A function–sheaf dictionary for tori over local fields

David Roe
joint with Clifton Cunningham

Department of Mathematics
University of Calgary/PIMS

Canadian Number Theory Association Meetings
June 17, 2014
Objective

Goal
Bring a geometric, categorical perspective to the study of the local Langlands correspondence for \( p \)-adic groups.

We approach this task from two directions:
1. Find geometric avatars for objects in local Langlands,
2. Find \( p \)-adic analogues of objects in geometric Langlands.

Key Idea
The representation theory of \( G(\mathbb{Q}_p) \) depends on its structure as a topological group, not as a scheme. A scheme \( \mathcal{G} \) over \( \mathbb{F}_p \) with \( \mathcal{G}(\mathbb{F}_p) = G(\mathbb{Q}_p) \) offers a new perspective.
Quasicharacters to sheaves

\(K\) – a non-archimedean local field with residue field \(k\),

\(R\) – the ring of integers of \(K\) with uniformizer \(\pi\),

\(T\) – an algebraic torus over \(K\),

\(X^*\) – for a group \(X\), notation for \(\text{Hom}(X, \overline{\mathbb{Q}}_\ell)\).

1. We construct a commutative group scheme \(\mathfrak{T}\) over \(k\) with \(\mathfrak{T}(k) \cong T(K)\).

2. For any smooth commutative group scheme \(G\) over \(k\) we define a category \(\mathcal{QC}(G)\) of quasicharacter sheaves on \(G\) and show

**Theorem (Cunningham-R.)**

*Trace of Frobenius defines an isomorphism of groups*

\[\mathcal{QC}(G)/_{iso} \cong G(k)^*\]
The Néron model of a torus

The Néron model $T_R$ of $T$ is a separated, smooth commutative group scheme over $R$ so that

**Néron mapping property**

For any smooth $R$-scheme $Z$ and morphism $f : Z_K \to T$, $f$ extends uniquely to $Z \to T_R$.

As a consequence,

$$T_R(R) = T(K).$$

Note that $T_R$ is not necessarily finite type.
Examples of Néron models

Example \((\mathbb{G}_m)\)

If \(T = \mathbb{G}_m\), then the Néron model for \(T\) is

\[
T_R = \bigcup_{n \in \mathbb{Z}} \mathbb{G}_m, R,
\]

with gluing along generic fibers:

\[
\begin{align*}
\mathbb{G}_m, R & \xrightarrow{\cong} \mathbb{G}_m, R \\
\mathbb{G}_m & \xrightarrow{\cong} \mathbb{G}_m \\
R[x_0, x_0^{-1}] & \xrightarrow{\text{iso}} R[x_n, x_n^{-1}] \\
K[x_0, x_0^{-1}] & \xleftarrow{\text{iso}} K[x_n, x_n^{-1}]
\end{align*}
\]

given by:

\[
\pi^n x_0 \leftarrow x_n
\]
Example (SO₂)

Let \( T = \text{SO}_2 \) over \( K \), split over \( E = K(\sqrt{\pi}) \). Then

\[
K[T] = K[x, y]/(x^2 - \pi y^2 - 1).
\]

The Néron model for \( T \) is given by

\[
R[T_R] = R[x, y]/(x^2 - \pi y^2 - 1).
\]

Here \( T_R \) is finite type, but not connected: the special fiber \( T_k \) of \( T_R \) is given by

\[
k[T_k] = k[x, y]/(x^2 - 1),
\]

two disjoint lines.
The Greenberg functor

Greenberg defines a functor

\[(\text{Sch} / R) \rightarrow (\text{Sch} / k). \]

\[X \rightarrow \text{Gr}(X)\]

**Proposition (Greenberg)**

- **If** \(X\) **is separated and locally of finite type then**
  \[\text{Gr}(X)(k) = X(R).\]

- **This functor respects open and closed immersions, étale and smooth morphisms and geometric components.**

- **There are finite level Greenberg functors** \(\text{Gr}_n\) **with**
  \[\text{Gr}(X) = \lim_{\leftarrow} \text{Gr}_n(X).\]
Greenberg of Néron

**Definition**

\[ \mathcal{U} := \text{Gr}(T_R). \]

**Proposition**

1. \( \mathcal{U}(k) = T(K) \)
2. \( \mathcal{U} \) is a smooth commutative group scheme over \( k \)
3. \( \pi_0(\mathcal{U}) = X_*(T)_I \)
Greenberg of Néron for $\mathbb{G}_m$

Set $\mathbb{W}_k^\times$ as the group of units in the Witt ring scheme $\mathbb{W}_k$.

**Example**

If $T = \mathbb{G}_m$, then

$$\mathcal{T} = \bigsqcup_{n \in \mathbb{Z}} \mathbb{W}_k^\times.$$

The component group for $\mathcal{T}$ is

$$X_* (T)_I = \mathbb{Z},$$

with the trivial $\text{Gal}(\bar{k}/k)$ action.
From now on, $G$ will denote a smooth, commutative group scheme over $k$. We will write $m : G \times G \to G$ for multiplication.

