Overconvergent Modular Symbols in Sage

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Outline

Modular Symbols

For $k > 1$, computation of modular forms made possible by *modular symbols*.

Δ_0 – $\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q}))$: formal sums $\sum_{\alpha \in \mathbb{Q} \cup \{\infty\}} a_\alpha \alpha$ with $\sum_\alpha a_\alpha = 0$.

$S_0(p)$ – $\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid (a, p) = 1, p \mid c \text{ and } ad - bc \neq 0 \right\}$.

V – a $\mathbb{Z}$-module (e.g. $\mathbb{C}$ or $\mathrm{Sym}^{k-2}(\mathbb{C})$) with right actions of Γ and $S_0(p)$.

Γ – acts on $\mathrm{Hom}(\Delta_0, V)$ by $(\varphi|\gamma)(D) = \varphi(\gamma D)|\gamma$.

$\mathrm{Smb}_\Gamma(V)$ – $\{\varphi \in \mathrm{Hom}(\Delta_0, V) \mid \varphi = \varphi|\gamma\}$.

T_ℓ – acts by $\varphi|T_\ell = \varphi|\begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix} + \sum_{a=0}^{\ell-1} \varphi|\begin{pmatrix} 1 & a \\ 0 & \ell \end{pmatrix}$ for $\ell \nmid N$.

U_q – acts by $\varphi|U_q = \sum_{a=0}^{q-1} \varphi|\begin{pmatrix} 1 & a \\ 0 & q \end{pmatrix}$ for $q \mid N$.

Manin Relations

G – $\mathrm{PSL}_2(\mathbb{Z})$

$[\gamma]$ – $\frac{b}{d} - \frac{a}{c} \in \Delta_0$ when $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$.

σ – $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, a two-torsion element.

τ – $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$, a three-torsion element.

I – the left ideal $\mathbb{Z}[G](1+\sigma) + \mathbb{Z}[G](1+\tau+\tau^2)$.

$\{g_i\}$ – right coset reps for $\Gamma\backslash G$, generate $\mathbb{Z}[G]$ as a free $\mathbb{Z}[\Gamma]$-module.

Using continued fractions, every element of Δ_0 is the sum of elements $[\gamma]$, so get surjective map

$$\mathbb{Z}[G] \to \Delta_0.$$

Manin showed that the kernel is I. Therefore Δ_0 is generated by the g_i, with relations given by I. For instance,

$$g_i(1+\sigma) = g_i + g_i\sigma = g_i + \gamma_{ij} g_j.$$

Modular Symbols to Modular Forms

Theorem (Eichler-Shimura)

$\mathrm{Smb}_\Gamma(\mathrm{Sym}^{k-2}(\mathbb{C})) \cong M_k(\Gamma) \oplus S_k(\Gamma)$ *as Hecke-modules.*

So to compute $M_k(\Gamma)$, we

1. Using Manin relations, write down a basis for $\mathrm{Smb}_\Gamma(\mathrm{Sym}^{k-2}(\mathbb{C}))$.
2. Compute matrices for action of U_q and T_ℓ for small ℓ.
3. Diagonalize to get *systems of Hecke eigenvalues* $\{a_\ell\}$.
4. These systems provide the Fourier coefficients for a basis of eigenforms in $M_k(\Gamma)$.

p-adic Distributions

- A – $\left\{f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathbb{Q}_p[\![z]\!] \ : \ |a_n| \to 0\right\}$; $\|f\| = \sup_{z\in\mathbb{Z}_p} |f(z)|$.
- D – $\mathrm{Hom}(\mathrm{A}, \mathbb{Q}_p)$; $\|\mu\| = \sup_{0\neq f\in \mathrm{A}} \frac{|\mu(f)|}{\|f\|}$.
- A_k – A with $(\gamma \cdot_k f)(z) = (a+cz)^k \cdot f\left(\frac{b+dz}{a+cz}\right)$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in S_0(p)$.
- D_k – D with $(\mu|_k\gamma)(f) = \mu(\gamma \cdot_k f)$.
- V_k – $\mathrm{Sym}^k(\mathbb{Q}_p^2) = \mathbb{Q}_p[X,Y]_k$ with $(P|\gamma)(X,Y) = P(dX - cY, -bX + aY)$.

