Geometrizing Characters of Tori

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Outline

Objective

K – a finite extension of $\mathbb{Q}_p$,
$\mathbf{T}$ – an algebraic torus over K (e.g. $\mathbb{G}_m$),
ℓ – a prime different from p,
X^* – for a group X, notation for $\mathrm{Hom}(X, \overline{\mathbb{Q}}_\ell^\times)$.

Goal

Construct "geometric avatars" for characters in

$$\mathbf{T}(K)^* :$$

sheaves on some space functorially associated to $\mathbf{T}$.

- Try to push characters forward along maps such as $\mathbf{T} \hookrightarrow \mathbf{G}$;
- Deligne-Lusztig representations $\Longrightarrow$ character sheaves;
- Give a new perspective on class field theory.

Approach

1. For commutative group schemes G, locally of finite type over the residue field k of K we define a category $\mathcal{CS}(G)$ of character sheaves on G.
2. We show

Main Result

$$\mathcal{CS}(G)_{/\,\mathrm{iso}} \cong G(k)^*.$$

3. Given a torus T over K we construct a commutative group scheme $\mathfrak{T}$ over k with $T(K) \cong \mathfrak{T}(k)$.

Character Sheaves (G connected)

Two definitions of character sheaves for a connected (commutative) algebraic groups G over k:

Definition

- An *ℓ-adic local system* is a constructible sheaf of $\overline{\mathbb{Q}}_\ell$-vector spaces on the étale site of G that becomes trivial after pulling back along a finite étale map $H \to G$.
- A *geometric character sheaf* on G is an ℓ-adic local system $\mathcal{E}^\circ$ on G equipped with an isomorphism $m^*\mathcal{E}^\circ \cong \mathcal{E}^\circ \boxtimes \mathcal{E}^\circ$, where $m\colon G \times G \to G$ is multiplication.

Character Sheaves 2 (G connected)

Definition

Alternatively, a character sheaf on G is a short exact sequence

$$1 \to A \to H \to G \to 1$$

together with a character $A \to \overline{\mathbb{Q}}_\ell^\times$, so that

1. $H \to G$ is a finite étale cover,
2. Fr_q acts trivially on A.

Rationality of character sheaves

Base change to $\overline{k}$ yields a pair $(\overline{\mathcal{E}}^{\circ}, \mathrm{Fr}_{\mathcal{E}^{\circ}})$, where $\overline{\mathcal{E}}^{\circ}$ is a character sheaf on $\overline{G}$ and $\mathrm{Fr}_{\mathcal{E}^{\circ}} : \mathrm{Fr}_q^* \overline{\mathcal{E}}^{\circ} \xrightarrow{\sim} \overline{\mathcal{E}}^{\circ}$.

Proposition

In general this functor is faithful; when G is connected, base change defines an equivalence of categories

$$\left\{ \begin{array}{c} \textit{character sheaves} \\ \textit{on } G \end{array} \right\} \to \left\{ \textit{pairs } (\overline{\mathcal{E}}^{\circ}, \mathrm{Fr}_{\mathcal{E}^{\circ}}) \right\}$$

Characters in the connected case

- Suppose $(\overline{\mathcal{E}}^{\circ}, \mathrm{Fr}_{\mathcal{E}^{\circ}})$ is a character sheaf on G. Define a character $\chi^{\circ}_{\mathcal{E}}$ of $G(k)$ by

$$\chi^{\circ}_{\mathcal{E}}(x) = \mathrm{Tr}(\mathrm{Fr}_{\mathcal{E}^{\circ}}, \overline{\mathcal{E}}^{\circ}_{x})$$

  for $x \in G(k)$.
- Suppose χ is a character of $G(k)$. Define a character sheaf on G using the Lang isogeny $L(x) = x^{-1}\,\mathrm{Fr}_q(x)$,

$$1 \to G(k) \to G \xrightarrow{L} G \to 1,$$

  together with the character χ of $G(k)$.

Theorem (Deligne, SGA 4.5)

The maps defined above are mutually inverse isomorphisms between character sheaves on G and $G(k)^*$.

