FORMAL GROUPS, COMPLEX COBORDISM AND QUILLEN'S THEOREM

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1. I

Quillen's theorem, giving an explicit isomorphism between the Lazard ring and the homotopy ring of the **MU** spectrum, has provided a guiding direction for my studies over the past month. Along the way, I have solidified and expanded my knowledge about formal groups. Though I did not think about the applications of formal groups to local class field theory or elliptic curves, what I picked up will certainly prove useful when I encounter them in the future. Learning about CW-spectra and beginning to think about the possible generalizations has proved quite interesting. Working with spectra is more to my taste than directly with CW-complexes. And learning the proof of Quillen's theorem itself has been rewarding.

The main source for Quillen's theorem has been Adams [1], and his book has proved a valuable resource in learning about spectra as well. Switzer's textbook [6] served to fill in some of the details that Adams glossed over. Finally, Strickland's course notes [5] provided a more modern perspective on the topic of formal groups and the symmetric 2-cocycle lemma.

The proof of Quillen's theorem is by no means self contained. We have chosen to focus on the theory of spectra and formal groups rather than learning enough about the Adams spectral sequence and the Steenrod algebra to completely understand certain calculations in Adam's proof. As a result some of the details of the proof are left as references to his book.

In Section 2, we present results about formal groups. Beginning with the definition of a formal group law and examples, we proceed to offer a possible definition of a formal group as a group object in a category of filtered algebras. The rest of Section 2 deals with the Lazard ring L, which has the universal property that giving a commutative formal group law over a ring R is the same as giving a ring homomorphism from L to R. We compute the structure of the Lazard ring, following a mixture of Adams [1, pp. 31-74] and Strickland [5, pp. 11-17], together with one modification in the proof of 2.17. Instead of a tedious analysis of $A = \mathbb{Z}/p\mathbb{Z}$, we are able to instead use the p-adic ring $\mathbb{Z}_p$ and leverage the fact that the desired result is easier to prove for torsion-free A. We retain enough of Adams notation and specific lemmas to make the proofs in Section 4 easier, but incorporate the notion of symmetric cocycles from Strickland in order to simplify and guide some of the lemmas needed. By the end of Section 2 we have a good description of L and a specific injection $L \to \mathbb{Z}[b_1, b_2, \ldots]$ that will prove useful in Section 4.

We take an interlude in Section 3 to give some of the definitions and results that we use from the theory of CW-complexes and CW-spectra. Adams [1], Hatcher [3], Milnor-Stasheff [4] and Switzer [6] provide the source for the material in this section. We begin with the definition of a CW-complex and outline some of the common geometric constructions such as suspension and smash product. We proceed in Section 3.2 to motivate the passage from CW-complexes to spectra. Two kinds of results motivate this change. The first is a number of theorems that suggest there should be a category in which suspension is invertible. In addition, Browns represntability theorem

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shows that any homology theory gives a way to generate a collection of spaces, indexed by the integers, and related by maps from the suspension of each to the next in line. This result provides the second source of inspiration for the definition of a spectra, which we then give, together with various associated definitions. Section 3.3 then consists of examples of spectra that appear in the rest of this work. The rest of the section is devoted to outlining some of the constructions done with spectra. We give a sketch of the construction of the smash product of spectra, the definition of homotopy, homology and cohomology groups and the definition of ring and module spectra. Finally, we give Adam's definition of an orientation on a ring spectra and include a primer on the Atiyah-Hirzebruch spectral sequence.

The final section consists of a proof of Quillens theorem, following Adams [1]. As mentioned above, some details are left out related to the use of the Adams spectral sequence.

2.1. **Formal Group Laws.**

Definition 2.1. Let R be a commutative ring with unit. A *formal group law* over R is a power series $\mu(x,y) = \sum_{i,j\geq 0} a_{ij}x^iy^j \in R[\![x,y]\!]$, satisfying

$$\mu(x,0) = x, \quad \mu(0,y) = y, \text{ and} \tag{2.1.1}$$

$$\mu(x,\mu(y,z)) = \mu(\mu(x,y),z). \tag{2.1.2}$$

If, in addition, $\mu(x,y) = \mu(y,x)$, then we say that the formal group law is commutative.

