The inverse Galois problem for p-adic fields

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Inverse Galois Problem

- Classic Problem: determine if a finite G is a Galois group.
- Depends on base field: every G is a Galois group over $\mathbb{C}(t)$.
- Most work focused on $L/\mathbb{Q}$: S_n and A_n, every solvable group, every sporadic group except possibly $M_{23}, \ldots$
- Generic polynomials $f_G(t_1, \ldots, t_r, X)$ are known for some (G, K): every L/K with group G is a specialization.

Computational Problems

Given a finite group G, find algorithms for

1. Existence problem: exist $L/\mathbb{Q}_p$ with Gal$(L/\mathbb{Q}_p) \cong G$?
2. Counting problem: how many such L exist (always finite)?
3. Enumeration problem: list the L.

Ramification Groups

Suppose

- L/K is an extension of p-adic fields, $G = \mathrm{Gal}(L/K)$,
- π is a uniformizer of L,
- $G_i = \{\sigma \in G : v_L(\sigma(x) - x) \geq i + 1 \forall x \in \mathcal{O}_L\}$ for $i \geq -1$,
- $U_L^{(0)} = \mathcal{O}_L^\times$ and $U_L^{(i)} = 1 + \pi^i \mathcal{O}_L$ for $i \geq 1$.

Proposition ([2, Prop. IV.2.7])

For $i \geq 0$, the map $\theta_i : G_i/G_{i+1} \to U_L^{(i)}/U_L^{(i+1)}$ defined by $\theta_i(\sigma) = \sigma(\pi)/\pi$ is injective and independent of π.

Corollary

- *G/G_0 is cyclic,*
- *G_0/G_1 is cyclic of order prime to p,*
- *G_i/G_{i+1} is an elementary abelian p-group for $i \geq 1$.*

p-realizable groups

Definition

A group G is *potentially p-realizable* if it has a filtration $G \supseteq G_0 \supseteq G_1$ so that

1. G_0 and G_1 are normal in G,
2. G/G_0 is cyclic, generated by some $\sigma \in G$,
3. G_0/G_1 is cyclic, generated by some $\tau \in G_0$,
4. $\tau^\sigma = \tau^p$,
5. G_1 is a p-group.

It is *p-realizable* if there exists $L/\mathbb{Q}_p$ with $\mathrm{Gal}(L/\mathbb{Q}_p) \cong G$.
It is *minimally unrealizable* if it is not p-realizable, but every proper quotient is.

Counting for p-groups

When G is a p-group, complete answer available. Suppose $K/\mathbb{Q}_p$ has degree n and $K \not\supset \mu_p$.

Theorem ([3])

The maximal pro-p quotient of Gal(K) *is a free pro-p group on $n+1$ generators.*

Corollary

If G is a p-group generated by d elements (minimally), the number of extensions L/K with Galois group G is

$$\frac{1}{|\mathrm{Aut}(G)|}\left(\frac{|G|}{p^d}\right)^{n+1}\prod_{i=0}^{d-1}(p^{n+1}-p^i).$$

Criterion for p-groups

We get an easy condition on when a p-group is p-realizable. Let $W = G^p G'$ be the Frattini subgroup; G/W is the maximal elementary abelian quotient of G. A set of elements generates G if and only if its projection onto G/W spans G/W as an $\mathbb{F}_p$-vector space.

Corollary

If $p > 2$ and G is a p-group then G is p-realizable if and only if G/W has dimension less than 3.

Presentation of the absolute Galois group

For $p > 2$, Gal$(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ is the profinite group generated by σ, τ, x_0, x_1 with x_0, x_1 pro-p and the following relations (see [1])

$$\tau^\sigma = \tau^p$$

$$\langle x_0, \tau\rangle^{-1} x_0^\sigma = x_1^p \left[x_1, x_1^{\tau_2^{p+1}} \left\{x_1, \tau_2^{p+1}\right\}^{\sigma_2 \tau_2^{(p-1)/2}} \left\{ \left\{x_1, \tau_2^{p+1}\right\}, \sigma_2 \tau_2^{(p-1)/2} \right\}^{\sigma_2 \tau_2^{(p+1)/2} + \tau_2^{(p+1)/2}} \right]$$

