), \quad n = 5, 6, 8, 10, 20, 30$$

etc.

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Let $\Gamma$ be the orientation-preserving subgroup of the group of reflections in its sides.
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- Only sequences of an even number of reflections are orientation preserving automorphisms.
- Two reflections give rotation around an angle of $\pi$. This is order 2. There are five such elements of $\Gamma$.
- The signature of the group $\Gamma$ is $(2, 2, 2, 2, 2)$.
- The Riemann surface $S = \Gamma \backslash \mathcal{H}$ is of genus zero.
Subdivide the pentagon into 10 congruent triangles:

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- The triangle has angles $\pi/2$, $\pi/4$ and $\pi/5$. So $\Gamma'$ is the (2, 4, 5) triangle group, which is arithmetic.
- But $\Gamma$ is a subgroup of $\Gamma'$ of index 10. Hence the two groups are commensurable, and so $\Gamma$ is arithmetic.
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### Lemma

Let \( \{S_g\}_{g \in G} \) be an infinite sequence of arithmetic surfaces of different genera \( g \), such that for each \( g \in G \), the group of automorphisms of \( S_g \) has order \( a(g + b) \) for some fixed \( a \) and \( b \). Then \( b = -1 \).
Lemma

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Proof. Let \( S \) be a surface from the given sequence.

Then \( \text{Aut}(S) \cong N(\Gamma_S)/\Gamma_S \), where \( \Gamma_S \) is the surface group corresponding to \( S \).
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The Riemann-Hurwitz formula yields

\[
\mu(N(\Gamma_S)) = \frac{\mu(\Gamma_S)}{|\text{Aut}(S)|} = \frac{2\pi(2g - 2)}{a(g + b)},
\]

so \( \mu(N(\Gamma_S)) \to 4\pi/a \) as \( g \to \infty \).
\( \Gamma_S \text{ arithmetic } \Rightarrow N(\Gamma_S) \text{ arithmetic.} \)
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The measures of arithmetic groups form a discrete subset of \( \mathbb{R} \) (Borel).
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So for all but finitely many $g \in G$,

$$\frac{2\pi(2g - 2)}{a(g + b)} = \mu(\mathcal{N}(\Gamma_S)) = \frac{4\pi}{a}.$$

Therefore $b = -1$. 
It follows from Lemma 1 that the Accola-Maclachlan lower bound for $N(g), 8(g + 1)$, cannot be attained by infinitely many arithmetic surfaces.
It follows from Lemma 1 that the Accola-Maclachlan lower bound for $N(g), 8(g + 1)$, cannot be attained by infinitely many arithmetic surfaces.

In fact it is never attained by arithmetic surfaces, since the extremal surfaces for this bound are uniformized by surface subgroups of $(2, 4, 2(g+1))$-groups with $g \geq 24$ (Maclachlan), and these are not arithmetic (Takeuchi).
Lemma

$$N_{ar}(g) \geq 4(g - 1) \text{ for all } g \geq 2.$$
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Proof. Let \( \Gamma = \langle \gamma_1, \ldots, \gamma_5 \mid \gamma_j^2 = \gamma_1 \cdots \gamma_5 = 1 \rangle \) be an arithmetic group with signature (2, 2, 2, 2, 2).
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Let \( G = D_{2(g-1)} = \langle a, b | a^{2(g-1)} = b^2 = (ab)^2 = 1 \rangle \).
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Define \( \theta : \Gamma \rightarrow G \) by \( \gamma_j \mapsto ab, b, a^{g-2}b, b, a^{g-1} \).
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\( \theta \) is a SKE and thus \( K = \ker(\theta) \) is a surface group.
The surface $\mathcal{S} = \mathcal{H}/K$ is arithmetic and $\text{Aut}(\mathcal{S}) \geq \Gamma/K \cong G.$
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$\mu(\Gamma) = \pi$ and $|G| = 4(g - 1)$, so by Riemann-Hurwitz

$$\mu(K) = \mu(\Gamma)|G| = 2\pi(2g - 2).$$
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Then $N_{ar}(g) \geq |\text{Aut}(S)| \geq |G| = 4(g - 1)$ as required.
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- Imposing specific conditions on \( g \) we a get a contradiction.
Theorem

$N_{ar}(g) \geq 4(g - 1)$ for all $g \geq 2$, and this bound is attained for infinitely many values of $g$.