The first condition places obvious restrictions on the allowable a_{ij}: we must have $a_{i0} = \delta_{i1}$ and $a_{0j} = \delta_{1j}$, where δ is the Kronecker δ. How the second condition affects the a_{ij} is more difficult to determine. We will consider this question more in section 2.4.

Of course, we are missing one axiom of a group. This omission is justified by the following definition and lemma.

Definition 2.2. Given a formal group law μ over R, a *formal inverse* for μ is a power series $\iota(x) = \sum_{i\geq 1} a'_1x^i \in R[\![x]\!]$, satisfying

$$\mu(x,\iota(x)) = \mu(\iota(x),x) = 0. \tag{2.1.3}$$

Lemma 2.3. *Given any formal group law μ, there is a unique formal inverse ι for μ.*

Proof. Substituting $\iota(x)$ for y in μ, we get $x + \sum_{i\geq 1} a'_ix^i + \sum_{i,j\geq 1} a_{ij}x^i\left(\sum_{k\geq 1} a'_kx^k\right)^j = 0$. We have that a'_i appears as a coefficient of x^i in the first sum, and since i and j are both at least 1, only a'_k for $k < i$ appear in the coefficient of i in the second sum. So we can solve for the a'_i, thus proving existence of a right inverse, and uniqueness.

Suppose that $\iota(x)$ is a right inverse. Then, by associativity,

$$x = \mu(0,x) = \mu(\mu(x,\iota(x)),x) = \mu(x,\mu(\iota(x),x)) = x + \mu(\iota(x),x) + \sum_{i,j\geq 1} a_{ij}x^i\mu(\iota(x),x)^j.$$

Suppose that the coefficient of x^i in $\mu(\iota(x),x)$ is zero for $0 \leq i < n$, and equal to a'_n for $i = n$. Then reducing the above equation modulo x^{n+1} yields $a'_n = 0$. □

All of our formal group laws will be commutative; the third equation then translates to $a_{ij} = a_{ji}$.

If R is graded, then we give x, y, and $\mu(x,y)$ degree -2 (the topological reason for the factor of 2 is that the generator of $\mathbf{H}^*(\mathbb{C}P^\infty)$ lies in degree 2), and thus we ask that a_{ij} lie in degree $2(i + j - 1)$.

One can define formal group laws in more variables (see Fröhlich [2] or Strickland [5]), but we will only use the one dimensional case. We will, however, provide a more categorical approach in section 2.3 and define a formal group, rather than just a formal group law.

But before proceeding, we first define homomorphisms between formal group laws.

Definition 2.4. A *homomorphism* from a formal group law μ over R to a formal group law ν over R is a power series $f(x) \in R[\![x]\!]$ satisfying

$$\nu(f(x), f(y)) = f(\mu(x,y)) \tag{2.1.4}$$

An *isomorphism* of formal groups is a homomorphism with an inverse, as usual.

Lemma 2.5. *If $f = \sum_{i>=1} c_ix^i$, then f is an isomorphism if and only if c_1 is invertible in R.*

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2.2. **Examples.** We begin with two simple examples, then proceed with a number of situations in number theory and geometry where formal groups arise.

(i) The additive formal group law is defined by

$$\mu(x,y) = x + y \tag{2.2.1}$$

The inverse is then given by $\iota(x) = -x$.

(ii) The multiplicative formal group law is defined by

$$\mu(x,y) = x + y + xy \tag{2.2.2}$$

The inverse is given by $\iota(x) = -x + x^2 - x^3 + \cdots$.