$h \in \mathbb{Z}_p$ with mult. order $p - 1$, $\quad \mathrm{proj}_p : \hat{\mathbb{Z}} \to \mathbb{Z}_p$

$$\langle x_0, \tau\rangle := \left(x_0 \tau x_0^{h^{p-2}} \tau \ldots x_0^h \tau\right)^{\mathrm{proj}_p/(p-1)}$$

$$\beta : \mathrm{Gal}(\mathbb{Q}_p^t/\mathbb{Q}_p) \to \mathbb{Z}_p^\times \qquad \beta(\tau) = h \qquad \beta(\sigma) = 1$$

$$\{x, \rho\} := \left(x^{\beta(1)} \rho^2 x^{\beta(\rho)} \rho^2 \ldots x^{\beta(\rho^{p-2})} \rho^2\right)^{\mathrm{proj}_p/(p-1)}$$

$$\sigma_2 := \mathrm{proj}_2(\sigma) \qquad\qquad \tau_2 := \mathrm{proj}_2(\tau)$$

Counting in general

By the Galois correspondence, Galois extensions of $\mathbb{Q}_p$ correspond to finite index normal subgroups of Gal$(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$. Thus the number of extensions $L/\mathbb{Q}_p$ with Gal$(L/\mathbb{Q}_p) \cong G$ is

$$\frac{1}{\#\,\mathrm{Aut}(G)}\#\left\{\varphi : \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \twoheadrightarrow G\right\}$$

So we count the tuples $\sigma, \tau, x_0, x_1 \in G$ (up to automorphism) that

1. satisfy the relations from Gal$(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$,
2. generate G.

Basic Strategy

Loop over σ generating the unramified quotient and τ generating the tame inertia (with $\tau^\sigma = \tau^p$). For each such (σ, τ) up to automorphism, count the valid x_0, x_1.

Iterative approach

Counting for many G, so we can build up from quotients.

Iterative Strategy

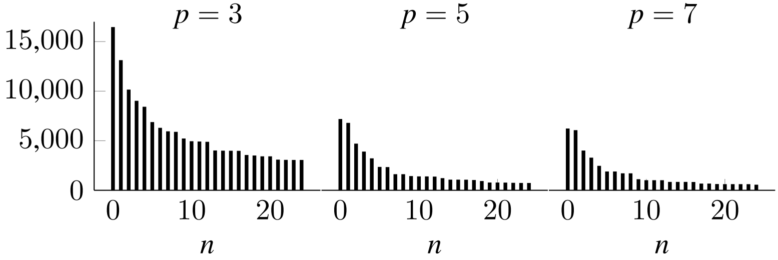
- Pick a minimal normal subgroup $N \triangleleft G$, then try to lift (σ, τ, x_0, x_1) from G/N to G.
- Tame G form a base case.

Two subtleties.

- If N is not characteristic, it will not be preserved by Aut(G) so not all automorphisms descend;
- The map $\mathrm{Stab}_{\mathrm{Aut}(G)}(N) \to \mathrm{Aut}(G/N)$ may not be surjective, so equivalent quadruples may become inequivalent.

Counts

Potentially p-realizable G with the count of $L/\mathbb{Q}_p$ at least n.



The largest counts occurred for cyclic groups or products of large cyclic groups with small nonabelian groups:

- C_{1458} ($p = 3$) with 2916,
- C_{1210} ($p = 11$) with 2376,
- $C_{243} \times S_3$ ($p = 3$) with 1944.

But also 1458G553, $(C_{27} \rtimes C_{27}) \rtimes C_2$ ($p = 3$) with 1323.

Realizability Criteria

Given potentially p-realizable G, let V be it's p-core and $W = V^p V'$. Then V/W is an $\mathbb{F}_p$ vector space with action of G/V. Let T_G be the set of pairs $(\sigma, \tau) \in G^2$ generating G/V and satisfying $\tau^\sigma = \tau^p$.

Definition

G is *strongly-split* if $\mathrm{ord}_G(\sigma) = \mathrm{ord}_{G/V}(\sigma)$ for all $(\sigma, \tau) \in T_G$.
G is *tame-decoupled* if τ acts trivially on V/W for all $(\sigma, \tau) \in T_G$.
G is *x_0-constrained* if $x_0^\sigma \langle x_0, \tau \rangle^{-1} \in W \Rightarrow x_0 \in W$ for all $(\sigma, \tau) \in T_G$.

Set $n_{G,ss} = 0$ if strongly-split, 1 o/w; $n_{G,xc} = 0$ if x_0-constrained, 1 o/w.

Theorem

Let n be the largest multiplicity of an indecomposable factor of V/W.

