The Local Langlands Correspondence and character sheaves

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Outline

1. Introduction to Local Langlands
   - Local Langlands for $\text{GL}_n$
   - Beyond $\text{GL}_n$
   - DeBacker-Reeder

2. Local Langlands for Tamely Ramified Unitary Groups
   - The Torus
   - The Character
   - Embeddings and Induction

3. Character Sheaves
   - A Different Induction Process
   - Greenberg of Néron
   - Character Sheaves
What is the Langlands Correspondence?

- A generalization of class field theory to non-abelian extensions.
- A tool for studying $L$-functions.
- A correspondence between representations of Galois groups and representations of algebraic groups.
The 1-dimensional case of local Langlands is local class field theory.
Conjecture

Irreducible $n$-dimensional representations of $\mathcal{W}_{\mathbb{Q}_p}$

$\downarrow$

Irreducible representations of $\text{GL}_n(\mathbb{Q}_p)$

In order to make this conjecture precise, we need to modify both sides a bit.
Smooth Representations

For $n > 1$, the representations of $\text{GL}_n(\mathbb{Q}_p)$ that appear are usually infinite dimensional.

**Definition**

A *smooth* $\mathbb{C}$-*representation* of $\text{GL}_n(\mathbb{Q}_p)$ is a pair $(\pi, V)$, where

- $V$ is a $\mathbb{C}$-vector space (possibly infinite dimensional),
- $\pi : \text{GL}_n(\mathbb{Q}_p) \to \text{GL}(V)$ is a homomorphism,
- The stabilizer of each $v \in V$ is open in $\text{GL}_n(\mathbb{Q}_p)$.

The only finite-dimensional irreducible smooth $\pi$ are

$$g \mapsto \chi(\det(g))$$

for some character $\chi : \mathbb{Q}_p^\times \to \mathbb{C}^\times$. 
We also need to clarify what kinds of representations of $\mathcal{W}_{\mathbb{Q}_p}$ to focus on.

**Definition**

A *Langlands parameter* is a pair $(\varphi, V)$ with

$$\varphi : \mathcal{W}_{\mathbb{Q}_p} \to \text{GL}(V) \quad \text{dim}_\mathbb{C} V = n$$

such that $\varphi$ is continuous and semisimple.
Parabolic Subgroups

Given a number of Langlands parameters $\varphi_i : W_{Q_p} \to GL(V_i)$, one can form their direct sum. There should be a corresponding operation on the $GL_n(Q_p)$ side.

**Definition**

A parabolic subgroup of $GL_n$ is a subgroup $P$ conjugate to one consisting of block triangular matrices of a given pattern. For example:

$$
\begin{pmatrix}
* & * & * & * & * \\
0 & * & * & * & * \\
0 & * & * & * & * \\
0 & 0 & 0 & * & * \\
0 & 0 & 0 & * & * \\
\end{pmatrix}
$$

Such a subgroup has a Levi decomposition $P = M \times N$, where $M$ is conjugate to the corresponding subgroup of block diagonal matrices, and $N$ consists of the subgroup of $P$ with identity blocks on the diagonal.
Parabolic Induction

Since each Levi subgroup $M$ is just a direct product of $GL_{n_i}$, a collection of representations $\pi_i: GL_{n_i}(\mathbb{Q}_p) \to GL(V_i)$ yields a representation $\boxtimes_i \pi_i$ of $M$. We can pull this back to $P$ and then induce to obtain

$$\pi = \text{Ind}_P^{GL_n(\mathbb{Q}_p)} \boxtimes_i \pi_i.$$

**Definition**

We say that $\pi$ is the *parabolic induction* of the $\pi_i$. We say that $\pi$ is *supercuspidal* if $\pi$ is not parabolically induced from any proper parabolic subgroup of $GL_n(\mathbb{Q}_p)$. 
There is a natural bijection

\[
\text{Supercuspidal representations of } \text{GL}_n(\mathbb{Q}_p) \leftrightarrow \text{n-dimensional irreducible representations of } \mathcal{W}_p.
\]

