Algebraic tori and a computational inverse Galois problem

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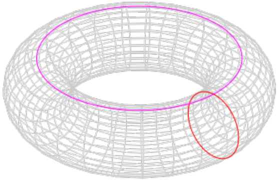
Outline

Tori over $\mathbb{R}$

When you hear torus, you probably think



Today: an algebraic version. Define three basic tori over $\mathbb{R}$:

- **U**, with $\mathbf{U}(\mathbb{R}) = \{z \in \mathbb{C}^\times \ : \ z\bar{z} = 1\}$,
- $\mathbb{G}_m$, with $\mathbb{G}_m(\mathbb{R}) = \mathbb{R}^\times$,
- **S**, with $\mathbf{S}(\mathbb{R}) = \mathbb{C}^\times$.

Theorem (c.f. [1, Thm 2])

Every algebraic torus over $\mathbb{R}$ is a product of these tori.

Algebraic tori

- $\mathbb{G}_m$ is the variety defined by $xy - 1$: for any ring R its points are the units $R^\times$.
- **U** is the variety defined by $x^2 + y^2 - 1$; after tensoring with $\mathbb{C}$ can factor as $(x + iy)(x - iy) - 1$.
- Both are in fact *group schemes*: the set of points has a group structure.

Definition

An *algebraic torus* over a field K is a group scheme, isomorphic to $(\mathbb{G}_m)^n$ after tensoring with a finite extension.

Can also give $T(\bar{K})$ plus a continuous action of $\mathrm{Gal}(\bar{K}/K)$ on it.

Character lattices

Definition

The *character lattice* of T is $X^*(T) = \mathrm{Hom}_{\bar{K}}(T, \mathbb{G}_m)$,

$X^*(T)$ is a free rank-n $\mathbb{Z}$-module with a $\mathrm{Gal}(\bar{K}/K)$ action.
Can take $\{\chi_i : (z_1, \ldots, z_n) \mapsto z_i\}$ as a basis for $X^*(\mathbb{G}_m^n)$.

- $X^*(\mathbb{G}_m) = \mathbb{Z}$ with trivial action,
- $X^*(\mathbf{U}) = \mathbb{Z}$ with conjugation acting as $x \mapsto -x$,
- $X^*(\mathbf{S}) = \mathbb{Z}v \oplus \mathbb{Z}w$ with conjugation exchanging v and w.

Theorem

*The functor $T \mapsto X^*(T)$ defines a contravariant equivalence of categories K-**Tori** $\to$ $\mathrm{Gal}(\bar{K}/K)$-**Lattices**.*

Finding tori

Goal

1. *Create a database of algebraic tori over p-adic fields (`www.lmfdb.org`)*
2. *Use to study structure of algebraic groups, p-adic representation theory and local Langlands, especially for exceptional groups.*

Some will apply to other fields and to Galois representations.

Strategy

We break up the task of finding tori into two pieces:

1. For each dimension n, list all finite groups G that act (faithfully) on $\mathbb{Z}^n$. For fixed n, the set of G is finite.
2. For each G and p, list all Galois extensions $L/\mathbb{Q}_p$ with $\mathrm{Gal}(L/\mathbb{Q}_p) \cong G$. For fixed G and p, the set of L is finite. Moreover, when p does not divide $|G|$, this question is easy.

Finite Subgroups of $\mathrm{GL}_n(\mathbb{Z})$

- With a choice of basis, a faithful action of G on $\mathbb{Z}^n$ is the same as an embedding $G \subset \mathrm{GL}_n(\mathbb{Z})$.
- Two G-lattices are isomorphic if and only if the corresponding subgroups are conjugate within $\mathrm{GL}_n(\mathbb{Z})$.
- Two G-lattices are *isogenous* if the corresponding subgroups are conjugate within $\mathrm{GL}_n(\mathbb{Q})$.

$\mathbb{G}_m \times$ **U** and **S** are isogenous but not isomorphic, since $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ are conjugate in $\mathrm{GL}_n(\mathbb{Q})$ but not in $\mathrm{GL}_n(\mathbb{Z})$.

