A function–sheaf dictionary for tori over local fields

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## Outline

1. Introduction
2. Greenberg of Néron
3. Quasicharacter Sheaves
4. Applications and Further Work
Objective

\( K \) – a non-archimedean local field,
\( T \) – an algebraic torus over \( K \),
\( \ell \) – a prime different from \( p \),
\( X^* \) – for a group \( X \), notation for \( \text{Hom}(X, \mathbb{Q}_\ell^\times) \).

Goal

Attach a space \( \mathcal{T} \) to \( T \) and find a dictionary that translates
\[ \{ \text{characters of } T(K) \} \leftrightarrow \{ \text{sheaves on } \mathcal{T} \}. \]

- Try to push characters forward along maps such as \( T \leftrightarrow G \);
- Deligne-Lusztig representations \( \leftrightarrow \) character sheaves;
- Give a new perspective on class field theory.
Approach

1. For commutative group schemes $G$, locally of finite type over the residue field $k$ of $K$ we define a category $\mathcal{QC}(G)$ of quasicharacter sheaves on $G$.

2. We show

Main Result over $k$

$$\mathcal{QC}(G)/_{\text{iso}} \cong G(k)^*.$$ 

3. Given a torus $T$ over $K$ we construct a commutative group scheme $\mathfrak{T}$ over $k$ with $T(K) \cong \mathfrak{T}(k)$. 
Main Result for Tori

**Theorem (Cunningham & R., [CR13])**

For every torus $T$ over $K$, there is a pro-algebraic group $\mathcal{T}/k$ with $\mathcal{T}(k) = T(K)$ and a monoidal category $\mathcal{QC}(\mathcal{T})$ of Weil local systems on $\mathcal{T}$ so that $\mathcal{QC}(\mathcal{T})/\text{iso} \cong T(K)^*$. 
Let $R$ be the ring of integers of $K$ with uniformizer $\pi$. The Néron model $T_R$ of $T$ is a separated, smooth commutative group scheme over $R$, locally of finite type with the Néron mapping property:
For
As a consequence,
\[ T_R(R) = T(K). \]
Examples of Néron models

Example \((\mathbb{G}_m)\)

If \(T = \mathbb{G}_m\), then the Néron model for \(T\) is

\[
T_R = \bigcup_{n \in \mathbb{Z}} \mathbb{G}_{m,R},
\]

with gluing along generic fibers:

\[
\mathbb{G}_{m,R} \xrightarrow{\approx} \mathbb{G}_m \quad \xrightarrow{\pi^n x_0} x_n
\]

given by:

\[
\pi^n x_0 \leftarrow x_n
\]
Examples of Néron models

Example (SO₂)

Let $T = \text{SO}_2$ over $K$, split over $E = K(\sqrt{\pi})$. Then

$$K[T] = K[x, y]/(x^2 - \pi y^2 - 1).$$

The Néron model for $T$ is given by

$$\mathcal{R}[T_\mathcal{R}] = \mathcal{R}[x, y]/(x^2 - \pi y^2 - 1).$$

Here $T_\mathcal{R}$ is finite type, but not connected: the special fiber $T_k$ of $T_\mathcal{R}$ is given by

$$k[T_k] = k[x, y]/(x^2 - 1),$$

two disjoint lines.
The Greenberg functor

Proposition ([DG70, V, §4, no. 1; BLR80, Ch. 9, §6; SN08, §2.2; AC13, §5])

The Greenberg functor

\[(\text{Sch} / R) \to (\text{Sch} / k)\]
\[X \to \text{Gr}(X)\]

has the property that, if \(X\) is separated and locally of finite type then

\[\text{Gr}(X)(k) = X(R)\.]\]
**Greenberg of Néron**

**Definition**

\[ \mathcal{T} := \text{Gr}(T_R). \]

**Proposition**

1. \( \mathcal{T}(k) = T(K) \)
2. \( \mathcal{T} \) is a smooth commutative group scheme over \( k \)
3. \( \mathcal{T} \) is locally of finite type over \( k \)
4. \( \pi_0(\mathcal{T}) = X^*_*(T)_I \)
Set $\mathbb{W}_k^\times$ as the group of units in the Witt ring scheme $\mathbb{W}_k$.