Character Sheaves (G non-connected)

Definition

A character sheaf on G is a triple $\mathcal{E} = (\bar{\mathcal{E}}, \mu, F)$, where

1. $\bar{\mathcal{E}}$ is a constructible ℓ-adic sheaf on $\bar{G}$, locally constant of rank 1 on each connected component;
2. $\mu : m^*\bar{\mathcal{E}} \to \bar{\mathcal{E}} \boxtimes \bar{\mathcal{E}}$ is an isomorphism of sheaves on $\bar{G} \times \bar{G}$;
3. $F : \mathrm{Fr}_G^* \bar{\mathcal{E}} \to \bar{\mathcal{E}}$ is an isomorphism of sheaves on $\bar{G}$.

μ and F are required to satisfy various compatibility diagrams. We write $\mathcal{CS}(G)$ for the category of character sheaves on G.

Trace of Frobenius

For any G, trace of Frobenius defines a map

$$\mathcal{CS}(G)_{/\,\mathrm{iso}} \to G(k)^*.$$

Pullback then gives a diagram

$$\begin{tikzcd}
1 \arrow[r] & \mathcal{CS}(G/G^\circ)_{/\,\mathrm{iso}} \arrow[r] \arrow[d] & \mathcal{CS}(G)_{/\,\mathrm{iso}} \arrow[r] \arrow[d] & \mathcal{CS}(G^\circ)_{/\,\mathrm{iso}} \arrow[d] & \\
1 \arrow[r] & (G/G^\circ)(k)^* \arrow[r] & G(k)^* \arrow[r] & G^\circ(k)^* \arrow[r] & 1
\end{tikzcd}$$

Character Sheaves 2 (G non-connected)

In the non-connected case, not every character sheaf can be realized in the second manner.

Definition

A *bounded character sheaf* on G is a short exact sequence

$$1 \to A \to H \to G \to 1$$

together with a character $A \to \overline{\mathbb{Q}}_\ell^\times$, so that

1. $H \to G$ is a finite étale cover, inducing an isomorphism on component groups
2. Fr_q acts trivially on A.

Extending character sheaves

Theorem

Every character sheaf on G° extends to a (bounded) character sheaf on G.

Proof.

Suppose that $1 \to A \to H \to G \to 1$ and $\chi\colon A \to \overline{\mathbb{Q}}_\ell^\times$ defines a bounded character sheaf. Suppose that $\mathrm{Gal}(\bar{k}/k)$ acts on H and G through the finite quotient Γ. Restriction to $H^\circ \to G^\circ$ then defines a character sheaf on G°.

Extending character sheaves

Proof.

On extension classes, this map is the first in

$$\mathrm{Ext}^1_{\mathbb{Z}[\Gamma]}(G, A) \to \mathrm{Ext}^1_{\mathbb{Z}[\Gamma]}(G^\circ, A) \to \mathrm{Ext}^2_{\mathbb{Z}[\Gamma]}(G/G^\circ, A).$$

Since $\mathbb{Z}[\Gamma]$ is a product of Dedekind domains it has cohomological dimension 1 and thus $\mathrm{Ext}^2_{\mathbb{Z}[\Gamma]}(G/G^\circ, A)$ vanishes. So $\mathrm{Ext}^1_{\mathbb{Z}[\Gamma]}(G, A) \to \mathrm{Ext}^1_{\mathbb{Z}[\Gamma]}(G^\circ, A)$ is surjective. □

Character Sheaves (G étale)

- The category of étale group schemes is equivalent to the category of groups with with Galois action.
- A character sheaf on an étale group scheme G is a collection of 1-dimensional $\overline{\mathbb{Q}}_\ell$-vectors spaces $\mathcal{E}_x$ for $x \in G(\bar{k})$ together with $F_x : \mathcal{E}_{\mathrm{Fr}(x)} \xrightarrow{\sim} \mathcal{E}_x$ and $\mu_{x,y} : \mathcal{E}_x \otimes \mathcal{E}_y \xrightarrow{\sim} \mathcal{E}_{x+y}$.