- $G := \text{Aut}(S)$ has order $|G| > 4(g - 1)$ for some compact arithmetic surface $S$ of genus $g \geq 2$.
- Imposing specific conditions on $g$ we get a contradiction.
- Show that infinitely many values of $g$ satisfy these conditions. For these $N_{ar}(g) = 4(g - 1)$.
By our hypothesis, $G \cong \Gamma/K$ for some co-compact arithmetic group $\Gamma$ and normal surface subgroup $K = \Gamma_S$ of $\Gamma$, with

$$4\pi(g - 1) = \mu(K) = |G|\mu(\Gamma) > 4(g - 1)\mu(\Gamma),$$  \hspace{1cm} (1)
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Borel’s discreteness theorem implies that there are only finitely many measures of co-compact arithmetic groups $\mu(\Gamma) < \pi$. 
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Hurwitz’s formula and (1) show that these correspond to a finite set $\Sigma$ of signatures.

For each $\sigma \in \Sigma$, the number $q = \frac{\mu(\Gamma)}{4\pi}$ is rational and depends only on the signature $\sigma$ of $\Sigma$, so writing $q = r/s = r_\sigma/s_\sigma$ in reduced form, we have $|G| = (g - 1)/q = (g - 1)s/r$. 
Restrictions on $g$

Let $R = \text{lcm}\{r_\sigma | \sigma \in \Sigma\}$, and $S = \max\{s_\sigma | \sigma \in \Sigma, r_\sigma = 1\}$. 
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Let $p$ be a prime such that $p \notin \Pi$, $(p, R) = 1$ and $p > S$. Suppose $g = p + 1$. 
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Then $|G| = ps$ with $(s, p) = 1$ and $s < p + 1$. By Sylow's Theorems there is a $P \cong \mathbb{Z}/p\mathbb{Z}$ with $P \trianglelefteq G$. 
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Let $\Delta$ denote the inverse image of $P$ in $\Gamma$, a normal subgroup of $\Gamma$ with $\Gamma/\Delta \cong Q := G/P$. 
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Let $\Delta$ denote the inverse image of $P$ in $\Gamma$, a normal subgroup of $\Gamma$ with $\Gamma/\Delta \cong Q := G/P$.

Since $|Q|$ is coprime to $p$, the natural epimorphism $G \to Q$ preserves the orders of the images of all elliptic generators of $\Gamma$. 
The inclusions $K \subseteq \Delta \subseteq \Gamma$ induce an étale $\mathbb{Z}/p\mathbb{Z}$-covering of Riemann surfaces
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$$S \cong K \setminus \mathcal{H} \quad \xrightarrow{P \cong \mathbb{Z}/p\mathbb{Z}} \quad T \cong \Delta \setminus \mathcal{H} \quad \xrightarrow{G} \quad \Gamma \setminus \mathcal{H}$$

In particular we have that $Q \leq \text{Aut}(T)$, and $T$ has genus $1 + (g - 1)/p = 2$. 
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$$Q$$

$$\Gamma \setminus \mathcal{H}$$

In particular we have that $Q \leq Aut(T)$, and $T$ has genus $1 + (g - 1)/p = 2$.

Then $Q$ is a group of automorphisms of a Riemann surface $T$ of genus 2.
Notice that $|\text{Aut}(\mathcal{T})| \leq 84$, thus there are just finitely many possibilities for $\text{Aut}(\mathcal{T})$ and hence for $Q$. 
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Let $E$ be the least common multiple of the exponents of all the groups of automorphisms of Riemann surfaces of genus 2.
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Riemann surfaces of genus 2 are hyperelliptic, therefore their automorphism groups always contain an element of order 2.
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In particular $E \equiv 0 \pmod{2}$. 
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Outline of remainder of proof

- Consider $H_1(\mathcal{I}, \mathbb{F}_p)$. 
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- Consider $H_1(\mathcal{T}, \mathbb{F}_p)$.
- We give an action of $Q$ on this $\mathbb{F}_p$-vector space.
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- Consider $H_1(\mathcal{I}, \mathbb{F}_p)$.
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- It decomposes into 1-dimensional submodules.
• Consider $H_1(T, \mathbb{F}_p)$.
• We give an action of $Q$ on this $\mathbb{F}_p$-vector space.
• It decomposes into 1-dimensional submodules.
• $Q$ acts faithfully.
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- Consider $H_1(\mathcal{T}, \mathbb{F}_p)$.
- We give an action of $Q$ on this $\mathbb{F}_p$-vector space.
- It decomposes into 1-dimensional submodules.
- $Q$ acts faithfully.
- We find $Q \subset \text{GL}_1(\mathbb{F}_p)^4$, which constrains the exponent $\epsilon$ of $Q$. 