But the parabolic induction of irreducible representations does not always remain irreducible. To extend this bijection from supercuspidal representations of \( \text{GL}_n(\mathbb{Q}_p) \) to all smooth irreducible representations of \( \text{GL}_n(\mathbb{Q}_p) \), one enlarges the right hand side using the following group:

\[ \text{WD}_{\mathbb{Q}_p} := \mathcal{W}_p \times \text{SL}_2(\mathbb{C}). \]
Theorem (Local Langlands for GL$_n$: Harris-Taylor, Henniart)

There is a unique system of bijections

$$\begin{align*}
\text{Irreducible representations} & \quad \text{of } \text{GL}_n(\mathbb{Q}_p) \\
\text{rec}_n & \quad \mapsto \quad \text{n-dimensional irreducible representations of } \text{WD}_{\mathbb{Q}_p}
\end{align*}$$

- $\text{rec}_1$ is induced by the Artin map of local class field theory.
- $\text{rec}_n$ is compatible with 1-dimensional characters:
  $$\text{rec}_n(\pi \otimes \chi \circ \det) = \text{rec}_n(\pi) \otimes \text{rec}_1(\chi).$$
- The central character $\omega_\pi$ of $\pi$ corresponds to $\det \circ \text{rec}_n$:
  $$\text{rec}_1(\omega_\pi) = \det(\text{rec}_n(\pi)).$$
- $\text{rec}_n(\pi^\vee) = \text{rec}_n(\pi)^\vee$
- $\text{rec}_n$ respects natural invariants associated to each side, namely L-factors and $\epsilon$-factors of pairs.
Now suppose $\mathbf{G}$ is some other connected reductive group defined over $\mathbb{Q}_p$, such as $\text{SO}_n$, $\text{Sp}_n$ or $\text{U}_n$. We’d like to use a Langlands correspondence to understand representations of $\mathbf{G}(\mathbb{Q}_p)$ in terms of Galois representations. Something like

$$\varphi : \text{WD}_{\mathbb{Q}_p} \rightarrow \mathbf{G}(\mathbb{C}) \iff \text{Irreducible representations of } \mathbf{G}(\mathbb{Q}_p).$$

We need to modify this guess in two ways:

- change $\mathbf{G}(\mathbb{C})$ to a related group, $^L\mathbf{G}(\mathbb{C})$,

- and account for the fact that our correspondence is no longer a bijection.
Root Data

Reductive groups over algebraically closed fields are classified by root data

\[(X^*(S), \Phi(G, S), X_*(S), \Phi^\vee(G, S)),\]

where

- \(S \subset G\) is a maximal torus,
- \(X^*(S)\) is the lattice of characters \(\chi: S \to \mathbb{G}_m\),
- \(X_*(S)\) is the lattice of cocharacters \(\lambda: \mathbb{G}_m \to S\),
- \(\Phi(G, S)\) is the set of roots (eigenvalues of the adjoint action of \(S\) on \(g\)),
- \(\Phi^\vee(G, S)\) is the set of coroots \(\langle \alpha, \alpha^\vee \rangle = 2\).
Given $\mathbf{G} \supset \mathbf{S}$, the connected Langlands dual group $\hat{\mathbf{G}}$ is defined to be the algebraic group over $\mathbb{C}$ with root datum

$$(X_\ast(\mathbf{S}), \Phi^\vee(\mathbf{G}, \mathbf{S}), X^\ast(\mathbf{S}), \Phi(\mathbf{G}, \mathbf{S})).$$

For semisimple groups, this has the effect of exchanging the long and short roots (as well as interchanging the simply connected and adjoint forms).