(iii) If c is an invertible element in R, the Lorentz formal group law is given by

$$\mu(x,y) = \frac{x+y}{1 + xy/c^2} \tag{2.2.3}$$

This formal group law is related to the addition of parallel velocities in special relativity.

(iv) Let p be a prime and $f(x) \in \mathbb{Z}[x]$ be monic such that $f(x) \equiv px \pmod{x^2}$ and $f(x) \equiv x^{p^n} \pmod{p}$ for some $n > 0$. Then Lubin-Tate theory gives that there is a unique formal group law F over $\mathbb{Z}_p$ such that $f(F(x,y)) = F(f(x), f(y))$, and that for this group law if we add x to itself p times using the group law, we get $f(x)$.

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2.3. **A Categorical Approach.** There are a number of ways to make formal group laws into a category. The simplest way is to merely define a category $\mathcal{C}_{\mathrm{fgl}}$whose objects are formal group laws over R, and define the morphisms to be exactly the homomorphisms of formal group laws defined in section 2.1. But there are many other ways to put formal group laws into a categorical framework. We will show that it is reasonable to define a formal group to be a group object in the opposite category to a category of filtered R-algebras.

Definition 2.6. Suppose that $\mathcal{C}$ is a category with finite products and a terminal object $\mathbf{1}$. We define a group object in $\mathcal{C}$ to an object G in $\mathcal{C}$ together with morphisms

- $m\colon G \times G \to G$
- $e\colon \mathbf{1} \to G$
- $i\colon G \to G$

such that

(i) $m \circ (m \times \mathrm{Id}_G) = m \circ (\mathrm{Id}_G \times m)$

(ii) $m \circ (e \times \mathrm{Id}_G) = p_1$ and $m \circ (\mathrm{Id}_G \times e) = p_2$ where $p_1\colon \mathbf{1} \times G \to G$ and $p_2\colon G \times \mathbf{1} \to G$ are the canonical projection maps.

(iii) $m \circ (\mathrm{Id}_G \times i) \circ d = m \circ (i \times \mathrm{Id}_G) \circ d = e_G$, where $d\colon G \to G \times G$ is the diagonal map and e_G is the composition of the unique morphism $G \to \mathbf{1}$ with e.

We define $\mathrm{Grp}(\mathcal{C})$ to be the category whose objects are group objects in G and whose morphisms $(G, m_G, e_G, i_G) \xrightarrow{f} (H, m_H, e_H, i_H)$ are morphisms $f\colon G \to H$ in $\mathcal{C}$ that satisfy

$$\begin{array}{ccc} G \times G & \xrightarrow{m_G} & G \\ {\scriptstyle f\times f}\downarrow & & \downarrow{\scriptstyle f} \\ H \times H & \xrightarrow[m_H]{} & H \end{array}$$

$$\begin{array}{ccc} G & \xrightarrow{i_G} & G \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ H & \xrightarrow[i_H]{} & H \end{array}$$

and

$$\begin{array}{ccc} \mathbf{1} & \xrightarrow{e_G} & G \\ \Vert & & \downarrow{\scriptstyle f} \\ \mathbf{1} & \xrightarrow[e_H]{} & H \end{array}$$

For example, groups are group objects in the category of sets, Lie groups are group objects in the category of smooth manifolds, topological groups are group objects in the category of topological spaces and algebraic groups are group objects in the category of algebraic varieties.

Consider the category whose objects are filtered commutative algebras over R that are complete and Hausdorff for the filtration topology, and whose morphisms are filtration preserving homomorphisms. Let C_{alg}be the opposite category.