Automorphisms of Arithmetic Riemann Surfaces
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- Thus $\epsilon$ divides $\gcd(E, p - 1)$, which we can force to be 2.
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- This gives a contradiction using the area formula.
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- We find $Q \subset GL_1(\mathbb{F}_p)^4$, which constrains the exponent $\epsilon$ of $Q$.
- Thus $\epsilon$ divides $\gcd(E, p - 1)$, which we can force to be 2.
- This gives a contradiction using the area formula.
- We have infinitely many $p$ satisfying our conditions.
We consider first the module structure of $H_1(\mathcal{T})$.

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$H_0(\mathcal{T}, \mathbb{Z}) \cong \mathbb{Z}$, so $\text{Tor}(H_0(\mathcal{T}, \mathbb{Z}), G) = 0$ for all $G$. 
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$\mathcal{T}$ has genus 2, so $H_1(\mathcal{T}, \mathbb{Z}) \cong \mathbb{Z}^4$.

$H_0(\mathcal{T}, \mathbb{Z}) \cong \mathbb{Z}$, so $\text{Tor}(H_0(\mathcal{T}, \mathbb{Z}), G) = 0$ for all $G$.

By the Universal Coefficient Theorem,

$$H_1(\mathcal{T}, \mathbb{F}_p) \cong H_1(\mathcal{T}, \mathbb{Z}) \otimes \mathbb{F}_p \cong \mathbb{F}_p^4.$$
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$$H_1(\mathcal{T}, \mathbb{C}) \cong H_1(\mathcal{T}, \mathbb{Z}) \otimes \mathbb{C} \cong \mathbb{C}^4.$$
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Thus $\Delta/\Delta' \cong H_1(\mathcal{T}, \mathbb{Z})$ and $\Delta/\Delta' \Delta^p \cong H_1(\mathcal{T}, \mathbb{F}_p)$. 
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In fact, these isomorphisms are $Q$-equivariant.
\[ H^1(\mathcal{I}, \mathbb{C}) \cong H^{1,0}(\mathcal{I}, \mathbb{C}) \oplus H^{0,1}(\mathcal{I}, \mathbb{C}) . \]
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These spaces give complex conjugate representations of \( Q \).
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These spaces give complex conjugate representations of \( Q \).

After Poincaré duality, \( H_1(\mathcal{T}, \mathbb{C}) \) decomposes into a pair of two dimensional \( Q \)-invariant subspaces.
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So \( H_1(\mathcal{T}, \mathbb{F}_p) \) decomposes into a pair of two dimensional subspaces, both irreducible or both reducible.
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Thus 

$$V = H_1(\mathcal{T}, \mathbb{F}_p) \cong \bigoplus_{i=1}^{4} V_i,$$

with each $V_i$ a 1-dimensional $Q$-invariant subspace of $V$. 
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with each $V_i$ a 1-dimensional $Q$-invariant subspace of $V$.

Therefore we have a map $Q \rightarrow \text{GL}_1(\mathbb{F}_p)^4$. 
Lemma (Farkas & Kra, V.3.4) If $A \in \text{SL}_k(\mathbb{Z})$ has finite order $m > 1$ and $A \equiv I \pmod{n}$ then $m = n = 2$. 
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$\epsilon$ thus divides $\gcd(E, p - 1)$. 

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Terminology  SKEs  The Lower Bound  Sharpness pt 1  Sharpness pt 2  Effective Version
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This contradicts \( 0 < \mu(\Gamma) < \pi \).
In summary, we have required that $g - 1 = p$ is prime, $p > S$, $p \notin \Pi$, $p$ is coprime to $R$ and $\gcd(p - 1, E) = 2$. 
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All but finitely many satisfy the other required properties.

