A function–sheaf dictionary for tori over local fields

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Objective

Goal

Bring a geometric, categorical perspective to the study of the local Langlands correspondence for \( p \)-adic groups.

We approach this task from two directions:

1. Find geometric avatars for objects in local Langlands,
2. Find \( p \)-adic analogues of objects in geometric Langlands.

Key Idea

The representation theory of \( G(\mathbb{Q}_p) \) depends on its structure as a topological group, not as a scheme. A scheme \( \mathcal{G} \) over \( \mathbb{F}_p \) with \( \mathcal{G}(\mathbb{F}_p) = G(\mathbb{Q}_p) \) offers a new perspective.
Quasicharacters to sheaves

$K$ – a non-archimedean local field with residue field $k$,
$R$ – the ring of integers of $K$ with uniformizer $\pi$,
$T$ – an algebraic torus over $K$,
$X^*$ – for a group $X$, notation for $\text{Hom}(X, \overline{\mathbb{Q}}_\ell^\times)$.

1. We construct a commutative group scheme $\mathfrak{T}$ over $k$ with $\mathfrak{T}(k) \cong T(K)$.

2. For any smooth commutative group scheme $G$ over $k$ we define a category $\mathcal{QC}(G)$ of quasicharacter sheaves on $G$ and show

Theorem (Cunningham-R.)

*Trace of Frobenius defines an isomorphism of groups*

\[ \mathcal{QC}(G)/\text{iso} \cong G(k)^*. \]
The Néron model of a torus

The Néron model $T_R$ of $T$ is a separated, smooth commutative group scheme over $R$ so that

**Néron mapping property**

For any smooth $R$-scheme $Z$ and morphism $f : Z_K \rightarrow T$, $f$ extends uniquely to $Z \rightarrow T_R$.

As a consequence,

$$T_R(R) = T(K).$$

Note that $T_R$ is not necessarily finite type.
Examples of Néron models

Example ($\mathbb{G}_m$)

If $T = \mathbb{G}_m$, then the Néron model for $T$ is

$$T_R = \bigcup_{n \in \mathbb{Z}} \mathbb{G}_{m,R},$$

with gluing along generic fibers:

$$
\begin{array}{ccc}
\mathbb{G}_{m,R} & \ni & \mathbb{G}_{m,R} \\
\uparrow & & \uparrow \\
\mathbb{G}_m & \cong & \mathbb{G}_m
\end{array}
\quad
\begin{array}{ccc}
R[x_0, x_0^{-1}] & \leftarrow & R[x_n, x_n^{-1}] \\
\downarrow & & \downarrow \\
K[x_0, x_0^{-1}] & \cong & K[x_n, x_n^{-1}]
\end{array}
$$

given by:

$$\pi^n x_0 \leftarrow x_n$$
Examples of Néron models

Example (SO$_2$)

Let $T = \text{SO}_2$ over $K$, split over $E = K(\sqrt{\pi})$. Then

$$K[T] = K[x, y]/(x^2 - \pi y^2 - 1).$$

The Néron model for $T$ is given by

$$R[T_R] = R[x, y]/(x^2 - \pi y^2 - 1).$$

Here $T_R$ is finite type, but not connected: the special fiber $T_k$ of $T_R$ is given by

$$k[T_k] = k[x, y]/(x^2 - 1),$$

two disjoint lines.
The Greenberg functor

Greenberg defines a functor

$$(\text{Sch} / R) \rightarrow (\text{Sch} / k).$$

$X \rightarrow \text{Gr}(X)$

Proposition (Greenberg)

- If $X$ is separated and locally of finite type then
  $$\text{Gr}(X)(k) = X(R).$$

- This functor respects open and closed immersions, étale and smooth morphisms and geometric components.

- There are finite level Greenberg functors $\text{Gr}_n$ with
  $$\text{Gr}(X) = \lim_{\leftarrow} \text{Gr}_n(X).$$
**Definition**

\[ \mathcal{T} := \text{Gr}(T_R). \]

**Proposition**

1. \( \mathcal{T}(k) = T(K) \)
2. \( \mathcal{T} \) is a smooth commutative group scheme over \( k \)
3. \( \pi_0(\mathcal{T}) = X_*(T)_I \)
Greenberg of Néron for $\mathbb{G}_m$

Set $\mathbb{W}_k^\times$ as the group of units in the Witt ring scheme $\mathbb{W}_k$.

**Example**

If $T = \mathbb{G}_m$, then

$$\mathcal{T} = \bigsqcup_{n \in \mathbb{Z}} \mathbb{W}_k^\times.$$

The component group for $\mathcal{T}$ is

$$X_\ast(T)_I = \mathbb{Z},$$

with the trivial $\text{Gal}(\bar{k}/k)$ action.
From now on, $G$ will denote a smooth, commutative group scheme over $k$. We will write $m : G \times G \to G$ for multiplication.