Previous Computations

CARAT [2]

Up to dimension 6, the software package CARAT lists all of the finite subgroups of $\mathrm{GL}_n(\mathbb{Z})$, up to $\mathbb{Z}$- and $\mathbb{Q}$-conjugacy.

IMF GAP Library [4]

The group theory software package GAP has a library for maximal finite subgroups where the corresponding lattice is irreducible as a G-module. The $\mathbb{Q}$-classes are known for $n \leq 31$, the $\mathbb{Z}$-classes for $n \leq 11$ and $n \in \{13, 17, 19, 23\}$.

Indecomposible subgroups

- A G-lattice is *indecomposible* if it does not split as a direct sum of G-submodules.
- For example, $X^*(\mathbf{S})$ is not irreducible, since $\langle a + b \rangle$ is a stable submodule, as is $\langle a - b \rangle$.
- But it is indecomposible: the sum of these submodules has index 2.

For $n > 6$, work remains to recover a list of indecomposible subgroups. Note that the decomposition into indecomposible submodules is NOT unique.

Interlude: p-adic fields

- For each prime p, define $v_p : \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}$ by $v_p(p^k \alpha) = k$ when α is relatively prime to p.
- Set $|x|_p = p^{-v_p(x)}$, and $\mathbb{Q}_p$ as the completion.
- $\mathbb{Z}_p = \{x \in \mathbb{Q}_p \ : \ v(x) \geq 0\}$ and $\mathcal{P}_p = \{x \in \mathbb{Z}_p \ : \ v(x) > 0\}$ is the unique maximal ideal in $\mathbb{Z}_p$, with quotient $\mathbb{F}_p$ (residue field). A *uniformizer* is an element of valuation 1, ie $p \cdot u$.
- $\mathbb{Q}_p^\times \cong \mathbb{Z}_p^\times \times p^{\mathbb{Z}}$ and $\mathbb{Z}_p^\times \cong \mathbb{F}_p^\times \times (1 + \mathcal{P}_p)$.

For example, $\frac{2}{5} + 3 + 5^2 + 2 \cdot 5^3 + 4 \cdot 5^4 + \cdots$ is an element of $\mathbb{Q}_5$.

Interlude: p-adic extensions

Algebraic extensions of $\mathbb{Q}_p$ are much richer than those of $\mathbb{R}$. Let $K/\mathbb{Q}_p$ be a finite extension. There is a unique extension of v to a valuation $v_K : K \to \mathbb{Q} \cup \{\infty\}$.

- L/K is *unramified* if the image of v_K is the same as v_L. There is a unique unramified extension of each degree (comes from the residue field).
- L/K is *totally ramified* if the corresponding extension of residue fields is trivial.
- A totally ramified extension is *tame* if $[L : K]$ is prime to p. These are obtained by adjoining roots of uniformizers.
- A totally ramified extension is *wild* if $[L : K]$ is a power of p.

Any extension L/K can be split as $L/L_t/L_u/K$, with L_u/K unramified, L_t/L_u tame and L/L_t wild.

Number of Subgroups (up to $\mathrm{GL}_n(\mathbb{Z})$-conjugacy)

Dimension	1	2	3	4	5	6
Real	2	4	6	9	12	16
Unramified	2	7	16	45	96	240
Tame	2	13	51	298	1300	6661
7-adic	2	10	38	192	802	3767
5-adic	2	11	41	222	890	4286
3-adic	2	13	51	348	1572	9593
2-adic	2	11	60	536	4820	65823
Local	2	13	67	633	5260	69584
All	2	13	73	710	6079	85308

Note that each subgroup corresponds to multiple tori, since there are multiple field extensions with that Galois group.