- *If G is tame-decoupled then it is x_0-constrained.*
- *If $n > 1 + n_{G,ss} + n_{G,xc}$ then G is not p-realizable.*
- *If $W = 1$ and V is a sum of distinct irreducibles, G is p-realizable.*

Minimally unrealizable G with abelian V, $p = 3$

Label	Description	V	SS	TD	XC	$1 + n_{G,ss} + n_{G,xc}$
27G5	$\mathbb{F}_3^3$	1^3	N	Y	Y	2
36G7	$\mathbb{F}_3^2 \rtimes C_4$	1^2	Y	Y	Y	1
54G14	$\mathbb{F}_3^3 \rtimes C_2$	1^3	Y	N	N	2
72G33	$\mathbb{F}_3^2 \rtimes D_8$	1^2	Y	Y	Y	1
162G16	$C_9^2 \rtimes C_2$	1^2	Y	N	N	2
324G164	$\mathbb{F}_3^4 \rtimes C_4$	2^2	Y	N	Y	1
324G169	$\mathbb{F}_3^4 \rtimes (C_2 \times C_2)$	$1^2 \oplus 1^2$	Y	N	N	2
378G51	$\mathbb{F}_3^2 \rtimes (C_7 \rtimes C_6)$	1^2	Y	Y	Y	1
648G711	$\mathbb{F}_3^4 \rtimes C_8$	2^2	Y	N	Y	1

162G16 and 324G169

There are two instances not explained by the theorem.

- For 324G169, $V \cong 1^2 \oplus 1^2$. There are nontrivial x_0 satisfying $x_0^\sigma \langle x_0, \tau \rangle^{-1} = 1$, but they all lie in a 1-dimensional indecomposable subrepresentation. The other subrepresentation can't be spanned by x_1 on its own.
- For 162G16, the quotient by W is p-realizable. Here V is abelian but has exponent 9 rather than 3, so the wild relation takes the form

  $$x_0^\sigma \langle x_0, \tau \rangle^{-1} = x_1^p.$$

  In order to get a nontrivial x_1, we need to find x_0 with $x_0^\sigma \langle x_0, \tau \rangle^{-1}$ of order 3. Such x_0 exist, but they all have the property that $x_0^\sigma \langle x_0, \tau \rangle^{-1}$ is a multiple of x_0, preventing x_1 from spanning the rest of V.

Minimally unrealizable G with nonabelian V, $p = 3$

Label	Description	G/W	V/W
486G146	$(\mathbb{F}_3^4 \rtimes C_3) \rtimes C_2$	54G13	$1^2 \oplus 1$
648G218	$(C_{27} \rtimes C_3) \times D_8$	72G37	1^2
648G219	$(\mathbb{F}_3^3 \rtimes C_3) \times D_8$	72G37	1^2
648G220	$((C_9 \times C_3) \rtimes C_3) \times D_8$	72G37	1^2
648G221	$((C_9 \times C_3) \rtimes C_3) \times D_8$	72G37	1^2
972G816	$(\mathbb{F}_3^2 \times (\mathbb{F}_3^2 \rtimes C_3)) \rtimes (C_2^2)$	324G170	$1^2 \oplus 1 \oplus 1$
1458G613	$((C_{81} \times C_3) \rtimes C_3) \rtimes C_2$	18G4	1^2
1458G640	$(C_9^2 \rtimes C_9) \rtimes C_2$	18G4	1^2

Sketch for G not p-realizable

Proof.

To prove that G is not p-realizable, we show that a map $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \to G$ cannot possibly be surjective. In any attempt at surjectivity, we need to choose $(\sigma, \tau) \in T_G$. Having done so, we must generate all of V, which is equivalent to generating V/W. There are only three ways to produce elements of V: the image of x_0, the image of x_1 and a power of σ. When G is x_0-constrained, x_0 must map to $0 \in V/W$. When G is strongly split, every power of σ lying in V also lies in W. So there are $1 + n_{G,ss} + n_{G,xc}$ generators available.

The action of G/V on V/W spreads out these generators: we can get anything in the G/V submodule spanned by them. But when $n > 1 + n_{G,ss} + n_{G,xc}$, this submodule can't possibly be everything. □

Sketch for G p-realizable

Proof.

We show that when $W = 1$ and V is a sum of distinct irreducibles, then G is p-realizable. In this case, the relations simplify and we can just choose to map x_0 to 1 and x_1 to an element projecting nontrivially on each irreducible G/V-submodule of V. The resulting homomorphism is surjective. $\square$

References

[1] J. Neukirch, A. Schmidt, K. Wingberg. *Cohomology of number fields*. Springer, Berlin, 2015, pg 419.

[2] J.-P. Serre. *Local fields*. Springer, Berlin, 1979, pg 67.

[3] I. Shafarevich. *On p-extensions*. Mat. Sb. **20** (1947), no. 62, 351–363.