Moments

The map

$$M : \mathrm{D} \to \prod_{j=0}^{\infty} \mathbb{Q}_p$$
$$\mu \mapsto \left(\mu(z^j)\right)_{j=0}^{\infty}$$

is injective, with image the bounded sequences.

The map

$$\rho_k : \mathrm{D}_k \to V_k$$
$$\mu \mapsto \int (Y - zX)^k d\mu(z) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \mu(z^j) X^j Y^{k-j}$$

is $S_0(p)$-equivariant.

Computing with Distributions

D^0 – $\{\mu \in \mathrm{D} : \mu(z^j) \in \mathbb{Z}_p$ for all $j\}$.

Fil^m – $\{\mu \in \mathrm{D}^0 : v_p(\mu(z^j)) \geq m - j\}$.

$\mathcal{F}^m$ – $\mathrm{D}^0 / \mathrm{Fil}^m$, a finite $\mathbb{Z}_p$-module.

We will define Hecke operators via the action of $S_0(p)$ and Fil^m is chosen to be stable under this action.

Overconvergent Modular Symbols

- Let N be prime to p and $\Gamma = \Gamma_0(Np) \subset S_0(p)$.
- An overconvergent modular symbol is an element of $\mathrm{Smb}_\Gamma(\mathrm{D}_k)$. Have Hecke operators.
- Approximate by elements of $\mathrm{Smb}_\Gamma(\mathcal{F}_k^m)$, Hecke operators descend.
- The *slope* of an eigensymbol φ is the valuation of the U_p-eigenvalue.
- Specialization map $\rho^* : \mathrm{Smb}_\Gamma(\mathrm{D}_k) \to \mathrm{Smb}_\Gamma(V_k)$ is surjective, isomorphism on the slope $< (k+1)$ piece.

Overconvergent Modular Symbols in Sage

Sage

Break for Sage demo: `https://cloud.sagemath.com`

Application: p-adic L-functions

Classically, $\zeta(1-k)$ *p-adically interpolates* for positive integers k.
Kummer congruences:
if $h \equiv k \pmod{\phi(p^m)}$ then $\frac{B_h}{h} \equiv \frac{B_k}{k} \pmod{p^m}$.

Can do the same for other L-functions. For example, if $f \in S_{k+2}(\Gamma, \bar{\mathbb{Q}})$ is a slope $h < k+1$ eigenform, define the p-adic L-function of f to be the unique distribution μ_f on $\mathbb{Z}_p^\times$ so that if χ is a character of $\mathbb{Z}_p^\times$ with conductor p^n and $0 \le j \le k$, then

$$\mu_f(z^j \cdot \chi) = \frac{1}{\alpha^n} \cdot \frac{p^{n(j+1)}}{(-2\pi i)^j} \cdot \frac{j!}{\tau(\chi^{-1})} \cdot \frac{L(f, \chi^{-1}, j+1)}{\Omega_f^\pm}.$$

Here α is the U_p-eigenvalue of f, $\tau(\chi^{-1})$ is a Gauss sum and $\Omega_f^\pm$ are complex periods.

Computation of p-adic L-functions

The classical construction of μ_f involves an integral, the computation of which requires a Riemann sum. The resulting algorithm for computing μ_f is exponential in the desired precision.

Pollack and Stevens show that there is an overconvergent eigensymbol Φ_f, lifting the symbol φ_f, so that

$$\mu_f = \Phi_f(\{\infty\} - \{0\})|_{\mathbb{Z}_p^\times}.$$

The resulting algorithm for computing μ_f is polynomial in the desired precision.

p-adic *L*-functions in Sage

Sage

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