**Definition (Local System)**

An \(\ell\)-adic local system on $G$ is a constructible sheaf of \(\overline{\mathbb{Q}}_\ell\)-vector spaces on the étale site of $G$, locally constant on each connected component.
Rigid Quasicharacter Sheaves

Definition (Rigid quasicharacter sheaf)

A *rigid quasicharacter sheaf* on $G$ is a triple $\mathcal{L} := (\mathcal{L}, \mu, \phi)$.

1. $\mathcal{L}$ is a rank-one local system on $\mathcal{G}$,
2. $\mu : \mathcal{m}^* \mathcal{L} \to \mathcal{L} \boxtimes \mathcal{L}$ is an isomorphism of sheaves on $\mathcal{G} \times \mathcal{G}$, satisfying an associativity diagram.
3. $\phi : \mathcal{F}_G^* \mathcal{L} \to \mathcal{L}$ is an isomorphism of sheaves on $\mathcal{G}$ compatible with $\mu$.

A morphism of quasicharacter sheaves is a morphism of constructible $\ell$-adic sheaves on $\mathcal{G}$ commuting with $\mu$ and $\phi$.

Tensor product makes $\mathcal{QC}_{rig}(G)$ into a rigid monoidal category and $\mathcal{QC}_{rig}(G)/_{iso}$ into a group.
Bounded and Finite Rigid Quasicharacter Sheaves

Definition

- A **bounded rigid quasicharacter sheaf** on $G$ is a pair $(\mathcal{L}_0, \mu_0)$, where $\mathcal{L}_0$ is a rank-one local system on $G$ and $\mu_0$ is as before.

- A **finite rigid quasicharacter sheaf** on $G$ is a pair $(f, \psi)$, where $f : H \to G$ is a finite, surjective, étale morphism of group schemes and $\psi : \ker f \to \overline{\mathbb{Q}}_\ell^\times$.

- Have full and faithful functors
  
  $QC_f(G) \to QC_0(G) \to QC_{rig}(G)$,

- these are equivalences when $G$ is connected,

- Bounded rigid quasicharacter sheaves will correspond to bounded characters, and finite to ones with finite image.
Étale Group Schemes

\( \mathcal{L} \mapsto (\text{stalks } \overline{\mathcal{L}}_x \text{ and indexed isomorphisms } \mu_{x,y} \text{ and } \phi_x). \)

Choice of basis for \( \overline{\mathcal{L}}_x \Rightarrow a \in C^2(\overline{G}, \mathbb{Q}_\ell^\times) \) and \( b \in C^1(\overline{G}, \mathbb{Q}_\ell^\times) \).

\[
\overline{\mathcal{L}}_{x+y+z} \xrightarrow{\mu_{x+y,z}} \overline{\mathcal{L}}_{x+y} \otimes \overline{\mathcal{L}}_z \\
\mu_{x,y+z} \downarrow \quad \downarrow \mu_{x,y} \otimes \text{id} \\
\overline{\mathcal{L}}_x \otimes \overline{\mathcal{L}}_{y+z} \xrightarrow{\text{id} \otimes \mu_{y,z}} \overline{\mathcal{L}}_x \otimes \overline{\mathcal{L}}_y \otimes \overline{\mathcal{L}}_z
\]

\[
\overline{\mathcal{L}}_{F(x)+F(y)} \xrightarrow{\mu_{F(x),F(y)}} \overline{\mathcal{L}}_{F(x)} \otimes \overline{\mathcal{L}}_{F(y)} \\
\phi_{x+y} \downarrow \quad \downarrow \phi_x \otimes \phi_y \\
\overline{\mathcal{L}}_{x+y} \xrightarrow{\mu_{x,y}} \overline{\mathcal{L}}_x \otimes \overline{\mathcal{L}}_y
\]

\[
\Rightarrow a(F(x),F(y)) = \frac{a(x,y)}{b(x)} \cdot \frac{b(x+y)}{b(x) b(y)}
\]
Hochschild-Serre Spectral Sequence

\( \mathcal{W} \) – the Weil group of \( k \),
\( a \leadsto \alpha \in C^0(\mathcal{W}, Z^2(\tilde{G}, \mathbb{Q}_\ell^\times)) \),
\( b \leadsto \beta \in Z^1(\mathcal{W}, C^1(\tilde{G}, \mathbb{Q}_\ell^\times)) \) with \( \beta(F) = b \),
\( E_{0}^{i,j} = C^i(\mathcal{W}, C^j(\tilde{G}, \mathbb{Q}_\ell^\times)) \).

