Overconvergent Modular Symbols in Sage

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Towards new development of mathematics via computational algebra systems
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Outline

1. Modular Forms

2. Modular Symbols

3. Overconvergent Modular Symbols

4. Positive Slope Families
Classical Modular Forms

\( \mathcal{H} \) – upper half plane: complex numbers \( z = x + iy \) with \( y > 0 \).

\( \Gamma_0(N) \subseteq \text{SL}_2(\mathbb{Z}) \) consists of \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) with \( N \mid c \).

Acts on \( \mathcal{H} \) by \( \gamma \cdot z = \frac{az+b}{cz+d} \). Example of a level \( \Gamma \).

\( k \) – an integer, the weight.

\( M_k(\Gamma) \) – holomorphic functions \( f : \mathcal{H} \to \mathbb{C} \) with
\[
f(\gamma \cdot z) = (cz+d)^k f(z) \text{ for } \gamma \in \Gamma.
\]
These are modular forms.

Since \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma_0(N) \), \( f(z + 1) = f(z) \). Get a Fourier expansion around \( i\infty \): if \( q = e^{2\pi i z} \),
\[
f(z) = \sum_{n=0}^{\infty} a_n q^n.
\]

Note: \( a_n = 0 \) for \( n < 0 \) is an additional condition on \( f \).
Examples: Eisenstein Series

For $k > 2$ even, $G_k(z) = \sum_{(m,n) \neq (0,0)} \frac{1}{(mz+n)^k} \in M_k(\text{SL}_2(\mathbb{Z}))$.

$$G_k(z) = 2\zeta(k) \left(1 + \frac{2}{\zeta(1-k)} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n\right),$$

where $\sigma_{k-1}(n) = \sum_{0<d|n} d^{k-1}$.

- For other $\Gamma$, a cusp is a $\Gamma$-orbit on $\mathbb{Q} \cup \{\infty\}$.
- Basis for Eisenstein series of forms that take value 1 on one cusp and zero on others.
- Cusp forms $S_k(\Gamma) \subset M_k(\Gamma$) are those vanishing on all cusps.
Examples: Modular Forms from Elliptic Curves

If $E$ is an elliptic curve $y^2 = x^3 + ax + b$ over $\mathbb{Q}$ with discriminant $-16(4a^3 + 27b^2)$ and conductor $N$ (same prime factors),

$$a_p = (p + 1) - \#E(\mathbb{F}_p) \text{ if } p \nmid N$$
$$a_p = 0 \text{ if } E \text{ has additive reduction}$$
$$a_p = 1 \text{ if } E \text{ has split multiplicative reduction}$$
$$a_p = -1 \text{ if } E \text{ has non-split multiplicative reduction}$$
$$a_{pr} = a_{pr-1} \cdot a_p - p \cdot a_{pr-2} \text{ if } p \nmid N$$
$$a_{pr} = a_p^r \text{ if } p \mid N$$
$$a_{mn} = a_m \cdot a_n \text{ if } (m, n) = 1.$$
Hecke Operators

- For fixed $k$ and $\Gamma$, the space $M_k(\Gamma)$ is finite dimensional (with explicit dimensions via Riemann-Roch).
- For each $n \geq 1$ there is a linear operator $T_n$ on $M_k(\Gamma)$, and the $T_n$ commute with each other.
- An eigenform is a simultaneous eigenvector for these operators (e.g. $f_E$).
L-functions and Modular Forms Database

Modular Symbols

For $k > 1$, computation made possible by *modular symbols*.

$\Delta_0 = \text{Div}^0(\mathbb{P}^1(\mathbb{Q}))$: formal sums $\sum_{\alpha \in \mathbb{Q} \cup \{\infty\}} a_\alpha \alpha$ with $\sum_\alpha a_\alpha = 0$.

$S_0(p) = \left\{(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \mid (a, p) = 1, p \mid c \text{ and } ad - bc \neq 0\right\}$.

$V$ – a $\mathbb{Z}$-module (e.g. $\mathbb{C}$ or $\text{Sym}^{k-2}(\mathbb{C})$) with right actions of $\Gamma$ and $S_0(p)$.

$\Gamma$ – acts on $\text{Hom}(\Delta_0, V)$ by $(\varphi|\gamma)(D) = \varphi(\gamma D)|\gamma$.

$\text{Smb}_\Gamma(V) = \{\varphi \in \text{Hom}(\Delta_0, V) \mid \varphi = \varphi|\gamma\}$.

$T_\ell$ – acts by $\varphi|T_\ell = \varphi|\left(\begin{smallmatrix} \ell & 0 \\ 0 & 1 \end{smallmatrix}\right) + \sum_{a=0}^{\ell-1} \varphi|\left(\begin{smallmatrix} 1 & a \\ 0 & \ell \end{smallmatrix}\right)$ for $\ell \nmid N$.