The R-algebra $R[\![x_1 \ldots x_n]\!]$, with the obvious filtration, is an object of C_{alg}. R is a terminal object of C_{alg}, and the Cartesian product of $R[\![x_1 \ldots x_n]\!]$ and $R[\![y_1 \ldots y_m]\!]$ is $R[\![x_1 \ldots x_n, y_1 \ldots y_m]\!]$. If G is the object $R[\![x]\!]$ in C_{alg}, then a map $m\colon G \times G \to G$ is a filtration preserving homomorphism $m\colon R[\![x]\!] \to R[\![x, y]\!]$, which is specified by the image $\mu(x, y) \in R[\![x, y]\!]$ of $x \in R[\![x]\!]$. So, given any formal group law μ, we have a corresponding morphism m_μ in C_{alg}. We also define $R \xrightarrow{e} R[\![x]\!]$ by $\sum_{i\geq 0} c_i x^i \mapsto c_0$ and $R[\![x]\!] \xrightarrow{i_\mu} R[\![x]\!]$ by $x \mapsto \iota(x)$, where ι_μ is the inverse associated to μ by Lemma 2.3. So for every formal group law over R we get a group object in C_{alg}.

Moreover, a morphism $f \in R[\![x]\!]$ in $C_{\text{fgl}}$of formal group laws specifies a morphism in $\mathrm{Grp}(C_{\text{alg}})$ from $R[\![x]\!]$ to $R[\![y]\!]$ by $y \mapsto f(y)$. So we have a functor $F_{\text{alg}}\colon C_{\text{fgl}} \to \mathrm{Grp}(C_{\text{alg}})$. In fact, the following proposition holds.

Proposition 2.7. *The functor F_{alg} is fully faithful, though not essentially surjective. The objects in the image consist of exactly the group objects in* $\mathrm{Grp}(C_{\text{alg}})$ *with $G = R[\![x]\!]$.*

One might define a formal group as a group object in C_{alg}. We will mainly consider formal group laws rather than formal groups because explicit power series are easier to work with. However, there are many other interpretations of formal groups. See Adams [1, pp. 44-46] for a reformulation in terms of Hopf algebras, and Fröhlich [2, pp. 29-42] for a more extensive treatment in the same direction. See Strickland [5, pp. 6-11] for a description of formal groups as formal schemes with an Abelian group structure on the fibers.

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2.4. **Lazard's Ring.** In this section we answer a question from Section 2.1: what constraints does associativity place on the coefficients a_{ij} of a formal group law? In particular, we define a universal ring L and compute its structure.

Theorem 2.8. *There is a commutative ring L with unit, and a commutative formal group law μ^L defined over L such that for any commutative ring R with unit, and any commutative formal group law μ^R defined over R there is a unique homomorphism $\theta_R\colon L \to R$ with $\theta_{R*}\mu^L = \mu^R$.*

Proof. We can define L by giving it as the quotient of a polynomial ring by an ideal of relations. Consider the polynomial ring defined on the variables $\{a_{ij} : i, j \geq 1\}$,

$$P = \mathbb{Z}[a_{11}, a_{12}, a_{21}, a_{13}, a_{22}, a_{31}, \ldots].$$

Define

$$\mu^L(x, y) = x + y + \sum_{i,j \geq 1} a_{ij} x^i y^j,$$

and define elements b_{ijk} of P by

$$\mu^L(x, \mu^L(y, z)) - \mu^L(\mu^L(x, y), z) = \sum_{i,j,k} b_{ijk} x^i y^j z^k.$$

Let I be the ideal generated by the b_{ijk} and $a_{ij} - a_{ji}$. Set $L = P/I$. Since we have quotiented out by exactly the relations giving associativity and commutativity, μ^L is a commutative formal group law. Conversely, given any R and μ^R, the fact that polynomial rings over $\mathbb{Z}$ are the free in the category of rings allow us to define a unique ring homomorphism $\phi \colon L \to R$ sending a_{ij} to the corresponding coefficient of μ^R. Since μ^R is associatiative and commutative, this homomorphism descends to a homomorphism from $L = P/I$. $\square$

Yet this description of L is not totally satisfactory: we cannot specify a morphism from L by just arbitrarily giving the images of the a_{ij}, because there are relations among them. In the rest of this section we work to give a more useful description of L.