| $\mathbf{G}$ | $\text{GL}_n$ | $\text{SL}_n$ | $\text{PGL}_n$ | $\text{Sp}_{2n}$ | $\text{SO}_{2n}$ | $\text{U}_n$ |
| $\hat{\mathbf{G}}$ | $\text{GL}_n$ | $\text{PGL}_n$ | $\text{SL}_n$ | $\text{SO}_{2n+1}$ | $\text{SO}_{2n}$ | $\text{GL}_n$ |
For non-split $G$, such as $U_n$, we need to work a little harder. Suppose that $G$ is quasi-split with Borel $B \supset S$, splitting over a finite extension $E/\mathbb{Q}_p$. The fact that $B$ is defined over $\mathbb{Q}_p$ implies that $\text{Gal}(E/\mathbb{Q}_p)$ acts on the root datum. The connected dual group $\hat{G}$ comes equipped with maximal torus $\hat{S}$ canonically dual to $S$. By choosing basis vectors for each (1-dimensional) root space in the Lie algebra of $\hat{G}$, we can extend the action of $\text{Gal}(E/\mathbb{Q}_p)$ from the root datum to an action on $\hat{G}$. Define

\[ ^L G := \hat{G} \rtimes \text{Gal}(E/\mathbb{Q}_p), \]

the L-group of $G$. 

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Unitary Groups

A unitary group over $\mathbb{Q}_p$ is specified by the following data:

- $E/\mathbb{Q}_p$ a quadratic extension (so for $p \neq 2$ there are three possibilities),
- set $\tau \in \text{Gal}(E/\mathbb{Q}_p)$ the nontrivial element,
- $V$ an $n$-dimensional $E$-vector space,
- Non-degenerate Hermitian form $\langle , \rangle$ (so $\langle x, y \rangle = \tau \langle y, x \rangle$).

Then $U(V)$ is the group of automorphisms of $V$ preserving $\langle , \rangle$. Over $\bar{\mathbb{Q}}_p$, $U$ becomes isomorphic to $GL_n$, so $\hat{U}_n$ is $GL_n$, but $^L \mathbf{G}$ is non-connected: $\tau$ acts on $GL_n(\mathbb{C})$ by the outer automorphism

$$g \mapsto (g^{-1})^T.$$
A Langlands parameter is now an equivalence class of homomorphisms

\[ \varphi : \mathcal{W}_{\mathbb{Q}_p} \to \mathcal{L}\mathcal{G}. \]

- We require that the composition of \( \varphi \) with the projection \( \mathcal{L}\mathcal{G} \to \text{Gal}(E/\mathbb{Q}_p) \) agrees with the standard projection \( \mathcal{W}_{\mathbb{Q}_p} \to \text{Gal}(E/\mathbb{Q}_p) \).

- We consider two parameters to be equivalent they are conjugate by an element of \( \hat{\mathcal{G}} \). This definition of equivalence is chosen to match up with the notion of isomorphic representations on the \( \mathcal{G}(\mathbb{Q}_p) \) side.
There is a natural map

\[ \text{Irreducible representations of } G \rightarrow \text{Langlands parameters } \varphi: WD_{\mathbb{Q}_p} \rightarrow {}^L G \]

It is surjective and finite-to-one; the fibers are called \emph{L-packets}. 
Moreover, we can naturally parameterize these fibers. Given a Langlands parameter \( \varphi \), let \( Z_{\hat{G}}(\varphi) \) be the centralizer in \( \hat{G} \) of \( \varphi \), and let \( {}^LZ \) be the center of \( {}^LG \). Define

\[
A_{\varphi} = \pi_0(Z_{\hat{G}}(\varphi)/{}^LZ).
\]

The fibers should be in bijection with

\[
A_{\varphi}^\vee = \{ \text{irreducible representations of } A_{\varphi} \}.
\]

So we get a natural bijection

\[
\text{Irreducible representations of } G \leftrightarrow (\varphi, \rho) \text{ with } \varphi: WD_{Qp} \to {}^LG \quad \text{and } \rho \in A_{\varphi}^\vee.
\]
Approaches to Local Langlands

- One approach to proving the local Langlands correspondence for general $G$ is to try to reduce to the $GL_n$ case: the recent book of Jim Arthur for example.

- Another approach is that of Stephen DeBacker and Mark Reeder, outlined below.
Let $G$ be a connected reductive group defined over $\mathbb{Q}_p$, and assume that $G$ splits over an unramified extension $E/\mathbb{Q}_p$.