Therefore we have an infinitely many $g$ that lead to a contradiction.
Outline

1. Terminology and Riemann-Hurwitz (Ying Zong)
2. Surface Kernel Epimorphisms and an Example (Kate Stange)
3. The Lower Bound on $N_{ar}(g)$ (Dermot McCarthy)
4. Sharpness of Bound, part 1 (Guillermo Mantilla)
5. Sharpness of Bound, part 2 (David Roe)
6. An Effective Version (Linda Gruendken)
Main Theorem

- **Main Theorem**: Let $\Sigma$ be the set of all signatures of cocompact arithmetic Fuchsian groups with volume strictly less than $\pi$. 
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Main Theorem

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Assume that $g - 1 =: p$ is a prime such that $\gcd(p, R) = 1$, $p \not\in S$, $p > S$ and such that $\gcd(p - 1, E) = 2$, where $E$ is the least common multiple of the exponents of all automorphism groups of Riemann surfaces of genus 2.
Main Theorem: Let $\Sigma$ be the set of all signatures of cocompact arithmetic Fuchsian groups with volume strictly less than $\pi$. Writing $\frac{\mu(\Gamma_\sigma)}{4\pi}$ as a fraction $r_\sigma/s_\sigma$ in lowest terms for every $\sigma \in \Sigma$, let $R = \text{lcm}\{r_\sigma\}$, let $\Pi$ be the list of primes that divide the period of an elliptic element of one of the $\Gamma_\sigma$, and $S = \max\{s_\sigma\}$.

Assume that $g - 1 =: p$ is a prime such that $\gcd(p, R) = 1$, $p \not\in S$, $p > S$ and such that $\gcd(p - 1, E) = 2$, where $E$ is the least common multiple of the exponents of all automorphism groups of Riemann surfaces of genus 2. Then the size of the automorphism group of any surface of genus $g$ cannot be greater than $4(g - 1)$, so we have to have equality.
Explicit Sequence Theorem

Goal

Construct a specific sequence of genera $g$ such that $N_{ar}$ attains the lower bound.
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Theorem (Main Theorem)

_For all primes \( p \equiv 23, 47, 59 \pmod{60} \), we have \( N_{ar}(g) = 4(g - 1) \). The least genus \( g \) for which the the lower bound \( N_{ar}(g) = 4(g - 1) \) is attained is \( g = 24 \)._
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Theorem (Main Theorem)

For all primes $p \equiv 23, 47, 59 \pmod{60}$, we have $N_{ar}(g) = 4(g - 1)$. The least genus $g$ for which the lower bound $N_{ar}(g) = 4(g - 1)$ is attained is $g = 24$.

Idea

Construct primes $p$ satisfying the hypotheses of the Main Theorem. Then $g = p + 1$ will be such that:

$$N_{ar}(g) = 4(g - 1).$$
Strategy

1. Listing all Arithmetic Fuchsian Signatures
2. The Conditions on Sufficiently Large Primes $p$
3. Smaller Primes
List of Possible Signatures

- Want to find the set $\Sigma$ of all signatures of cocompact arithmetic Fuchsian groups with volume strictly less than $\pi$. 
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- Writing $\mu(\Gamma_\sigma)$ as a fraction $r_\sigma/s_\sigma$ in lowest terms for every $\sigma \in \Sigma$, we need to determine $R = \text{lcm}\{r_\sigma\}$, the list $\Pi$ of primes that divide an elliptic period $m_k$, and $S = \max\{s_\sigma\}$.
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• Then by the proof of the Main Theorem, for any prime $p$ not dividing $R$, not contained in $\Pi$ and greater than $S$, we cannot have

$$|G| > 4(g - 1)$$

if we impose the additional condition that $\gcd(p - 1, E) = 2$. 
Let \((g; m_1; \ldots; m_r)\) be the signature of a Fuchsian group \(\Gamma\). Then
\[
\frac{1}{\pi} \mu(\Gamma) = 4(g - 1) + \sum_{k=1}^{r} \left( 1 - \frac{1}{m_k} \right) < 1
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has no solution unless \(g = 0\).
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- If \(g = 0\), then since \(m_k \geq 2\), we must have \(r < 5\), so all signatures have length 3 or 4.
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• Takeuchi gave a complete list of cocompact arithmetic triangle groups; almost all of these have volume less than \(\pi\).
The only other possible candidates are 
\((2, 2, 3, 3),(2, 2, 3, 4),(2, 2, 3, 5)\) and \((2, 2, 2, n), \text{ for } n \geq 3.\)
The only other possible candidates are 
$(2, 2, 3, 3), (2, 2, 3, 4), (2, 2, 3, 5)$ and $(2, 2, 2, n)$, for $n \geq 3$.