Proposition

Suppose that G is an étale commutative group scheme and $G(\bar{k})$ is finitely generated. Then there is a canonical isomorphism

$$\mathcal{CS}(G)_{/\mathrm{iso}} \cong \mathrm{H}^1(W_k, G(\bar{k})^*).$$

Trace of Frobenius Diagram

$$
\begin{tikzcd}
 & & & 1 \arrow[d] & \\
1 \arrow[r] & \mathcal{CS}(G/G^\circ)_{/\mathrm{iso}} \arrow[r] \arrow[d] & \mathcal{CS}(G)_{/\mathrm{iso}} \arrow[r] \arrow[d] & \mathcal{CS}(G^\circ)_{/\mathrm{iso}} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & (G/G^\circ)(k)^* \arrow[r] & G(k)^* \arrow[r] & G^\circ(k)^* \arrow[r] \arrow[d] & 1 \\
 & & & 1 &
\end{tikzcd}
$$

A Galois cohomology result

Theorem

If X is a finitely generated abelian group with a continuous $\mathrm{Gal}(\bar{k}/k)$ *action then*

$$\mathrm{H}^1(k, X^*) \xrightarrow{\rho} \mathrm{H}^0(k, X)^*$$
$$\rho([z])(x) = z(\mathrm{Fr})(x)$$

is an isomorphism

Corollary

If G is a commutative group scheme with finitely generated component group then trace of Frobenius gives an isomorphism

$$\mathcal{CS}(G)_{/\mathrm{iso}} \cong G(k)^*.$$

Surjectivity of ρ

Proof.

Since $\overline{\mathbb{Q}}_\ell^\times$ is divisible it is injective as a $\mathbb{Z}$-module and thus $\mathrm{Ext}^1_{\mathbb{Z}}(X/X^{\mathrm{Fr}}, \overline{\mathbb{Q}}_\ell^\times) = 0$ so restriction

$$X^* \to (X^{\mathrm{Fr}})^*$$

is surjective. Given a character on X^{Fr} we define a cocycle by setting a value on Frobenius and extending via the cocycle relation. $\square$

Injectivity of ρ

Proof.

Suppose $[z]$ is in the kernel of ρ and set $\phi = z(\mathrm{Fr})$. By assumption $\phi(x) = 1$ for $x \in X^{\mathrm{Fr}}$; it suffices to construct $\psi \in X^*$ with $\phi(x) = \frac{\psi(\mathrm{Fr}(x))}{\psi(x)}$ for all $x \in X$. Reduce to the case that $X = \mathbb{Z}[\zeta]/P^s$ as a $\mathbb{Z}[\zeta]$-module:

- $\mathbb{Z}[\Gamma]$ is a product of Dedekind domains $\mathbb{Z}[\zeta]$,
- Modules over products of Dedekind domains decompose as direct sums of cyclic modules.

Injectivity of ρ

Proof.

Now define ψ on generators ζ^i by

$$\psi(\zeta^i) = \alpha \prod_{j=0}^{i-1} \phi(\zeta^j).$$

Now use properties of cyclotomic polynomials to find an α so that ψ is well-defined and has the property

$$\phi(x) = \frac{\psi(\zeta \cdot x)}{\psi(x)}.$$

$\square$

The Néron model of a torus

R – ring of integers of K with uniformizer π

R_d – $R/\pi^{d+1}R$

$\mathbf{T}_R$ – The Néron model of $\mathbf{T}$: a separated, smooth commutative group scheme over R, locally of finite type with the Néron mapping property.

$$\mathbf{T}_R(R) = \mathbf{T}(K)$$

In the $\mathbb{G}_m$ case the Néron model is a union of copies of $\mathbb{G}_m/R$, glued along the generic fiber.

$\mathbf{T}_d$ – $\mathbf{T}_R \times_R R_d$.

The Greenberg functor

The Greenberg functor Gr takes a group scheme over an Artinian local ring A (locally of finite type) and produces a group scheme over the residue field k whose k points are canonically identified with the A-points of the original scheme. We set

$$\mathfrak{T}_d = \mathrm{Gr}(\mathbf{T}_d)$$

and

$$\mathfrak{T} = \varprojlim \mathfrak{T}_d.$$

$\mathfrak{T}$ is a commutative group scheme over k with

$$\mathfrak{T}(k) = \mathbf{T}(K).$$

Character sheaves on $\mathfrak{T}$

We write $\mathcal{CS}(\mathfrak{T})$ for the projective limit of the categories $\mathcal{CS}(\mathfrak{T}_d)$.

Theorem

$$T(K)^* \cong \mathcal{CS}(\mathfrak{T})_{/\,\mathrm{iso}}$$

and this isomorphism preserves depth.

Thank you.