**Example**

If $\mathbf{T} = \mathbb{G}_m$, then

$$\mathcal{T} = \bigsqcup_{n \in \mathbb{Z}} \mathbb{W}_k^\times.$$ 

The component group for $\mathcal{T}$ is

$$X_*(\mathbf{T})_{\mathcal{I}} = \mathbb{Z},$$

with the trivial $\text{Gal}(\bar{k}/k)$ action.
Local Systems

From now on, $G$ will denote a smooth, commutative group scheme, locally of finite type over $k$ with finitely generated geometric component group. We will write $m : G \times G \to G$ for multiplication.

Definition (Local System)

An $\ell$-adic local system on $G$ is a constructible sheaf of $\overline{\mathbb{Q}}_\ell$-vector spaces on the étale site of $G$, locally constant on each connected component.
Quasicharacter Sheaves

**Definition (Quasicharacter sheaf)**

A *quasicharacter sheaf* on $G$ is a triple $\mathcal{L} := (\bar{\mathcal{L}}, \mu, \phi)$, where

1. $\bar{\mathcal{L}}$ is a rank-one local system on $\bar{G}$,
2. $\mu : \bar{m}^* \bar{\mathcal{L}} \rightarrow \bar{\mathcal{L}} \boxtimes \bar{\mathcal{L}}$ is an isomorphism of sheaves on $\bar{G} \times \bar{G}$, satisfying an associativity diagram.
3. $\phi : F^*_G \bar{\mathcal{L}} \rightarrow \bar{\mathcal{L}}$ is an isomorphism of sheaves on $\bar{G}$ compatible with $\mu$.

A morphism of quasicharacter sheaves is a morphism of constructible $\ell$-adic sheaves on $\bar{G}$ commuting with $\mu$ and $\phi$.

Tensor product makes $\mathcal{QC}(G)$ into a rigid monoidal category and $\mathcal{QC}(G)/iso$ into a group.
Bounded Quasicharacter Sheaves

Definition (Bounded Quasicharacter Sheaf)

A bounded quasicharacter sheaf on $G$ is a pair $(\mathcal{L}_0, \mu_0)$, where

1. $\mathcal{L}_0$ is a rank-one local system on $G$,
2. $\mu_0 : m^* \mathcal{L}_0 \to \mathcal{L}_0 \boxtimes \mathcal{L}_0$ is an isomorphism of sheaves on $G \times G$, satisfying the same associativity diagram.

A morphism is a morphism of constructible sheaves on $G$ commuting with $\mu_0$. Write $\mathcal{QC}_0(G)$ for this category.

- Base change defines a full and faithful functor $B_G : \mathcal{QC}_0(G) \to \mathcal{QC}(G)$,
- $B_G$ is an equivalence when $G$ is connected.
- Under the isomorphism $\mathcal{QC}(G)_{/iso} \cong G(k)^*$, bounded quasicharacter sheaves correspond to bounded characters.
Discrete Isogenies

**Definition (Discrete Isogeny)**

A *discrete isogeny* is a finite, surjective, étale morphism of group schemes $f : H \to G$ so that $\text{Gal}(\bar{k}/k)$ acts trivially on the kernel of $f$.

Write $C(G)$ for the category whose objects are pairs $(f, \psi)$, where

1. $f : H \to G$ is a discrete isogeny,
2. $\psi : \ker f \to \text{Aut}(V)$ is a representation on a $\mathbb{Q}_\ell$-vector space.

A morphism $(f, \psi) \to (f', \psi')$ is a pair $(g, T)$, where

1. $g : H' \to H$ is a morphism with $f' = f \circ g$,
2. $T : V \to V'$ is a linear transformation, equivariant for $\psi'$ and $\psi \circ g$. 
Finite Quasicharacter Sheaves

Let $C_1(G)$ be the subcategory where $V$ is one-dimensional.