$U_q$ – acts by $\varphi|U_q = \sum_{a=0}^{q-1} \varphi|\left(\begin{smallmatrix} 1 & a \\ 0 & q \end{smallmatrix}\right)$ for $q \mid N$. 


Manin Relations

\[ G = \text{PSL}_2(\mathbb{Z}) \]
\[ [\gamma] = \frac{b}{d} - \frac{a}{c} \in \Delta_0 \text{ when } \gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in G. \]
\[ \sigma = \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right), \text{ a two-torsion element.} \]
\[ \tau = \left( \begin{array}{cc} 0 & -1 \\ 1 & -1 \end{array} \right), \text{ a three-torsion element.} \]
\[ I = \text{the left ideal } \mathbb{Z}[G](1 + \sigma) + \mathbb{Z}[G](1 + \tau + \tau^2). \]
\[ \{g_i\} = \text{right coset reps for } \Gamma \backslash G, \text{ generate } \mathbb{Z}[G] \text{ as a free } \mathbb{Z}[\Gamma]-\text{module.} \]

Using continued fractions, every element of \( \Delta_0 \) is the sum of elements \( [\gamma] \), so get surjective map

\[ \mathbb{Z}[G] \rightarrow \Delta_0. \]

Manin showed that the kernel is \( I \). Therefore \( \Delta_0 \) is generated by the \( g_i \), with relations given by \( I \). For instance,

\[ g_i(1 + \sigma) = g_i + g_i\sigma = g_i + \gamma ij g_j. \]
Modular Symbols to Modular Forms

**Theorem (Eichler-Shimura)**

\[ \text{Smb}_\Gamma(\text{Sym}^{k-2}(\mathbb{C})) \cong M_k(\Gamma) \oplus S_k(\Gamma) \text{ as Hecke-modules.} \]

So to compute \( M_k(\Gamma) \), we

1. Using Manin relations, write down a basis for \( \text{Smb}_\Gamma(\text{Sym}^{k-2}(\mathbb{C})) \).
2. Compute matrices for action of \( U_q \) and \( T_\ell \) for small \( \ell \).
3. Diagonalize to get *systems of Hecke eigenvalues* \( \{a_\ell\} \).
4. These systems provide the Fourier coefficients for a basis of eigenforms in \( M_k(\Gamma) \).
$p$-adic Numbers

Fix $p$ prime. Recall:

- $v_p$ – For $a, b$ prime to $p$, set $v_p \left( p^v \cdot \frac{a}{b} \right) = v$ and $|p^v \cdot \frac{a}{b}|_p = p^{-v}$.
- $\mathbb{Q}_p$ – Completion of $\mathbb{Q}$ with norm $|\cdot|_p$. Then $\mathbb{Z}_p = \{ z \in \mathbb{Q}_p : |z| \leq 1 \}$.
- $\mathbb{Z}_p$ – Alternately, $\mathbb{Z}_p = \lim_{\leftarrow m} \mathbb{Z}/p^m\mathbb{Z}$ and $\mathbb{Q}_p = \mathbb{Z}_p \left[ \frac{1}{p} \right]$.
  - Concretely, of the form $\sum_{m=v}^{\infty} a_m p^m$ with $a_m \in \{0, \ldots, p-1\}$.
  - Computationally, represent as $p^v \cdot u$, where $u \in (\mathbb{Z}/p^m\mathbb{Z})^\times$. 
$p$-adic Distributions

\[ A = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathbb{Q}_p[z] : |a_n| \to 0 \right\}; \|f\| = \sup_{z \in \mathbb{Z}_p} |f(z)|. \]

\[ D = \text{Hom}(A, \mathbb{Q}_p); \|\mu\| = \sup_{0 \neq f \in A} \frac{|\mu(f)|}{\|f\|}. \]

\[ A_k = A \text{ with } (\gamma \cdot_k f)(z) = (a + cz)^k \cdot f\left(\frac{b + dz}{a + cz}\right) \text{ for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in S_0(p). \]

\[ D_k = D \text{ with } (\mu|_k \gamma)(f) = \mu(\gamma \cdot_k f). \]

\[ V_k = \text{Sym}^k(\mathbb{Q}_p^2) = \mathbb{Q}_p[X, Y]_k \text{ with } (P|\gamma)(X, Y) = P(dX - cY, -bX + aY). \]
Moments

The map

\[ M : D \to \prod_{j=0}^{\infty} \mathbb{Q}_p \]

\[ \mu \mapsto (\mu(z^j))_{j=0}^{\infty} \]

is injective, with image the bounded sequences.