It can be shown that there are only 12 signatures for which 
$(2, 2, 2, n)$ is arithmetic.
Note that the orders of the elliptic elements are either 2, 3, 4, 5 or 7, so \( \Pi = \{2, 3, 5, 7\} \).
• Note that the orders of the elliptic elements are either 2, 3, 4, 5 or 7, so $\Pi = \{2, 3, 5, 7\}$.

• Further examining the list of possible signatures, and putting $\frac{\mu(\Gamma)}{4\pi}$ into lowest terms, we find that $R = 4 \cdot 3 \cdot 5 \cdot 7$ is the least common multiple of the numerators of all $\frac{\mu(\Gamma_\sigma)}{4\pi}$ and $s = 84$ is the largest occurring denominator.
Sufficiently Large Primes

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- To deal with the last condition $\gcd(p - 1, E) = 2$, we need a lemma:
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- To deal with the last condition $\gcd(p - 1, E) = 2$, we need a lemma:

**Lemma**

*If S is a Riemann surface of genus $\gamma \geq 2$, then it has no automorphisms of prime order greater than $2\gamma + 1$.*
Sufficiently Large Primes

Proof.
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If $f$ is an automorphism of $S$ of order $p$, let $T$ be the Riemann surface corresponding to $S$ modulo $\langle f \rangle$, and $\gamma'$ its genus.
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If $f$ is an automorphism of $S$ of order $p$, let $T$ be the Riemann surface corresponding to $S$ modulo $< f >$, and $\gamma'$ its genus. Then $f : S \rightarrow T$ is a smooth $p$-sheeted covering of $T$, so the Riemann-Hurwitz formula reads:

$$2(\gamma - 1) = 2p(\gamma' - 1) + m(p - 1)$$
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where $m$ is the number of fixed points of $f$. 
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- for $\gamma' \geq 2$, $2(\gamma - 1) \geq 2p + m(p - 1) \geq 2p$, a contradiction
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If $f$ is an automorphism of $S$ of order $p$, let $T$ be the Riemann surface corresponding to $S$ modulo $\langle f \rangle$, and $\gamma'$ its genus. Then $f : S \rightarrow T$ is a smooth $p$-sheeted covering of $T$, so the Riemann-Hurwitz formula reads:

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- for $\gamma' \geq 2$, $2(\gamma - 1) \geq 2p + m(p - 1) \geq 2p$, a contradiction
- for $\gamma' = 1$, $2(\gamma - 1) = m(p - 1) \geq p - 1 \geq 2\gamma - 1$, a contradiction.
Sufficiently Large Primes

- For $\gamma' = 0$, $2(\gamma - 1) = -2pg + m(p - 1)$, we have $m = \frac{2\gamma}{p-1} + 2 \leq \frac{p}{p-1} + 2 \leq 3$, so $m = 3$. 
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- In this case, $2\gamma - 2 = -2p + 3(p - 1)$, so $p = 2\gamma + 1$. 

Automorphisms of Arithmetic Riemann Surfaces
Sufficiently Large Primes

- For \( \gamma' = 0 \), \( 2(\gamma - 1) = -2pg + m(p - 1) \), we have \( m = \frac{2\gamma}{p-1} + 2 \leq \frac{p}{p-1} + 2 \leq 3 \), so \( m = 3 \).

- In this case, \( 2\gamma - 2 = -2p + 3(p - 1) \), so \( p = 2\gamma + 1 \). Hence it follows that \( p \leq 2\gamma + 1 \).
For $\gamma' = 0$, $2(\gamma - 1) = -2pg + m(p - 1)$, we have $m = \frac{2\gamma}{p-1} + 2 \leq \frac{p}{p-1} + 2 \leq 3$, so $m = 3$.

In this case, $2\gamma - 2 = -2p + 3(p - 1)$, so $p = 2\gamma + 1$. Hence it follows that $p \leq 2\gamma + 1$.

So if $S$ is a surface of genus 2, it cannot have automorphisms of prime order $q$ for any $q > 5$. Thus the exponent of $\text{Aut}(S)$ is not divisible by any prime other than 2, 3 or 5.