Give a_{ij} degree $2(i + j - 1)$. Then b_{ijk} is a homogeneous polynomial of degree $2(i + j + k - 1)$. It turns out that, after quotienting by I, there is exactly one generator remaining in each positive even degree.

Set R to be the ring $\mathbb{Z}[b_1, b_2, \ldots]$, where b_i is given degree $2i$. Set $b_0 = 1$ in this ring. Define the power series $\exp(y) = \sum_{i \geq 0} b_i y^{i+1} \in R[\![y]\!]$, and $\log(x)$ as the power series inverse of exp. Define

$$\mu^R(x_1, x_2) = \exp(\log(x_1) + \log(x_2)). \tag{2.4.1}$$

This is just the image of the additive formal group law under the homomorphism of formal group laws defined by log. By Theorem 2.8, there is a unique homomorphism $\theta_R \colon L \to R$ carrying μ^L to μ^R. Note that θ_R preserves degree.

Theorem 2.9. *The homomorphism θ_R is injective. Moreover, L is isomorphic to a polynomial ring on generators in degree* $2, 4, \ldots,$ *though θ_R is not an isomorphism.*

Proof. We begin by defining the *indecomposible quotient* of a graded ring R, which will provide a tool for isolating generators of R.

Definition 2.10. Given a graded ring $S = \bigoplus_{n \geq 0} S_n$, we say that S is connected if $S_0 = \mathbb{Z}$. In this case, defined the *augmentation ideal* I_S to be the ideal

$$I_S = \bigoplus_{n > 0} S_n.$$

Elements of I_S^2 are called *decomposible elements* of S, because they can be written in terms of generators of lower degree. We can define a graded abelian group $Q_*(S)$, called the *indecomposible quotient*, by

$$Q_*(S) = I_S / I_S^2.$$

In fact, Q_* is a functor from connected graded rings to graded abelian groups, if we consider only morphisms that preserve degree. If $x \in I_S$, write $\overline{x}$ for the image of x in $Q_*(S)$.

Since R and L are nonzero only in nonnegative even degrees, $Q_m(R)$ and $Q_m(L)$ will both be zero unless $m = 2n$ with $n > 0$, in which case $Q_{2n}(R) \cong \mathbb{Z}$ will be generated by $\overline{b_n}$. The following lemmas give us information about $Q_*(L)$ and $Q_*(\theta_R)$ that will allow us to prove Theorem 2.9; the proofs of these lemmas will be delayed until after the proof of Theorem 2.9.

Lemma 2.11.

(i) $\theta_R(a_{ij}) \equiv \frac{(i+j)!}{i!j!} b_{i+j-1} \pmod{I_R^2}$ *for* $i, j \geq 1$.

(ii) *Set* T_n *to be the image of* $Q_{2n}(\theta_R)\colon Q_{2n}(L) \to Q_{2n}(R)$, *and*

$$\gamma(n) = \begin{cases} p & \text{if } n+1 = p^f \text{ for some } p \text{ prime and } f \geq 1, \\ 1 & \text{otherwise.} \end{cases}$$

Then T_n *consists of the multiples of* $\gamma(n)\overline{b_n}$.

Given any abelian group A, we can make $\mathbb{Z} \oplus A$ into a ring by setting $(n, a) \cdot (m, b) = (nm, nb + ma)$. Let $\pi_{\mathbb{Z}}\colon \mathbb{Z} \oplus A \to \mathbb{Z}$ be the projection map. Whenever we speak of a formal group law μ on $\mathbb{Z} \oplus A$, we require that $(\pi_{\mathbb{Z}}\mu)(x, y) = x + y$. In addition, if A is graded then we require the coefficient of $x^i y^j$ in $\mu(x, y)$ to lie in degree $2(i + j - 1)$.

There is a natural formal group law on $\mathbb{Z} \oplus Q_{2n}(L)$, namely

$$\mu^{Q_{2n}(L)}(x, y) = x + y + \sum_{\substack{i,j \geq 1 \\ i+j=n+1}} \overline{a_{ij}} x^i y^j.$$

Lemma 2.12.

