Overconvergent Modular Symbols in Sage

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Outline

1. Modular Symbols
2. Overconvergent Modular Symbols
3. $p$-adic $L$-functions
Modular Symbols

For $k > 1$, computation of modular forms made possible by *modular symbols*.

$\Delta_0$ – $\text{Div}^0(\mathbb{P}^1(\mathbb{Q}))$: formal sums $\sum_{\alpha \in \mathbb{Q} \cup \{\infty\}} a_\alpha \alpha$ with $\sum_\alpha a_\alpha = 0$.

$S_0(p)$ – $\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid (a, p) = 1, p \mid c \text{ and } ad - bc \neq 0\}$.

$V$ – a $\mathbb{Z}$-module (e.g. $\mathbb{C}$ or $\text{Sym}^{k-2}(\mathbb{C})$) with right actions of $\Gamma$ and $S_0(p)$.

$\Gamma$ – acts on $\text{Hom}(\Delta_0, V)$ by $(\varphi|\gamma)(D) = \varphi(\gamma D)|\gamma$.

$\text{Smb}_\Gamma(V)$ – $\{\varphi \in \text{Hom}(\Delta_0, V) \mid \varphi = \varphi|\gamma\}$.

$T_\ell$ – acts by $\varphi|T_\ell = \varphi|\begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix} + \sum_{a=0}^{\ell-1} \varphi|\begin{pmatrix} 1 & a \\ 0 & \ell \end{pmatrix}$ for $\ell \nmid N$.

$U_q$ – acts by $\varphi|U_q = \sum_{a=0}^{q-1} \varphi|\begin{pmatrix} 1 & a \\ 0 & q \end{pmatrix}$ for $q \mid N$. 
Manin Relations

\[ G = \text{PSL}_2(\mathbb{Z}) \]

\[ [\gamma] - \frac{b}{d} - \frac{a}{c} \in \Delta_0 \text{ when } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G. \]

\[ \sigma - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} , \text{ a two-torsion element.} \]

\[ \tau - \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} , \text{ a three-torsion element.} \]

\[ I - \text{the left ideal } \mathbb{Z}[G](1 + \sigma) + \mathbb{Z}[G](1 + \tau + \tau^2). \]

\[ \{g_i\} - \text{right coset reps for } \Gamma\backslash G, \text{ generate } \mathbb{Z}[G] \text{ as a free } \mathbb{Z}[\Gamma]-\text{module.} \]

Using continued fractions, every element of \( \Delta_0 \) is the sum of elements \([\gamma]\), so get surjective map

\[ \mathbb{Z}[G] \to \Delta_0. \]

Manin showed that the kernel is \( I \). Therefore \( \Delta_0 \) is generated by the \( g_i \), with relations given by \( I \). For instance,

\[ g_i(1 + \sigma) = g_i + g_i\sigma = g_i + \gamma_{ij} g_j. \]
Modular Symbols to Modular Forms

Theorem (Eichler-Shimura)

\[ \text{Smb}_\Gamma(\text{Sym}^{k-2}(\mathbb{C})) \cong M_k(\Gamma) \oplus S_k(\Gamma) \text{ as Hecke-modules}. \]

So to compute \( M_k(\Gamma) \), we

1. Using Manin relations, write down a basis for \( \text{Smb}_\Gamma(\text{Sym}^{k-2}(\mathbb{C})) \).
2. Compute matrices for action of \( U_q \) and \( T_\ell \) for small \( \ell \).
3. Diagonalize to get \textit{systems of Hecke eigenvalues} \( \{a_\ell\} \).
4. These systems provide the Fourier coefficients for a basis of eigenforms in \( M_k(\Gamma) \).
$p$-adic Distributions

$A - \{ f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathbb{Q}_p[[z]] : |a_n| \to 0 \}; \|f\| = \sup_{z \in \mathbb{Z}_p} |f(z)|.$

$D - \text{Hom}(A, \mathbb{Q}_p); \|\mu\| = \sup_{0 \neq f \in A} \frac{\|\mu(f)\|}{\|f\|}.$

$A_k - A$ with $(\gamma \cdot_k f)(z) = (a + cz)^k \cdot f \left( \frac{b+dz}{a+cz} \right)$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}_p).$

$D_k - D$ with $(\mu|_k \gamma)(f) = \mu(\gamma \cdot_k f).$

$V_k - \text{Sym}^k(\mathbb{Q}_p^2) = \mathbb{Q}_p[X, Y]_k$ with $(P|\gamma)(X, Y) = P(dX - cY, -bX + aY).$
Moments

The map

\[ M : D \to \prod_{j=0}^{\infty} \mathbb{Q}_p \]

\[ \mu \mapsto (\mu(z^j))_{j=0}^{\infty} \]

is injective, with image the bounded sequences.