Let $\varphi$ be a Langlands parameter vanishing on $\text{SL}_2(\mathbb{C})$.

Assume that $\varphi$ is *tame*: it vanishes on wild inertia.

Assume that $\varphi$ is *discrete*: the centralizer of $\varphi$ in $\hat{G}$ is finite modulo the center of $^L G$.

Assume that $\varphi$ is *regular*: the image of inertia is generated by a semisimple element of $\hat{G}$ whose centralizer is a maximal torus $\hat{S}$.

DeBacker-Reeder produce an L-packet that satisfies many of the properties expected of the local Langlands correspondence.
For each $\lambda \in X^*(\hat{S})$ they construct

- $F_{\lambda}$, a twisted action of Frobenius on $G(\overline{Q}_p)$, and
- $\pi_{\lambda}$, a representation of $G(\overline{Q}_p)^{F_{\lambda}}$.

They define an equivalence relation on such pairs, and prove that the equivalence class of $(\pi_{\lambda}, F_{\lambda})$ depends only on the class of $\lambda$ in

$$X^*(\hat{S})/(1 - w\theta)X^*(\hat{S}) \cong A^\vee_\varphi$$

where $w\theta$ is the automorphism of $X^*(\hat{S})$ induced by $\varphi(F) \in N_L G(\hat{S})$. The $\lambda$ with image in $A^\vee_\varphi$ are those with $G(\overline{Q}_p)^{F_{\lambda}} \cong G(\overline{Q}_p)$, and the corresponding equivalence classes of $\pi_{\lambda}$ form the L-packet associated to $\varphi$. 

The Construction of $\pi_\lambda$

- Using the Bruhat-Tits building they construct an anisotopic torus $T_\lambda$ in $G$,
- apply a canonical modification to $\varphi$ so that the image lies in a group isomorphic to $^LT_\lambda$,
- obtain a character of $T_\lambda(\mathbb{F}_p)$ using the (depth-preserving) local Langlands correspondence for tori,
- use Deligne-Lusztig theory to produce an irreducible representation of the parahoric subgroup $G_\lambda$, and
- compactly induce to $G(\overline{\mathbb{Q}}_p)^{F_\lambda}$, yielding a depth zero supercuspidal representation $\pi_\lambda$. 
They then prove that $G(\mathbb{Q}_p)$ acts on the pairs $(F_\lambda, \pi_\lambda)$, and the orbit of a given pair is independent of all choices. Moreover, two such pairs are equivalent if and only if the two $\lambda$s are equivalent modulo $(1 - w\theta)X^*(\hat{S})$. Much of their paper is then devoted to proving that this construction yields L-packets with desirable properties:

- The ratio of formal degrees $\text{deg}(\pi_\lambda)/\text{deg}(\text{St}_\lambda)$ is independent of $\lambda$.
- Generic representations in the L-packet correspond to hyperspecial vertices in the building.
- Their L-packet yields a stable class function on the set of strongly regular semisimple elements of $G(\mathbb{Q}_p)$. 
We say that a Langlands parameter $\varphi$ is

- **discrete** if $Z_{\hat{G}}(\varphi)$ is finite,
- **tame** if $\varphi$ factors through the maximal tame quotient (and thus $p \neq 2$).
- **regular** if $Z_{\hat{G}}(\varphi(\tilde{\tau}))$ is connected and minimum dimensional (here $\tilde{\tau}$ is a procyclic generator of tame inertia).

We will construct an L-packet of supercuspidal representations of $G(\mathbb{Q}_p)$ given a tame, discrete regular parameter.
Filtrations

$G(\mathbb{Q}_p)$ acts on the Bruhat-Tits building $B(G)$, and we can classify the compact subgroups of $G(\mathbb{Q}_p)$ as stabilizers of convex subsets of $B(G)$.