Sufficiently Large Primes

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- In this case, $2\gamma - 2 = -2p + 3(p - 1)$, so $p = 2\gamma + 1$. Hence it follows that $p \leq 2\gamma + 1$.
- So if $S$ is a surface of genus 2, it cannot have automorphisms of prime order $q$ for any $q > 5$. Thus the exponent of $Aut(S)$ is not divisible by any prime other than 2,3 or 5.
**Conclusion:** No prime other than \{2, 3, 5\} divides $E$, the least common multiple of the exponents of automorphism groups of surfaces of genus 2. Thus the condition that $\gcd(p - 1, E) = 2$ is satisfied by all $p$ such that $p - 1$ is not divisible by 3, 4, 5.
• **Conclusion:** No prime other than \(\{2, 3, 5\}\) divides \(E\), the least common multiple of the exponents of automorphism groups of surfaces of genus 2. Thus the condition that \(\gcd(p - 1, E) = 2\) is satisfied by all \(p\) such that \(p - 1\) is not divisible by 3, 4, 5.

• Since we also require that \(p \not\equiv 0 \mod q\) for \(q = 2, 3, 5\), this leaves the possibilities that \(p \equiv 2 \mod 3\), \(p \equiv 3 \mod 4\) and \(p \equiv 2, 3, 4 \mod 5\). The first two lift to the congruence \(p \equiv 11 \mod 12\); combining with the last one gives \(p \equiv 23, 47, 59 \mod 60\) as the equivalent congruence.
We have shown that any prime $p > 84$ congruent to one of 23, 47, 59 modulo 60 satisfies the conditions of the Main Theorem.
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Thus, surfaces of genus $p + 1$ for any such $p$ satisfy the lower bound: $N_g = 4(g - 1)$. 
We have shown that any prime $p > 84$ congruent to one of 23, 47, 59 modulo 60 satisfies the conditions of the Main Theorem.

Thus, surfaces of genus $p + 1$ for any such $p$ satisfy the lower bound: $N_g = 4(g – 1)$.

What about $p = 23, 47, 59$ or 83?
Smaller Primes: $p=59$

- $p = 59$, $S$ of genus $g = 60$: 


Smaller Primes: p=59

- $p = 59$, $S$ of genus $g = 60$:
- 59 is coprime to $R$, so $|Aut(S)| = |G| = (g - 1)s = 59s$ for some $s$. 
Smaller Primes: $p=59$

- $p = 59$, $S$ of genus $g = 60$:
- $59$ is coprime to $R$, so $|\text{Aut}(S)| = |G| = (g - 1)s = 59s$ for some $s$.
- By inspection, $s$ is coprime to $59$, so a $59$-Sylow subgroup is of order $59$. Letting $n_{59}$ be the number of $59$-Sylow subgroups, we must have $n_{59} | s$ and $n_{59} \equiv 1 \pmod{59} \Rightarrow n_{59} = 1$. So the $59$-Sylow subgroup $P_{59}$ is unique.
Smaller Primes: \( p=59 \)

- \( p = 59 \), \( S \) of genus \( g = 60 \):
- 59 is coprime to \( R \), so \(|Aut(S)| = |G| = (g - 1)s = 59s\) for some \( s \).
- By inspection, \( s \) is coprime to 59, so a 59-Sylow subgroup is of order 59. Letting \( n_{59} \) be the number of 59-Sylow subgroups, we must have \( n_{59}|s \) and \( n_{59} \equiv 1 \pmod{59} \Rightarrow n_{59} = 1 \). So the 59-Sylow subgroup \( P_{59} \) is unique.
- \( p \notin \Pi = \{2, 3, 5, 7\} \), the set of primes dividing an element of order in some \( \Gamma_\sigma \).
Smaller Primes: \( p = 59 \)

- \( p = 59 \), \( S \) of genus \( g = 60 \):
- 59 is coprime to \( R \), so \( |Aut(S)| = |G| = (g - 1)s = 59s \) for some \( s \).
- By inspection, \( s \) is coprime to 59, so a 59-Sylow subgroup is of order 59. Letting \( n_{59} \) be the number of 59-Sylow subgroups, we must have \( n_{59} | s \) and \( n_{59} \equiv 1 \pmod{59} \) \( \Rightarrow n_{59} = 1 \). So the 59-Sylow subgroup \( P_{59} \) is unique.
- \( p \notin \Pi = \{2, 3, 5, 7\} \), the set of primes dividing an element of order in some \( \Gamma_\sigma \).
- \( p - 1 = 58 = 2 \cdot 19 \), so \( \gcd(p - 1, E) = 2 \).
Smaller Primes: $p=59$