**Definition (Local System)**

An $\ell$-adic local system on $G$ is a constructible sheaf of $\overline{\mathbb{Q}}_\ell$-vector spaces on the étale site of $G$, locally constant on each connected component.
Rigid Quasicharacter Sheaves

Definition (Rigid quasicharacter sheaf)

A rigid quasicharacter sheaf on $G$ is a triple $\mathcal{L} := (\mathcal{L}, \mu, \phi)$. 

1. $\mathcal{L}$ is a rank-one local system on $\tilde{G}$,
2. $\mu : \tilde{m}^* \mathcal{L} \to \mathcal{L} \boxtimes \mathcal{L}$ is an isomorphism of sheaves on $\tilde{G} \times \tilde{G}$, satisfying an associativity diagram.
3. $\phi : F_G^* \mathcal{L} \to \mathcal{L}$ is an isomorphism of sheaves on $\tilde{G}$ compatible with $\mu$.

A morphism of quasicharacter sheaves is a morphism of constructible $\ell$-adic sheaves on $\tilde{G}$ commuting with $\mu$ and $\phi$. 

Tensor product makes $RQC(G)$ into a rigid monoidal category and $RQC(G)/iso$ into a group.
Bounded and Finite Rigid Quasicharacter Sheaves

**Definition**

- **A bounded rigid quasicharacter sheaf** on $G$ is a pair $(\mathcal{L}_0, \mu_0)$, where $\mathcal{L}_0$ is a rank-one local system on $G$ and $\mu_0$ is as before.

- **A finite rigid quasicharacter sheaf** on $G$ is a pair $(f, \psi)$, where $f : H \to G$ is a finite, surjective, étale morphism of group schemes and $\psi : \ker f \to \overline{\mathbb{Q}_\ell}^\times$.

- Have full and faithful functors $\mathcal{RQC}_f(G) \to \mathcal{RQC}_0(G) \to \mathcal{RQC}(G)$,

- these are equivalences when $G$ is connected,

- Bounded rigid quasicharacter sheaves will correspond to bounded characters, and finite to ones with finite image.
Étale Group Schemes

\( \mathcal{L} \leftrightarrow (\text{stalks } \bar{\mathcal{L}}_x \text{ and indexed isomorphisms } \mu_{x,y} \text{ and } \phi_x). \)

Choice of basis for \( \bar{\mathcal{L}}_x \leadsto a \in C^2(\bar{G}, \bar{\mathbb{Q}}_\ell^\times) \text{ and } b \in C^1(\bar{G}, \bar{\mathbb{Q}}_\ell^\times). \)

\[
\begin{align*}
\bar{\mathcal{L}}_{x+y+z} & \xrightarrow{\mu_{x+y,z}} \bar{\mathcal{L}}_{x+y} \otimes \bar{\mathcal{L}}_z \\
\mu_{x,y+z} \downarrow & \quad \downarrow \mu_{x,y} \otimes \text{id} \\
\bar{\mathcal{L}}_x \otimes \bar{\mathcal{L}}_{y+z} & \xrightarrow{\text{id} \otimes \mu_{y,z}} \bar{\mathcal{L}}_x \otimes \bar{\mathcal{L}}_y \otimes \bar{\mathcal{L}}_z \\
\end{align*}
\]

\[
\begin{align*}
\bar{\mathcal{L}}_{F(x)+F(y)} & \xrightarrow{\mu_{F(x),F(y)}} \bar{\mathcal{L}}_{F(x)} \otimes \bar{\mathcal{L}}_{F(y)} \\
\phi_{x+y} \downarrow & \quad \downarrow \phi_x \otimes \phi_y \\
\bar{\mathcal{L}}_{x+y} & \xrightarrow{\mu_{x,y}} \bar{\mathcal{L}}_x \otimes \bar{\mathcal{L}}_y \\
\end{align*}
\]

\[
\frac{a(F(x),F(y))}{a(x,y)} = \frac{b(x+y)}{b(x)b(y)}
\]
**Hochschild-Serre Spectral Sequence**

\( \mathcal{W} \) – the Weil group of \( k \),

\[ a \mapsto \alpha \in C^0(\mathcal{W}, Z^2(\bar{G}, \mathbb{Q}_\ell^\times)), \]

\[ b \mapsto \beta \in Z^1(\mathcal{W}, C^1(\bar{G}, \mathbb{Q}_\ell^\times)) \text{ with } \beta(F) = b, \]

\[ E^{i,j}_0 = C^i(\mathcal{W}, C^j(\bar{G}, \mathbb{Q}_\ell^\times)). \]

**Proposition**

- The map \( RQC(G)_{/iso} \to H^2(E^\bullet_0) \) to the cohomology of the total complex given by \( \mathcal{L} \mapsto (\alpha, \beta, 0) \) is an isomorphism.