- Any compact subgroup can be written as $H(\mathbb{Z}_p)$ for some $\mathbb{Z}_p$-scheme $H$.
- There is a decreasing filtration on each compact subgroup.
- $H^0$ is the schematic closure of the identity component on the special fiber and is of finite index in $H$.
- $H(\mathbb{F}_p)$ is given by $H/H^{0+}$.
- The filtration on $T$ is the one given by Moy and Prasad, coming from the filtration on $\mathbb{Q}_p^\times$.

We can thus obtain representations of compact subgroups of $G$ by pulling back representations of reductive groups over finite fields.
Our plan for constructing an L-packet from $\varphi$ is as follows. We construct:

- A maximal unramified anisotropic torus $T$, which embeds into $G$ in various ways,
- A character $\chi_\varphi$ on $T^0$ that vanishes on $T^{0+}$,
- For each $\rho \in A^\vee$, an embedding of $T$ into a maximal compact subgroup $H \subset G$.
- We get a Deligne-Lusztig representation of $H^0(F_p) = H^0/H^{0+}$ associated to the torus $T^0(F_p) = T^0/T^{0+}$ and the character $\chi_\varphi$.
- We induce this representation up to a representation of $G$. 

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The tame Weil group is topologically generated by two elements: an (arithmetic) Frobenius $F$ and a generator $\tilde{\tau}$ of the procyclic group

$$\mathcal{I}_{\mathbb{Q}_p} = \text{Gal}(\lim_{\to} \tilde{K}(p^{1/m})/\tilde{K}) \cong \prod_{\ell \neq p} \mathbb{Z}_\ell.$$

- The assumption that $E/\mathbb{Q}_p$ is totally ramified implies that $\varphi(F) \in \hat{G}$, while $\varphi(\tilde{\tau}) \in L^*G$ projects to $\tau \in \text{Gal}(E/\mathbb{Q}_p)$.
- Recall that we have a specified maximal torus $\hat{S}$ in $L^*G$. As Langlands parameters are defined only up to conjugacy, we may conjugate so that $\varphi(\tilde{\tau}) \in \hat{S}^{\tau} \rtimes \text{Gal}(E/\mathbb{Q}_p)$.
A Twisted Torus

- The equality
  \[ F \tilde{\tau} F = \tilde{\tau}^p \]
  implies that \( \varphi(F) \) lies in the normalizer of \( \varphi(\tilde{\tau}) \), and thus in the normalizer of \( \hat{S} \).
- Composing with the projection onto the Weyl group, we get a cocycle in
  \[ H^1(\langle F \rangle, W^I) \hookrightarrow H^1(\mathbb{Q}_p, W). \]
  Such a cocycle is precisely the data needed to define a torus over \( \mathbb{Q}_p \) as a twist of \( S \): here we’ve identified the Weyl groups of \( S \) and \( \hat{S} \). Write \( T \) for this torus.
Unramified and Anisotropic

- $T$ cannot literally be unramified, since no torus in $G$ splits over an unramified extension. But it does become isomorphic to the canonical torus $S$ after an unramified extension: we will call such tori in $G$ *unramified*.

- A torus $T$ is called *anisotropic* if $X_\ast(T)^{\text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)} = 0$, or equivalently if $T(\mathbb{Q}_p)$ is compact. The action of inertia on $T$ is the same as on $\hat{S}$, so any invariants in $X_\ast(T)$ would yield invariants in $X_\ast(S^T)$ under the action of $\varphi(F)$. But any such invariants would contradict our assumption that $\varphi$ is discrete, since

$$\left(\hat{g}^T\right)^F = 0.$$

Thus $T$ is anisotropic.
Since the tame Weil group is topologically generated by $F$ and $\tilde{\tau}$, the image of $\varphi$ is contained in $N_{\hat{G}}(\hat{S}) \times \text{Gal}(E/\mathbb{Q}_p)$. In fact, it is contained in the subgroup $D$ of $L^G$ generated by $\hat{S} \rtimes \text{Gal}(E/\mathbb{Q}_p)$ and $\varphi(F)$.