- $p = 59$, $S$ of genus $g = 60$:
- $59$ is coprime to $R$, so $|\text{Aut}(S)| = |G| = (g-1)s = 59s$ for some $s$.
- By inspection, $s$ is coprime to $59$, so a $59$-Sylow subgroup is of order $59$. Letting $n_{59}$ be the number of $59$-Sylow subgroups, we must have $n_{59}|s$ and $n_{59} \equiv 1 \pmod{59} \Rightarrow n_{59} = 1$. So the $59$-Sylow subgroup $P_{59}$ is unique.
- $p \not\in \Pi = \{2, 3, 5, 7\}$, the set of primes dividing an element of order in some $\Gamma_\sigma$.
- $p - 1 = 58 = 2 \cdot 19$, so $\gcd(p - 1, E) = 2$.
- Conclusion: $g = 60$ attains the lower bound.
Smaller Primes: \( p=83 \)

- \( p = 83 \), \( S \) of genus \( g = 84 \):
Smaller Primes: p=83

- $p = 83$, $S$ of genus $g = 84$:
- 83 is coprime to $R$, so $|Aut(S)| = |G| = (g - 1)s = 83s$ for some $s$. By inspection, $s$ is coprime to 83, so if $P_{83}$ is a 83-Sylow subgroup, then $|P_{83}| = 83$. Letting $n_{83}$ be the number of 83-Sylow subgroups, we must have $n_{83}|s$ and $n_{59} \equiv 1 \pmod{59}$. 
Smaller Primes: \( p = 83 \)

- \( p = 83 \), \( S \) of genus \( g = 84 \):
- 83 is coprime to \( R \), so \( |Aut(S)| = |G| = (g - 1)s = 83s \) for some \( s \). By inspection, \( s \) is coprime to 83, so if \( P_{83} \) is a 83-Sylow subgroup, then \( |P_{83}| = 83 \). Letting \( n_{83} \) be the number of 83-Sylow subgroups, we must have \( n_{83}|s \) and \( n_{59} \equiv 1 \pmod{59} \).
- Claim: \( P_{83} \) is normal in \( G \).
Smaller Primes: p=83

- $p = 83$, $S$ of genus $g = 84$:
- $83$ is coprime to $R$, so $|\text{Aut}(S)| = |G| = (g - 1)s = 83s$ for some $s$. By inspection, $s$ is coprime to $83$, so if $P_{83}$ is a 83-Sylow subgroup, then $|P_{83}| = 83$. Letting $n_{83}$ be the number of 83-Sylow subgroups, we must have $n_{83}|s$ and $n_{59} \equiv 1 \pmod{59}$.
- Claim: $P_{83}$ is normal in $G$.

Proof:
Smaller Primes: \( p=83 \)

- \( p = 83 \), \( S \) of genus \( g = 84 \):
- 83 is coprime to \( R \), so \( |Aut(S)| = |G| = (g - 1)s = 83s \) for some \( s \). By inspection, \( s \) is coprime to 83, so if \( P_{83} \) is a 83-Sylow subgroup, then \( |P_{83}| = 83 \). Letting \( n_{83} \) be the number of 83-Sylow subgroups, we must have \( n_{83}|s \) and \( n_{59} \equiv 1 \pmod{59} \).
- Claim: \( P_{83} \) is normal in \( G \).

**Proof:**
- The only possibility for the 83-Sylow subgroup \( P_{83} \) not being unique is if \( n_{83} = s = 84 \).
Smaller Primes: \( p=83 \)

- \( p = 83 \), \( S \) of genus \( g = 84 \):

- 83 is coprime to \( R \), so \( |Aut(S)| = |G| = (g - 1)s = 83s \) for some \( s \). By inspection, \( s \) is coprime to 83, so if \( P_{83} \) is a 83-Sylow subgroup, then \( |P_{83}| = 83 \). Letting \( n_{83} \) be the number of 83-Sylow subgroups, we must have \( n_{83}|s \) and \( n_{59} \equiv 1 \) (mod 59).