- The spectral sequence yields an exact sequence

\[ 1 \to H^0(\mathcal{W}, H^2(\bar{G}, \mathbb{Q}_\ell^\times)) \to H^2(E^\bullet_0) \to H^1(\mathcal{W}, H^1(\bar{G}, \mathbb{Q}_\ell^\times)) \to 1. \]

- \[ H^1(\mathcal{W}, H^1(\bar{G}, \mathbb{Q}_\ell^\times)) \to (G(\bar{k})^*)_{/\mathcal{W}} \to G(k)^* \]

is an isomorphism compatible with trace of Frobenius.
Quasicharacter Sheaves

So $\mathcal{QC}(G)/_{iso} \rightarrow G(k)^*$ has kernel $H^2(G(\bar{k}), \mathbb{Q}_\ell^\times)^F$ for étale $G$.

**Definition (Quasicharacter sheaf)**

For any smooth, commutative, group scheme $G$, a *quasicharacter sheaf* on $G$ is a Weil sheaf $\mathcal{L} := (\bar{\mathcal{L}}, \phi)$ so that $(\bar{\mathcal{L}}, \mu, \phi)$ is a rigid quasicharacter sheaf for some $\mu$.

**Proposition**

*For étale $G$, trace of Frobenius induces an isomorphism*

$$\mathcal{QC}(G)/_{iso} \rightarrow G(k)^*.$$
Snake Lemma

For any $G$, trace of Frobenius defines a map

$$t_G : \mathcal{QC}(G)/\text{iso} \to G(k)^*.$$ 

Pullback then gives the rows of

$$\begin{array}{cccc}
\mathcal{QC}(\pi_0(G))/\text{iso} & \longrightarrow & \mathcal{QC}(G)/\text{iso} & \longrightarrow & \mathcal{QC}(G^\circ)/\text{iso} \\
\downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & (\pi_0(G))(k)^* & \longrightarrow & G(k)^* & \longrightarrow & G^\circ(k)^* & \longrightarrow & 1 \\
\end{array}$$

- $t_{G^\circ}$ is an isomorphism by the classic function–sheaf dictionary (Deligne),
- $t_{\pi_0(G)}$ is an isomorphism as above,
- the snake lemma finishes the job.
Transfer of quasicharacter sheaves

Suppose $T$ and $T'$ are tori over local fields $K$ and $K'$. We say that $T$ and $T'$ are \textit{N-congruent} if there are isomorphisms

\[
\alpha : \mathcal{O}_L/\pi_K^N\mathcal{O}_L \rightarrow \mathcal{O}_{L'}/\pi_{K'}^N\mathcal{O}_{L'},
\]
\[
\beta : \text{Gal}(L/K) \rightarrow \text{Gal}(L'/K'),
\]
\[
\phi : X^*(T) \rightarrow X^*(T'),
\]

satisfying natural conditions. If $T$ and $T'$ are $N$-congruent then \(\text{Hom}_{<N}(T(K), \overline{\mathbb{Q}}_\ell^\times) \cong \text{Hom}_{<N}(T'(K'), \overline{\mathbb{Q}}_\ell^\times)\).

- Chai and Yu give an isomorphism of group schemes $T_n \cong T'_n$, for $n$ depending on $N$.
- This isomorphism induces an equivalence of categories $\mathcal{QC}(\mathcal{T}_n) \rightarrow \mathcal{QC}(\mathcal{T}'_n)$.
Class Field Theory

We have constructed the diagram

\[
\begin{array}{c}
QC(\mathfrak{T})/iso \\
\downarrow t_{\mathfrak{T}}
\end{array}
\quad \rightarrow \quad
\begin{array}{c}
\text{Hom}(T(K), \overline{\mathbb{Q}_\ell}^\times) \\
\downarrow \text{rec}_{\mathfrak{T}}
\end{array}
\quad \rightarrow \quad
\begin{array}{c}
H^1(K, \hat{T}_\ell)
\end{array}
\]

We are working with Takashi Suzuki to construct Langlands parameters directly from quasicharacter sheaves, which would give a different construction of the reciprocity map.
Non-commutative groups

If $\mathbf{G}$ is a connected reductive group over $K$, no Néron model. Instead, parahorics correspond to facets in the Bruhat-Tits building and give models for $\mathbf{G}$ over $\mathcal{O}_K$. After taking the Greenberg transform, we can glue the resulting $k$-schemes and try to build sheaves on the resulting space using some form of Lusztig induction from quasicharacter sheaves on a maximal torus. This work is still in progress.
Affine Grassmanians and Flag Varieties

- **K equal characteristic**
  Starting with $G$ over $k$, the affine Grassmanian $G(K)/G(O_K)$ and affine flag variety $G(K)/I$ ($I$ is the Iwahori) are ind-schemes over $k$. They play a large role in the geometric Langlands program.

- **K mixed characteristic**
  Now we need to start with a $G$ defined over $K$, and can no longer construct these directly as quotients. Martin Kreidl considers representability of $G(K)/G(O_K)$ for $G = \text{SL}_n$ but runs into complications with non-perfect rings. Again with Takashi Suzuki, we are working on representing this functor in a slightly modified category.