The minimal splitting field $M = \mathbb{Q}_p^s \cdot E$ of $T$ has Galois group

$$\text{Gal}(M/\mathbb{Q}_p) \cong \text{Gal}(E/\mathbb{Q}_p) \times \langle w \rangle,$$

where $w \in W^I$ is the image of $\varphi(F)$. Thus $D$ fits into an exact sequence

$$1 \to \hat{S} \to D \to \text{Gal}(M/\mathbb{Q}_p) \to 1.$$
Suppose that this sequence split and $D \cong \hat{T} \rtimes \text{Gal}(M/\mathbb{Q}_p)$. Then $\varphi$ would yield an element of $H^1(\mathbb{Q}_p, \hat{T})$, and the local Langlands correspondence for tori would give us a character of $T(\mathbb{Q}_p)$:

$$H^1(\mathbb{Q}_p, \hat{T}) \cong \text{Hom}(T(\mathbb{Q}_p), \mathbb{C}^\times).$$

In general the sequence for $D$ does not split. So our next task is to modify the Langlands correspondence for tori to obtain a character in the non-split case. We will get a character $\chi_\varphi$ of $T^0(\mathbb{Q}_p)$. 
Constructing $\chi_\varphi$

- Let $D_s$ be the subgroup of $D$ generated by $\hat{T}$ and $(1, \tau)$; the splitting $\text{Gal}(E/\mathbb{Q}_p) \to ^L\mathbb{G}$ implies $D_s \cong \hat{T} \rtimes \text{Gal}(M/\mathbb{Q}_p^s) \cong ^L\mathbb{T}$. Since $\varphi(\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p^s)) \subset D_s$, the local Langlands correspondence for tori gives a character $\chi$ of $T(\mathbb{Q}_p^s)$.

- Let $\Gamma = \text{Gal}(\mathbb{Q}_p^s/\mathbb{Q}_p)$. Since $\chi$ was determined by the restriction of a parameter on all of $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, it factors through the coinvariants $T(\mathbb{Q}_p^s)_{\text{Gal}(\mathbb{Q}_p^s/\mathbb{Q}_p)}$.

- From Tate cohomology we have

$$1 \to \hat{H}^{-1}(\Gamma, T) \to T(\mathbb{Q}_p^s)_{\Gamma} \to T(\mathbb{Q}_p) \to \hat{H}^0(\Gamma, T) \to 1$$

When $T^0(\mathbb{Q}_p) \neq T(\mathbb{Q}_p)$, the outer groups can be nontrivial.
Using Lang’s theorem on the cohomology of connected algebraic groups over finite fields, the corresponding outer terms for $T^0$ vanish. We define $\chi_\varphi$ as the restriction of $\chi$ to $T^0(\mathbb{Q}_p)^\Gamma \cong T^0(\mathbb{Q}_p)$.

Since $\varphi$ vanished on wild inertia, the depth-preservation properties of the local Langlands correspondence for tori imply that $\chi_\varphi$ vanishes on $T^{0+}(\mathbb{Q}_p)$, and thus induces a character of $T^0(\mathbb{F}_p)$.

The regularity of $\varphi$ implies that $\chi_\varphi$ is not fixed by any element of $W^I$: it is in “general position.”
Summary

From a Langlands parameter \( \varphi \) we’ve produced:

- An anisotropic unramified torus \( T \). Note that \( T \) is not yet provided with an embedding into \( G \).
- A character \( \chi_\varphi \) of \( T^0(\mathbb{F}_p) \).

In order to produce representations of \( G(\mathbb{Q}_p) \) we need to understand the embeddings of \( T \) into \( G \).
We classify unramified anisotropic twists of the “quasi-split” torus $S$. For each $s = 2r$, define $T_s = \{ x \in E_s : \text{Nm}_{E_s/L_r} x = 1 \}$,

![Diagram]

Every anisotropic unramified torus in $G$ is a product of such basic tori, together with at most one copy of $U_1$. 

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The Local Langlands Correspondence and character sheaves
Embeddings of Basic Tori

In order to get Deligne-Lustig representations, we need to embed $T$ into maximal compacts of $G$. We do so by building a Hermitian space around each basic torus in the product decomposition of $T$.