- Claim: \( P_{83} \) is normal in \( G \).

**Proof:**

- The only possibility for the 83-Sylow subgroup \( P_{83} \) not being unique is if \( n_{83} = s = 84 \).

- Then the normaliser of \( P_{83} \) is just \( P \), so \( G \) acts faithfully and transitively on \( P_{83} \) (Frobenius action).

\[ \Rightarrow \] There exists a normal subgroup \( N \) of \( G \) such that \( G \) is the semidirect product of \( N \) and \( P_{83} \).
Smaller Primes: \( p=83,47,23 \)

- In particular, there exists an epimorphism \( G \rightarrow \mathbb{Z}_{83} \).
Smaller Primes: $p=83,47,23$

- In particular, there exists an epimorphism $G \rightarrow \mathbb{Z}_{83}$.
- But since $s = 84$, $\Gamma = \Gamma(2, 3, 7)$ is a triangle group, this is impossible. Thus $P_{83}$ must be normal as required.
Smaller Primes: $p=83, 47, 23$

- In particular, there exists an epimorphism $G \rightarrow \mathbb{Z}_{83}$.
- But since $s = 84$, $\Gamma = \Gamma(2, 3, 7)$ is a triangle group, this is impossible. Thus $P_{83}$ must be normal as required.
- Also, $p - 1 = 82 = 2 \cdot 41$, so $\gcd(p - 1, E) = 2$. 
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- But since $s = 84$, $\Gamma = \Gamma(2, 3, 7)$ is a triangle group, this is impossible. Thus $P_{83}$ must be normal as required.
- Also, $p - 1 = 82 = 2 \cdot 41$, so $\gcd(p - 1, E) = 2$. Therefore, $p = 83$ satisfies all required conditions to exclude that $|G| > 4(g - 1) = 4 \cdot 83$. 
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- Conclusion: $g = 60$ attains the lower bound.
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- Conclusion: $g = 60$ attains the lower bound.
- Similarly, one can show that for $p = g - 1 = 47$, there exists a unique normal subgroup of order 47, and satisfies the other conditions of the Main Theorem as well.
Smaller Primes: $p=83,47,23$

- In particular, there exists an epimorphism $G \to \mathbb{Z}_{83}$.
- But since $s = 84$, $\Gamma = \Gamma(2, 3, 7)$ is a triangle group, this is impossible. Thus $P_{83}$ must be normal as required.
- Also, $p - 1 = 82 = 2 \cdot 41$, so $\gcd(p - 1, E) = 2$. Therefore, $p = 83$ satisfies all required conditions to exclude that $|G| > 4(g - 1) = 4 \cdot 83$.
- Conclusion: $g = 60$ attains the lower bound.
- Similarly, one can show that for $p = g - 1 = 47$, there exists a unique normal subgroup of order 47, and satisfies the other conditions of the Main Theorem as well.
- Using more results from group theory, one can show that $p = 23$ attains the lower bound as well.
Smaller Primes: $p=83, 47, 23$

- In particular, there exists an epimorphism $G \rightarrow \mathbb{Z}_{83}$.
- But since $s = 84$, $\Gamma = \Gamma(2, 3, 7)$ is a triangle group, this is impossible. Thus $P_{83}$ must be normal as required.
- Also, $p - 1 = 82 = 2 \cdot 41$, so $\gcd(p - 1, E) = 2$. Therefore, $p = 83$ satisfies all required conditions to exclude that $|G| > 4(g - 1) = 4 \cdot 83$.
- Conclusion: $g = 60$ attains the lower bound.
- Similarly, one can show that for $p = g - 1 = 47$, there exists a unique normal subgroup of order 47, and satisfies the other conditions of the Main Theorem as well.
- Using more results from group theory, one can show that $p = 23$ attains the lower bound as well.
- In fact, one can show that $g = 24$ is the smallest prime such that $N_{ar}(g) = 4(g - 1)$. 
Theorem (Explicit Sequence Theorem)

For all primes $p \equiv 23, 47, 59 \pmod{6} 0$, we have $N_{ar}(g) = 4(g - 1)$. The least genus $g$ for which the lower bound $N_{ar}(g) = 4(g - 1)$ is attained is $g = 24$. 

Explicit Sequence Theorem
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