Order of Largest Subgroup

Dimension	1	2	3	4	5	6
Real	2	2	2	2	2	2
Unramified	2	6	6	12	12	30
Tame	2	12	12	40	72	144
7-adic	2	8	12	40	40	120
5-adic	2	12	12	40	72	144
3-adic	2	12	12	72	72	432
2-adic	2	12	48	576	1152	2304
Irreducible	2	12	48	1152	3840	103680
Weyl	A_1	G_2	B_3	F_4	B_5	$2 \times E_6$

Dim	Largest Irreducible Subgroup
7	2903040 (E_7)
8	696729600 (E_8)
31	$17658411549989416133671730836395786240000000$ (B_{31})

Inverse Galois Problem

- Classic Problem: determine if a finite G is a Galois group.
- Depends on base field: every G is a Galois group over $\mathbb{C}(t)$.
- Most work focused on $L/\mathbb{Q}$: S_n and A_n, every solvable group, every sporadic group except possibly $M_{23}, \ldots$
- Generic polynomials $f_G(t_1, \ldots, t_r, X)$ are known for some (G, K): every L/K with group G is a specialization.

Computational Problem

Give an algorithm to find all of the field extensions of $K = \mathbb{Q}_p$ with a specified Galois group.

Database of p-adic Fields

Jones and Roberts [3] have created a database of p-adic fields.

- Lists all $L/\mathbb{Q}_p$ with a given degree, including non-Galois;
- Includes up to degree 10;
- Gives Galois group and other data about the extension;
- Biggest table is $[L : \mathbb{Q}_2] = 8$, of which there are 1823.
- We need G in degree up to 96 (tame) or 14, 60, 144, 144 (wild, $p = 7, 5, 3, 2$ resp.)

Their database solves the problem for small G, but most of the target G fall outside it.

Structure of p-adic Galois groups

The splitting of L/K into unramified, tame and wild pieces induces a filtration on Gal(L/K). We can refine this filtration to

$$G \trianglerighteq G_0 \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq \cdots \trianglerighteq G_r = 1.$$

- For every i, $G_i \trianglelefteq G$;
- $G/G_0 = \langle F \rangle$ is cyclic, and L^{G_0}/K is maximal unramified;
- $G_0/G_1 = \langle \tau \rangle$ is cyclic, order prime to p and $F\tau F^{-1} = \tau^p$;
- For $0 < i < r$, $G_i/G_{i+1} \cong \mathbb{F}_p^{k_i}$.

Finding such filtrations on an abstract group is not difficult.

Inductive Approach

For tame extensions: lift irreducible polynomials from residue field for unramified, then adjoin n^{th} roots of $p \cdot u$.

Thus, it suffices to solve:

Problem

Fix a Galois extension L/K, set $H = \mathrm{Gal}(L/K)$ and suppose G is an extension of H:

$$1 \to A \to G \to H \to 1,$$

with $A \cong \mathbb{F}_p^k$. Find all M/L s.t. M/K Galois and $\mathrm{Gal}(M/K) \cong G$.

Interlude: Local Class Field Theory

Let $M/L/\mathbb{Q}_p$ with $[M : L] = m$ and $\Gamma = \mathrm{Gal}(M/L)$.

Theorem (Local Class Field Theory [6, Part IV])

- $\mathrm{H}^2(\Gamma, M^\times) = \langle u_{M/L} \rangle \cong \frac{1}{m}\mathbb{Z}/\mathbb{Z}$
- $- \cup u_{M/L} : \Gamma^{\mathrm{ab}} = \hat{\mathrm{H}}^{-2}(\Gamma, \mathbb{Z}) \xrightarrow{\sim} \hat{\mathrm{H}}^0(\Gamma, M^\times) = L^\times / \mathrm{Nm}_{M/L} M^\times$.
- *The map $M \mapsto \mathrm{Nm}_{M/L} M^\times$ gives a bijection between abelian extensions M/L and finite index subgroups of $L^\times$.*

Pauli [5] gives algorithms for finding a defining polynomial of the extension associated to a given norm subgroup.

Upshot

Since $A = \mathbb{F}_p^k$ abelian, can use LCFT to find possible M/L in terms of subgroups of $L^\times$.

A Mod-p Representation

Given

$$1 \to A \to G \to H \to 1$$

and L/K, let $V = (1+\mathcal{P}_L)/(1+\mathcal{P}_L)^p$, an $\mathbb{F}_p[H]$-module.