For each $\kappa \in L_r^\times$, we define a Hermitian product on $E_s$

$$\phi_\kappa(x, y) = \text{Tr}_{E_s/E}(\frac{\kappa}{\pi_L} x \cdot \eta_s(y)).$$

This Hermitian space is quasi-split if and only if $v_L(\kappa)$ is even. By the definition of $T_s$ we have an embedding of $T_s$ into $U(E_s, \phi_\kappa)$.
Embeddings of General Tori

In general, we choose a $\kappa_i$ for each basic torus in the decomposition of $T$. This choice corresponds to a choice of $\rho \in A_\varphi^\vee$ as long as the sum of the valuations of the $\kappa_i$ is even.

We prove $T$ fixes a unique point on the building $B(G)$ and thus embeds in a unique maximal compact $H \subset G$. The reduction of $H$ is

$$O(m) \times \text{Sp}(m'),$$

where $m$ is the sum of the dimensions of basic tori whose $\kappa_i$ has even valuation and $m'$ is the sum of those with $v(\kappa_i)$ odd.
Modulo $p$, we have a maximal torus $T^0(\mathbb{F}_p)$ sitting in a connected reductive group $H^0(\mathbb{F}_p)$ and a character $\chi_\varphi$ of $T^0(\mathbb{F}_p)$. This situation was studied by Deligne and Lusztig, and they produce a representation of $H^0(\mathbb{F}_p)$ using étale cohomology. The irreducibility of this representation follows from the regularity condition on $\varphi$. We pull back to $H^0$ and the only wrinkle in the induction process occurs between $H^0$ and $H$. Once we have a representation of $H$, we define a representation on all of $G(\mathbb{Q}_p)$ by compact induction.
A Finite Induction

There are three cases for the induction from $H^0$ to $H$.

- $n$ even, $H(\mathbb{F}_p) = \text{Sp}(n)$. Here $H = H^0$ and there is no induction.

- $n$ even, otherwise. The fact that the normalizer of $T^0(\mathbb{F}_p)$ in $H(\mathbb{F}_p)$ contains the normalizer in $H^0(\mathbb{F}_p)$ with index 2 implies that the induction remains irreducible.

- $n$ odd. Now the induction from $H^0$ to $H$ splits into two irreducible components. We can pick one using a recipe for the central character, together with the fact that in the case that $n$ is odd the center of $O(m)$ is not contained in $SO(m)$. 

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The Local Langlands Correspondence and character sheaves
Two Paths

\[ T^0(\mathbb{Q}_p) \xrightarrow{\chi_\varphi} \mathbb{C}^* \]

Deligne-Lustzig representation

sheaf-function dictionary

H^0(F_p) \subseteq DL(\chi_\varphi, T^0/\mathbb{F}_p)

inflation

H^0(\mathbb{Q}_p) \subseteq DL(\chi_\varphi, T^0/\mathbb{F}_p)

finite induction

H(\mathbb{Q}_p) \subseteq \text{Ind} \ DL(\chi_\varphi, T^0/\mathbb{F}_p)

compact induction

G(\mathbb{Q}_p) \subseteq c\text{Ind} \ DL(\chi_\varphi, T^0/\mathbb{F}_p)

distribution on G(\mathbb{Q}_p)

character sheaf on \mathfrak{T}/\mathbb{F}_p

Deligne-Lusztig induction

perverse sheaf on \mathfrak{G}/\mathbb{F}_p

nearby cycles or trace

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The remainder of this talk is

- joint with Clifton Cunningham
- a summary of work in progress.

The right hand side of the diagram outlines an alternate construction of a distribution on $G(\mathbb{Q}_p)$ from a depth zero character on $T^0(\mathbb{Q}_p)$ and an embedding $T \hookrightarrow G$.