**Definition (Finite Quasicharacter Sheaf)**

The category $\mathcal{QC}_f(G)$ of *finite quasicharacter sheaves* is the localization of $C_1(G)$ at morphisms where $g$ is surjective and $T$ is an isomorphism.

Write $V_H$ for the constant sheaf $V$ on $H$.

- Taking the $\psi$-isotypic component of $f_* V_H$ defines a full and faithful functor $L_G : \mathcal{QC}_f(G) \to \mathcal{QC}_0(G)$.
- $L_G$ is an equivalence when $G$ is connected.
- Under the isomorphism $\mathcal{QC}(G)/_{iso} \cong G(k)^*$, finite quasicharacter sheaves correspond to characters with finite image.
**Sketch of Main Result**

For any $G$, trace of Frobenius defines a map

$$t_G : \mathcal{QC}(G)_{/iso} \to G(k)^*.$$ 

Pullback then gives the rows of

$$1 \longrightarrow \mathcal{QC}(\pi_0(G))_{/iso} \longrightarrow \mathcal{QC}(G)_{/iso} \longrightarrow \mathcal{QC}(G^\circ)_{/iso} \longrightarrow 1$$

$$1 \longrightarrow (\pi_0(G))(k)^* \longrightarrow G(k)^* \longrightarrow G^\circ(k)^* \longrightarrow 1$$

- $t_G$ is an isomorphism by [Del77]: the classic function–sheaf dictionary,
- one can build an isomorphism by hand for étale group schemes using stalks,
- the snake lemma finishes the job.
Transfer of quasi-character sheaves

Suppose \( T \) and \( T' \) are tori over local fields \( K \) and \( K' \). We say that \( T \) and \( T' \) are \( N \)-congruent if there are isomorphisms

\[
\begin{align*}
\alpha : \mathcal{O}_L / \pi_K^N \mathcal{O}_L & \to \mathcal{O}_{L'} / \pi_{K'}^N \mathcal{O}_{L'}, \\
\beta : \text{Gal}(L/K) & \to \text{Gal}(L'/K'), \\
\phi : X^*(T) & \to X^*(T'),
\end{align*}
\]

satisfying natural conditions. If \( T \) and \( T' \) are \( N \)-congruent then \( \text{Hom}_{<N}(T(K), \overline{\mathbb{Q}}_\ell^\times) \cong \text{Hom}_{<N}(T'(K'), \overline{\mathbb{Q}}_\ell^\times) \).

- Chai and Yu give an isomorphism of group schemes \( T_n \cong T'_n \), for \( n \) depending on \( N \).
- This isomorphism induces an equivalence of categories \( QC(T_n) \to QC(T'_n) \).
Class Field Theory

We have constructed the diagram

\[ \begin{array}{ccc}
QC(T)_{iso} & \xrightarrow{t_T} & \text{Hom}( T(K), \hat{\mathbb{Q}_\ell}^\times ) \\
 & \searrow & \downarrow \text{rec}_T \\
 & & H^1(K, \hat{T}_\ell)
\end{array} \]

We hope to be able to construct Langlands parameters directly from quasicharacter sheaves, which would give a different construction of the reciprocity map.
If $G$ is a connected reductive group over $K$, no Néron model. Instead, parahorics correspond to facets in the Bruhat-Tits building and give models for $G$ over $\mathcal{O}_K$. After taking the Greenberg transform, we can glue the resulting $k$-schemes and try to build sheaves on the resulting space using some form of Lusztig induction from quasicharacter sheaves on a maximal torus. This work is still in progress.
K equal characteristic
The affine Grassmanian $G(K)/G(\mathcal{O}_K)$ and affine flag variety $G(K)/I$ (here $I$ is the Iwahori) are ind-schemes over $k$. They play a large role in the geometric Langlands program.

K mixed characteristic
Can no longer construct these directly as quotients. We are working on defining analogues via gluing Schubert cells. As a test of the construction, we hope to give a geometric Satake transform in mixed characteristic.