The map

\[ \rho_k : D_k \to V_k \]

\[ \mu \mapsto \int (Y - zX)^k d\mu(z) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \mu(z^j) X^j Y^{k-j} \]

is \( S_0(p) \)-equivariant.
Computing with Distributions

\[ D^0 = \{ \mu \in D : \mu(z^j) \in \mathbb{Z}_p \text{ for all } j \}. \]

\[ \text{Fil}^m = \{ \mu \in D^0 : v_p(\mu(z^j)) \geq m - j \}. \]

\[ \mathcal{F}^m = D^0 / \text{Fil}^m, \text{ a finite } \mathbb{Z}_p\text{-module}. \]

We will define Hecke operators via the action of \( S_0(p) \) and \( \text{Fil}^m \) is chosen to be stable under this action.
Overconvergent Modular Symbols

- Let $N$ be prime to $p$ and $\Gamma = \Gamma_0(Np) \subset S_0(p)$.
- An overconvergent modular symbol is an element of $\text{Smb}_\Gamma(D_k)$. Have Hecke operators.
- Approximate by elements of $\text{Smb}_\Gamma(\mathcal{F}_k^m)$, Hecke operators descend.
- The slope of an eigensymbol $\varphi$ is the valuation of the $U_p$-eigenvalue.
- Specialization map $\rho^* : \text{Smb}_\Gamma(D_k) \to \text{Smb}_\Gamma(V_k)$ is surjective, isomorphism on the slope $< (k + 1)$ piece.
Overconvergent Modular Symbols in Sage

Break for Sage demo: https://cloud.sagemath.com
Application: \( p \)-adic \( L \)-functions

Classically, \( \zeta(1 - k) \) \( p \)-adically interpolates for positive integers \( k \). Kummer congruences:

if \( h \equiv k \pmod{\phi(p^m)} \) then \( \frac{B_h}{h} \equiv \frac{B_k}{k} \pmod{p^m} \).

Can do the same for other \( L \)-functions. For example, if \( f \in S_{k+2}(\Gamma, \bar{\mathbb{Q}}) \) is a slope \( h < k + 1 \) eigenform, define the \( p \)-adic \( L \)-function of \( f \) to be the unique distribution \( \mu_f \) on \( \mathbb{Z}_p^\times \) so that if \( \chi \) is a character of \( \mathbb{Z}_p^\times \) with conductor \( p^n \) and \( 0 \leq j \leq k \), then

\[
\mu_f(z^j \cdot \chi) = \frac{1}{\alpha^n} \cdot \frac{p^{n(j+1)}}{(-2\pi i)^j} \cdot \frac{j!}{\tau(\chi^{-1})} \cdot \frac{L(f, \chi^{-1}, j + 1)}{\Omega_f^\pm}.
\]

Here \( \alpha \) is the \( U_p \)-eigenvalue of \( f \), \( \tau(\chi^{-1}) \) is a Gauss sum and \( \Omega_f^\pm \) are complex periods.
Computation of $p$-adic $L$-functions

The classical construction of $\mu_f$ involves an integral, the computation of which requires a Riemann sum. The resulting algorithm for computing $\mu_f$ is exponential in the desired precision.

Pollack and Stevens show that there is an overconvergent eigensymbol $\Phi_f$, lifting the symbol $\varphi_f$, so that

$$\mu_f = \Phi_f(\{\infty\} - \{0\})|_{\mathbb{Z}_p^\times}.$$  

The resulting algorithm for computing $\mu_f$ is polynomial in the desired precision.
$p$-adic $L$-functions in Sage

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