Warning: no step on the right side is complete

For the remainder of this talk I will discuss the first arrow: the passage from a depth zero character of $T$ to a character sheaf on a related scheme $\mathcal{T}$. 
Now let $T = \mathbb{G}_m$. The Néron model of $T$ is a separated, smooth commutative group scheme $T_{\mathbb{Z}_p}$ locally of finite type over $\mathbb{Z}_p$ with the Néron mapping property. In particular,

$$T_{\mathbb{Z}_p}(\mathbb{Z}_p) = T(\mathbb{Q}_p) = \mathbb{Q}_p^\times.$$ 

The earlier $T^0$ is just the identity component of the Néron model, and in the $\mathbb{G}_m$ case the Néron model is a union of copies of $\mathbb{G}_m/\mathbb{Z}_p$, glued along the generic fiber. Set $T_d = T_{\mathbb{Z}_p} \times_{\mathbb{Z}_p} (\mathbb{Z}/p^{d+1}\mathbb{Z})$. 
The Greenberg functor

The Greenberg functor $\text{Gr}$ takes an affine group scheme over an Artinian local ring $A$ and produces an affine group scheme over the residue field $k$ whose $k$ points are canonically identified with the $A$-points of the original scheme. We set

$$\mathcal{T}_d = \text{Gr}(\mathcal{T}_d)$$

and

$$\mathcal{T} = \lim_{\leftarrow} \mathcal{T}_d.$$ 

$\mathcal{T}$ is a commutative group scheme over $\mathbb{F}_p$ with $\mathcal{T}(\mathbb{F}_p) = \mathbb{Q}_p^\times$, but it is neither connected nor locally of finite type.
An \(\ell\)-adic Weil local system on a scheme \(X\) over \(K\) is a pair \((\bar{\mathcal{L}}, \phi_{\mathcal{L}})\), where \(\bar{\mathcal{L}}\) is an \(\ell\)-adic local system on the étale site of \(X_{\bar{K}}\) and \(\phi_{\mathcal{L}}\) is an action of \(\text{Gal}(\bar{K}/K)\) on \(\bar{\mathcal{L}}\) compatible with the action on \(X_{\bar{K}}\).

An \(\ell\)-adic Weil character sheaf on a group scheme \(G\) is an \(\ell\)-adic Weil local system \(\mathcal{L}\) on \(G\) satisfying

\[
m^*(\bar{\mathcal{L}}) \cong \bar{\mathcal{L}} \boxtimes \bar{\mathcal{L}}
\]

as well as some compatibility conditions.

An \(\ell\)-adic Weil character sheaf on \(\mathfrak{T}\) is smooth of depth \(d\) if it arises as the pullback from \(\mathfrak{T}_d\) of an \(\ell\)-adic Weil character sheaf (with \(d\) minimal).
Theorem

There is a canonical, depth preserving isomorphism between smooth characters of $\mathbf{T}(\mathbb{Q}_p) = \mathbb{Q}_p^\times$ and smooth $\ell$-adic Weil character sheaves on $\mathbf{T}$. 
Summary

$\mathcal{W}_{\mathbb{Q}_p} \xrightarrow{\varphi} D \subset L\mathbf{G}$

$a$ twist $T$ of $S$

$H^0(\mathbb{F}_p) \preceq \text{DL}(\chi_\varphi, T^0/\mathbb{F}_p)$

$H(\mathbb{Q}_p) \preceq \text{Ind} \text{DL}(\chi_\varphi, T^0/\mathbb{F}_p)$

$G(\mathbb{Q}_p) \preceq \text{clInd} \text{DL}(\chi_\varphi, T^0/\mathbb{F}_p)$

$\mathcal{W}_{\mathbb{Q}_p} \xrightarrow{\varphi} \hat{S} \times \text{Gal}(E/\mathbb{Q}_p)$

$T^0(\mathbb{Q}_p) \xrightarrow{\chi_\varphi} \mathbb{C}^\times$

character sheaf on $\mathcal{T}/\mathbb{F}_p$

perverse sheaf on $\mathcal{G}/\mathbb{F}_p$

distribution on $G(\mathbb{Q}_p)$

David Roe
The Local Langlands Correspondence and character sheaves
References


