Geometrizing Quasi-characters of Tori

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Outline

1. Introduction

2. Greenberg of Néron

3. Character Sheaves
Objective

- $K$ – a finite extension of $\mathbb{Q}_p$,
- $\mathbf{T}$ – an algebraic torus over $K$ (e.g. $\mathbb{G}_m$),
- $\ell$ – a prime different from $p$.

Goal

Construct “geometric avatars” for (quasi)-characters in $\text{Hom}(\mathbf{T}(K), \overline{\mathbb{Q}}_\ell)$:

sheaves on some space functorially associated to $\mathbf{T}$.

- Try to push characters forward along maps such as $\mathbf{T} \hookrightarrow \mathbf{G}$;
- Deligne-Lusztig representations $\mapsto$ character sheaves;
- Give a new perspective on class field theory.
Approach

1. Associate to $\mathcal{T}$ a projective system $\mathcal{T}$ of commutative group schemes $\mathcal{T}_d$ over the residue field $k$ of $K$.
2. Define *character sheaves* on $\mathcal{T}$ following Deligne.
3. Map from character sheaves on $\mathcal{T}$ to characters on $T(K)$. 
The Néron model of $\mathbb{G}_m$

$R$ – ring of integers of $K$ with uniformizer $\pi$

$R_d = R/\pi^{d+1} R$

$T_R$ – The Néron model of $T$: a separated, smooth commutative group scheme over $R$, locally of finite type with the Néron mapping property.

$$T_R(R) = T(K)$$

In the $\mathbb{G}_m$ case the Néron model is a union of copies of $\mathbb{G}_m/R$, glued along the generic fiber.

$T_d = T_R \times_R R_d$. 
Components

- The geometric component group of $T_R$ is $X_*(T)_{I_K}$, where $I_K$ is the inertia group of $K$.
- $\pi_0(T_R)$ is a constant group scheme after base change to the maximal unramified extension of $K$, but Frobenius may act nontrivially.
- The sequence of commutative $R$-group schemes

$$1 \to T_R \to T \to \pi_0(T_R) \to 1$$

splits if $T$ is unramified.
The Greenberg functor

The Greenberg functor $\text{Gr}$ takes a group scheme over an Artinian local ring $A$ (locally of finite type) and produces a group scheme over the residue field $k$ whose $k$ points are canonically identified with the $A$-points of the original scheme. We set

$$\mathcal{T}_d = \text{Gr}(\mathcal{T}_d)$$

and

$$\mathcal{T} = \lim \mathcal{T}_d.$$ 

$\mathcal{T}$ is a commutative group scheme over $k$ with

$$\mathcal{T}(k) = \mathcal{T}(K).$$
Character Sheaves on $\mathcal{T}^\circ_d$

**Definition**

A *character sheaf* on $\mathcal{T}^\circ_d$ is an $\ell$-adic local system $\mathcal{E}^\circ$ on $\mathcal{T}^\circ_d$ so that $m^* \mathcal{E}^\circ \cong \mathcal{E}^\circ \boxtimes \mathcal{E}$, where $m: \mathcal{T}^\circ_d \times \mathcal{T}^\circ_d \to \mathcal{T}^\circ_d$ is multiplication.

Alternatively, a character sheaf on $\mathcal{T}^\circ_d$ is a short exact sequence

$$1 \to A \to H \to \mathcal{T}^\circ_d \to 1$$

together with a character $A \to \overline{\mathbb{Q}}\ell^\times$, so that

1. $H \to \mathcal{T}^\circ_d$ is a finite étale cover,
2. $\text{Fr}_q$ acts trivially on $A$.

**Proposition**

*Base change to $\overline{\mathcal{T}}^\circ_d$ defines an equivalence of categories*

$$\mathcal{E}^\circ \mapsto (\overline{\mathcal{E}}^\circ, \text{Fr}_{\mathcal{E}^\circ})$$
Characters of $\mathfrak{T}_d^\circ(k)$

Let $x \in \mathfrak{T}_d(k)$ and $\tilde{E}_x^\circ$ be the stalk of $\tilde{E}^\circ$ at $x$. Define a character $\chi_{\tilde{E}}^\circ$ of $\mathfrak{T}_d^\circ(k)$ by

$$\chi_{\tilde{E}}^\circ(x) = \text{Tr}(\text{Fr}_{E^\circ}, \tilde{E}_x^\circ).$$

Theorem (Deligne, SGA 4.5)

The map

$$\{\text{characters sheaves on } \mathfrak{T}_d^\circ\} \rightarrow \text{Hom}(\mathfrak{T}_d^\circ(k), \bar{\mathbb{Q}}_\ell^\times)$$

$$\mathcal{E}^\circ \mapsto \chi_{\mathcal{E}}^\circ$$

is an isomorphism.

Since $\mathfrak{T}_d^\circ(k) = \mathfrak{T}_d^\circ(R)$, this theorem gives a geometrization of depth $d$ characters of $\mathfrak{T}_R^\circ$. 
Character Sheaves on $\mathcal{T}_d$

**Definition**

A character sheaf $\mathcal{E}$ on $\mathcal{T}_d$ is a character sheaf $\mathcal{E}^\circ$ on $\mathcal{T}_d^\circ$ plus an action of $\mathcal{X}_*(\mathcal{T})\mathcal{I}_K \times \mathcal{W}_k$ on $\mathcal{E}^\circ$ compatible with $\text{Fr}\mathcal{E}^\circ$. 
Characters of $\mathfrak{T}_d(k)$

Suppose now that $T$ is unramified so that

$$1 \to T_R^0 \to T_R \to \pi_0(T_R) \to 1$$

splits. A splitting defines an extension of $\chi_{\mathcal{E}}^\circ$ to

$$\mathfrak{T}_d(k) = T(K)/T_R(R)_{d+}.$$ 

From the action of $X_*(T)_{I_K}$ on $\mathcal{E}^\circ$ one can produce a character of

$$(X_*(T)_{I_K})^{W_k} = T(K)/T_R^\circ(R).$$ 

Thus we may associate to $\mathcal{E}$ a depth $d$ character $\chi_{\mathcal{E}}$ of $T(K)$: the product of these two.
We define a character sheaf on $\mathcal{T}$ as the pullback of a character sheaf on $\mathcal{T}_d$ under the projection $\mathcal{T} \to \mathcal{T}_d$ for some $d$. A character of $\mathbf{T}(K)$ is *admissible* if it has depth $d$ for some $d$: factors through the quotient $\mathbf{T}(K)/\mathbf{T}(K)_{d^+}$.

**Theorem**

*The map*

$$\{\text{character sheaves on } \mathcal{T}\} \to \text{Hom}(\mathbf{T}(K), \bar{\mathbb{Q}}_\ell^\times)$$

*defined above is surjective*
Extra Character sheaves

Let $Y$ be the cokernel of the composition

$$(X_*(T)_{I_K})^{W_k} \hookrightarrow X_*(T)_{I_K} \twoheadrightarrow (X_*(T)_{I_K})^{W_k}$$

**Theorem**

The map

$$\{\text{character sheaves on } \mathcal{T}\} \rightarrow \text{Hom}(T(K), \overline{\mathbb{Q}}_\ell)$$

is surjective with fibers parameterized by $\text{Hom}(Y, \overline{\mathbb{Q}}_\ell)$. 
Questions

• Is

\[ 1 \rightarrow T_R^\circ \rightarrow T_R \rightarrow \pi_0(T_R) \rightarrow 1 \]

split for ramified tori?

• Is there some category containing all Néron models of tori on which the Greenberg functor is exact?