- Since $A = \mathrm{Gal}(M/L)$ has exponent p, it corresponds to a subgp $N \supseteq (1+\mathcal{P}_L)^p$ and $L^\times/N \cong (1+\mathcal{P}_L)/(N \cap (1+\mathcal{P}_L))$.
- Let $W = (N \cap (1+\mathcal{P}_L))/(1+\mathcal{P}_L)^p$, a subspace of V.
- M/K is Galois iff W is stable under $H = \mathrm{Gal}(L/K)$.
- The MeatAxe algorithm finds such subrepresentations.
- For each W, check $V/W \cong A$ as $\mathbb{F}_p[H]$-modules.
- The corresponding M/K are candidates for $\mathrm{Gal}(M/K) \cong G$.

Extension Classes

There may be multiple extensions

$$1 \to A \to G' \to H \to 1$$

yielding the same action of H on A. Use group cohomology to distinguish them.

- Choosing a section $s : H \to G'$, define a 2-cocycle by $(g,h) \mapsto s(g)s(h)s(gh)^{-1} \in A$.
- Get bijection $\mathrm{H}^2(H,A) \leftrightarrow \{1 \to A \to G' \to H \to 1\}/\sim$.

Two approaches to picking out G:

1. Just compute $\mathrm{Gal}(M/K)$,
2. Try to find the extension class, given W.

A Conjecture on the Fundamental Class

Conjecture

Let $N \subset L^\times$ correspond to M/L under LCFT and set $G = \mathrm{Gal}(M/K)$, $H = \mathrm{Gal}(L/K)$ and $A = \mathrm{Gal}(M/L)$. Then the image of $u_{L/K}$ under the natural map

$$\mathrm{H}^2(H, L^\times) \to \mathrm{H}^2(H, L^\times/N) \cong \mathrm{H}^2(H, A)$$

is the extension class for

$$1 \to \mathrm{Gal}(M/L) \to \mathrm{Gal}(M/K) \to \mathrm{Gal}(L/K) \to 1.$$

If this conjecture holds, can compute a 2-cocycle representing $u_{L/K}$ and use it for each W.

Summary of Algorithm

Data: $G \trianglerighteq G_0 \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq \cdots \trianglerighteq G_r = 1$
Result: List of all Galois $F/\mathbb{Q}_p$ with $\mathrm{Gal}(F/\mathbb{Q}_p) \cong G$
Find tame extensions $L_1/\mathbb{Q}_p$ with $\mathrm{Gal}(L_1/\mathbb{Q}_p) \cong G/G_1$;
for $0 < i < r$ **do**
    Find class σ_i of $1 \to G_i/G_{i+1} \to G/G_{i+1} \to G/G_i \to 1$;
    for *each* $L = L_i$ **do**
        Compute a 2-cocycle representing $u_{L/\mathbb{Q}_p}$;
        Find all stable submodules W with $L^\times/W \cong G_i/G_{i+1}$;
        for *each* W **do**
            if $u_{L/\mathbb{Q}_p} \mapsto \sigma_i \in \mathrm{H}^2(L/\mathbb{Q}_p, L^\times/W)$ **then**
                Add the M/L matching W to the list of L_{i+1};
            end
        end
    end
end

Future Work

1. Flesh out details of algorithm and implement it,
2. Extend group theoretic analysis to dimension 7 and 8,
3. Compute additional data for each torus: cohomology groups, embeddings into induced tori, Moy-Prasad filtrations, conductors, component groups of Néron models...
4. Put data online at `www.lmfdb.org`.

Thank you for your attention!

References

[1] B. Casselman. *Computations in real tori*, Representation theory of real groups, Contemporary Mathematics **472**, A.M.S. (2007).

[2] C. Cid, J. Opgenorth, W. Plesken, T. Schulz. *CARAT*. `wwwb.math.rwth-aachen.de/carat/`.

[3] J. Jones, D. Roberts. *A database of local fields*, J. Symbolic Comput **41** (2006), 80-97.

[4] G. Nebe, W. Pleskin, M. Pohst, B. Souvignier. *Irreducible maximal finite integral matrix groups*. GAP Library.

[5] S. Pauli. *Constructing class fields over local fields*, J. Théor. Nombres Bordeaux **18** (2006), 627-652.

[6] J.-P. Serre. *Local fields*. Springer, New York, 1979.