(i) *Given any abelian group concentrated in degree* $2n$ *and any commutative formal group law* μ^A *on* $\mathbb{Z} \oplus A$, *there is a unique homomorphism* $\varphi_A\colon Q_{2n}(L) \to A$ *such that* $\varphi_{A*}\mu^{Q_{2n}(L)} = \mu^A$.

(ii) *For any such A and any* μ^A, *the homomorphism* φ_A *factors through the homomorphism* $Q_{2n}(L) \to T_n$.

(iii) *The homomorphism* $Q_{2n}(L) \to T_n$ *is an isomorphism of groups.*

By Lemma 2, T_n is generated by a single element, and by Lemma 3 we can choose, for each $n > 0$, an element $t_n \in L_{2n}$ that projects to a generator of T_n. We get a map

$$\alpha\colon \mathbb{Z}[t_1, t_2, \ldots] \to L.$$

By Lemma 3, $Q_{2n}(\alpha)$ is an isomorphism for each n. In order to see that α is surjective, it is enough to say that every a_{ij} is in the image of α, since the a_{ij} generate L. We prove this by induction on $i + j$. If $i + j = 2$, then $\pm t_1$ must map to a_{11}, since $(I^2_{\mathbb{Z}[t_1,t_2,\ldots]})_2 = 0$ and $(I_L^2)_2 = 0$. Now consider some a_{ij} with $i + j > 2$. Since $Q_{2(i+j-1)}(\alpha)$ is an isomorphism, there exists $x \in \mathbb{Z}[t_1, t_2, \ldots]$ with $Q_{2(i+j-1)}(\alpha)(\overline{x}) = \overline{a_{ij}}$. Then $\alpha(x) \equiv a_{ij} \pmod{I_L^2}$. Write

$$\alpha(x) - a_{ij} = \sum_{k=1}^{\infty} \sum_{\substack{i_1,\ldots,i_k,j_1,\ldots,j_k \geq 1 \\ i_1+\cdots+i_k+j_1+\cdots+j_k-k=i+j-1}} c_{i_1,\ldots,i_k} \prod_{l=1}^{k} a_{i_l j_l}.$$

By induction, we can find x_{gh} with $\alpha(x_{gh}) = a_{gh}$ for $g + h < i + j$. Then

$$\alpha(x - \sum_{k=1}^{\infty} \sum_{\substack{i_1,\ldots,i_k,j_1,\ldots,j_k \geq 1 \\ i_1+\cdots+i_k+j_1+\cdots+j_k-k=i+j-1}} c_{i_1,\ldots,i_k} \prod_{l=1}^{k} x_{i_l j_l}) = a_{ij}.$$

But the composite $\theta_R \circ \alpha\colon \mathbb{Z}[t_1, t_2, \ldots] \xrightarrow{\alpha} L \xrightarrow{\theta_R} R = \mathbb{Z}[b_1, b_2, \ldots]$ is injective, since $\theta_R(\alpha(t_n)) \equiv \gamma(n)b_n \pmod{I_R^2}$. Therefore α is also injective, and thus an isomorphism. Since α is an isomorphism, the injectivity of the composition now implies the injectivity of θ_R. $\square$

We now need to fill in the gaps by giving proofs of Lemmas 2.11 and 2.12. We begin with two preliminary propositions. The first relates the coefficients of exp and log, which will be needed in the proof of Lemma 2.11. The second gives an elementary result giving the p-adic valuation of $\binom{n}{m}$, which will be used in the proofs of both lemmas.