The map

\[ \rho_k : D_k \to V_k \]

\[ \mu \mapsto \int (Y - zX)^k d\mu(z) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \mu(z^j) X^j Y^{k-j} \]

is \( S_0(p) \)-equivariant.
Computing with Distributions

\[ D^0 = \{ \mu \in D : \mu(z^j) \in \mathbb{Z}_p \text{ for all } j \}. \]

\[ \text{Fil}^m = \{ \mu \in D^0 : v_p(\mu(z^j)) \geq m - j \}. \]

\[ \mathcal{F}^m = D^0 / \text{Fil}^m, \text{ a finite } \mathbb{Z}_p\text{-module.} \]

We will define Hecke operators via the action of \( S_0(p) \) and \( \text{Fil}^m \) is chosen to be stable under this action.
Overconvergent Modular Symbols

- Let $N$ be prime to $p$ and $\Gamma = \Gamma_0(Np) \subset S_0(p)$.
- An overconvergent modular symbol is an element of $\text{Smb}_\Gamma(D_k)$. Have Hecke operators.
- Approximate by elements of $\text{Smb}_\Gamma(F^m_k)$, Hecke operators descend.
- The *slope* of an eigensymbol $\varphi$ is the valuation of the $U_p$-eigenvalue.
- Specialization map $\rho^*: \text{Smb}_\Gamma(D_k) \rightarrow \text{Smb}_\Gamma(V_k)$ is surjective, isomorphism on the slope $< (k + 1)$ piece.
Overconvergent Modular Symbols in Sage

Break for Sage demo: https://cloud.sagemath.com
Application: $p$-adic $L$-functions

Classically, $\zeta(1 - k)$ $p$-adically interpolates for positive integers $k$. Kummer congruences:

if $h \equiv k \pmod{\phi(p^m)}$ then $\frac{B_h}{h} \equiv \frac{B_k}{k} \pmod{p^m}$.

Can do the same for other $L$-functions. For example, if $f \in S_{k+2}(\Gamma, \overline{\mathbb{Q}})$ is a slope $h < k + 1$ eigenform, define the $p$-adic $L$-function of $f$ to be the unique distribution $\mu_f$ on $\mathbb{Z}_p^\times$ so that if $\chi$ is a character of $\mathbb{Z}_p^\times$ with conductor $p^n$ and $0 \leq j \leq k$, then

$$\mu_f(z^j \cdot \chi) = \frac{1}{\alpha^n} \cdot \frac{p^{n(j+1)}}{(-2\pi i)^j} \cdot \frac{j!}{\tau(\chi^{-1})} \cdot \frac{L(f, \chi^{-1}, j + 1)}{\Omega_f^\pm}.$$

Here $\alpha$ is the $U_p$-eigenvalue of $f$, $\tau(\chi^{-1})$ is a Gauss sum and $\Omega_f^\pm$ are complex periods.
Computation of $p$-adic $L$-functions

The classical construction of $\mu_f$ involves an integral, the computation of which requires a Riemann sum. The resulting algorithm for computing $\mu_f$ is exponential in the desired precision.

Pollack and Stevens show that there is an overconvergent eigensymbol $\Phi_f$, lifting the symbol $\varphi_f$, so that

$$\mu_f = \Phi_f(\{\infty\} - \{0\})|_{\mathbb{Z}_p^\times}.$$

The resulting algorithm for computing $\mu_f$ is polynomial in the desired precision.
$p$-adic $L$-functions in Sage

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Hida Families

Let $\mathcal{W} = \lim_{\leftarrow m} (\mathbb{Z}/\phi(p^m)\mathbb{Z}) \cong \mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}_p$.

Hida constructed families of overconvergent modular forms with varying \textit{weight}. These families

- consisted of \textit{ordinary forms}: slope 0,
- extended over all of weight space $\mathcal{W}$,
- have constant rank over weight space.

These families form a part of the \textit{eigencurve}, a rigid analytic object parameterizing overconvergent modular forms.
Ongoing work: positive slope families

The remainder of the eigencurve corresponds to families of overconvergent forms with *positive slope*. Want to compute power series that give the Hecke eigenvalues as a function of varying *weight*. These power series will be valid only in subsets of weight space (discs and annuli).

**Idea**

Use overconvergent modular symbols to find eigenvalues at specific weights and interpolate.

Overconvergent modular symbols are crucial since the weights will be *large*.
More details: positive slope families

Solved Problem

Need to match corresponding eigenvalues between different weights. Solution: since eigenvalues vary $p$-adically, their reduction modulo $p$ is constant over small discs. Can use reductions for varying $T_\ell$ as a signature to match between different weights.

Unsolved Problem

In higher slope, finding eigenvalues at a fixed weight involves iterating $\frac{U_p}{p^h}$. For positive $h$, we have been unable to avoid devastating precision loss.