**Proposition**

- The map \( QC_{\text{rig}}(G)/_{\text{iso}} \to H^2(E_0^\bullet) \) to the cohomology of the total complex given by \( \mathcal{L} \mapsto (\alpha, \beta, 0) \) is an isomorphism.
- The spectral sequence yields an exact sequence

\[
1 \to H^0(\mathcal{W}, H^2(\tilde{G}, \mathbb{Q}_\ell^\times)) \to H^2(E_0^\bullet) \to H^1(\mathcal{W}, H^1(\tilde{G}, \mathbb{Q}_\ell^\times)) \to 1.
\]

- \( H^1(\mathcal{W}, H^1(\tilde{G}, \mathbb{Q}_\ell^\times)) \to (G(k)^*)_{\mathcal{W}} \to G(k)^* \)

*is an isomorphism compatible with trace of Frobenius.*
Quasicharacter Sheaves

So $\mathcal{QC}_{\text{rig}}(G)_{/\text{iso}} \to G(k)^*$ has kernel $H^2(G(\bar{k}), \mathbb{Q}_\ell^\times)^F$ for étale $G$.

**Definition (Quasicharacter sheaf)**

For any smooth, commutative, group scheme $G$, a *quasicharacter sheaf* on $G$ is a Weil sheaf $\mathcal{L} := (\bar{\mathcal{L}}, \phi)$ so that $(\bar{\mathcal{L}}, \mu, \phi)$ is a rigid quasicharacter sheaf for some $\mu$.

**Proposition**

*For étale $G$, trace of Frobenius induces an isomorphism*

$$\mathcal{QC}(G)_{/\text{iso}} \to G(k)^*. $$
Snake Lemma

For any $G$, trace of Frobenius defines a map

$$t_G : QC(G)/_{iso} \rightarrow G(k)^*.$$ 

Pullback then gives the rows of

$$\begin{array}{cccccc}
QC(\pi_0(G))/_{iso} & \longrightarrow & QC(G)/_{iso} & \longrightarrow & QC(G^\circ)/_{iso} & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & (\pi_0(G))(k)^* & \longrightarrow & G(k)^* & \longrightarrow & G^\circ(k)^* & \longrightarrow & 1
\end{array}$$

- $t_{G^\circ}$ is an isomorphism by the classic function–sheaf dictionary (Deligne),
- $t_{\pi_0(G)}$ is an isomorphism as above,
- the snake lemma finishes the job.
Transfer of quasicharacter sheaves

Suppose $T$ and $T'$ are tori over local fields $K$ and $K'$. We say that $T$ and $T'$ are $N$-congruent if there are isomorphisms

$$
\alpha : \mathcal{O}_L / \pi_K^N \mathcal{O}_L \to \mathcal{O}_{L'} / \pi_K^N \mathcal{O}_{L'},$

$$
\beta : \text{Gal}(L/K) \to \text{Gal}(L'/K'),$

$$
\phi : X^*(T) \to X^*(T'),$

satisfying natural conditions. If $T$ and $T'$ are $N$-congruent then $\text{Hom}_{<N}(T(K), \mathbb{Q}_\ell^\times) \cong \text{Hom}_{<N}(T'(K'), \mathbb{Q}_\ell^\times)$.

- Chai and Yu give an isomorphism of group schemes $T_n \cong T'_n$, for $n$ depending on $N$.
- This isomorphism induces an equivalence of categories $\mathcal{QC}(\mathcal{T}_n) \to \mathcal{QC}(\mathcal{T}'_n)$. 
We have constructed the diagram

\[
\begin{array}{ccc}
QC(\mathfrak{T})/iso & \xrightarrow{t_T} & \text{Hom}(T(K), \overline{\mathbb{Q}}_\ell) \\
& & \xrightarrow{\text{rec}_T} \quad \text{H}^1(K, \hat{T}_\ell)
\end{array}
\]

We are working with Takashi Suzuki to construct Langlands parameters directly from quasicharacter sheaves, which would give a different construction of the reciprocity map.
Non-commutative groups

If $G$ is a connected reductive group over $K$, no Néron model. Instead, parahorics correspond to facets in the Bruhat-Tits building and give models for $G$ over $O_K$. After taking the Greenberg transform, we can glue the resulting $k$-schemes and try to build sheaves on the resulting space using some form of Lusztig induction from quasicharacter sheaves on a maximal torus. This work is still in progress.
Affine Grassmanians and Flag Varieties

- **$K$ equal characteristic**
  Starting with $G$ over $k$, the affine Grassmanian $G(K)/G(\mathcal{O}_K)$ and affine flag variety $G(K)/I$ ($I$ is the Iwahori) are ind-schemes over $k$. They play a large role in the geometric Langlands program.

- **$K$ mixed characteristic**
  Now we need to start with a $G$ defined over $K$, and can no longer construct these directly as quotients. Martin Kreidl considers representability of $G(K)/G(\mathcal{O}_K)$ for $G = \text{SL}_n$ but runs into complications with non-perfect rings. Again with Takashi Suzuki, we are working on representing this functor in a slightly modified category.