Proposition 2.13. *Set* $\exp(y) = \sum_{i\geq 0} b_i y^{i+1}$ *and* $\log(x) = \sum_{i\geq 0} m_i x^{i+1}$, *and denote the component in dimension* $2i$ *of an inhomogeneous sum* S *by* S_i *(after embedding* $\mathbb{Z}[b_1, b_2, \ldots]$ *into* $\mathbb{Z}[\![b_1, b_2, \ldots]\!]$*).*

(i) *The coefficients* b_n *and* m_n *of* exp *and* log *are related by*

$$m_n = \frac{1}{n+1}\left(\left(\sum_{i=0}^{\infty} b_i\right)^{-n-1}\right)_n \tag{2.4.2}$$

$$b_n = \frac{1}{n+1}\left(\left(\sum_{i=0}^{\infty} m_i\right)^{-n-1}\right)_n \tag{2.4.3}$$

(ii) $m_0 = 1$ *and* $m_n \equiv -b_n \pmod{I_R^2}$ *for* $n \geq 1$.

Proof. If

$$\omega = \sum_{i\geq -N} c_i y^i dy,$$

define $\operatorname{res}(\omega)$ to be c_{-1}, the residue of ω at $y = 0$. Then $\left((\sum_{i\geq 0} b_i)^{-n-1}\right)_n$ is the coefficient of y^n in $\left(\sum_{i\geq 0} b_i y^i\right)^{-n-1}$, which is the same as the coefficent of y^{-1} in $\left(\sum_{i\geq 0} b_i y^{i+1}\right)^{-n-1}$. So

$$\begin{aligned}
\left(\left(\sum_{i\geq 0} b_i\right)^{-n-1}\right)_n &= \operatorname{res}(x^{-n-1}dy)\\
&= \operatorname{res}(x^{-n-1}\frac{dy}{dx}dx)\\
&= \operatorname{res}(x^{-n-1}\left(\sum_{j\geq 0} m_j(j+1)x^j\right)dx)\\
&= (n+1)m_n.
\end{aligned}$$

The proof for (2.4.3) is entirely symmetric.

We now prove the second part. The fact that $m_0 = 1$ follows from the definition $b_0 = 1$. By the binomial theorem, for $n \geq 1$,

$$\begin{aligned}
\left(1 + \sum_{i=1}^{\infty} b_i\right)^{-n-1} &= \sum_{k=0}^{\infty}\binom{n+k}{k}(-1)^k\left(\sum_{i=1}^{\infty} b_i\right)^k\\
&\equiv 1 - (n+1)\left(\sum_{i=1}^{\infty} b_i\right) \pmod{I_R^2}.
\end{aligned}$$

The only term in this last sum that lies in degree $2n$ is $-b_n$, so by (2.4.2) we have

$$m_n \equiv -b_n \pmod{I_R^2}. \quad \square$$

Proposition 2.14. *Let q be any prime, and for $x \in \mathbb{Q}$ let $v_q(x)$ be the largest power of q dividing x. Then $v_q(\binom{i+j}{i})$ is equal to the number of carries that occur when i and j are added in base q.*

Proof. If $i = 0$ or $j = 0$ the result is clear. Otherwise, write $i = \kappa_0 + \kappa_1 q + \cdots + \kappa_r q^r$ and $j = \lambda_0 + \lambda_1 q + \cdots + \lambda_r q^r$ and $i + j = \tau_0 + \tau_1 q + \cdots + \tau_r q^r$ with all κ_s, λ_s and τ_s between 0 and $q-1$. If a carry occurs in position s, set $c(s) = 1$; otherwise set $c(s) = 0$. In addition, set $c(-1) = 0$ and $c(s) = 0$ for $s \geq r$. Then

$$\tau_s = \kappa_s + \lambda_s + c(s-1) - qc(s).$$

Counting the number of multiples of q^j occuring below i, we have

$$\begin{aligned} v_q(i!) &= \left\lfloor \frac{i}{q} \right\rfloor + \left\lfloor \frac{i}{q^2} \right\rfloor + \cdots \\ &= \sum_{s=1}^{r} \sum_{t=s}^{r} \kappa_t q^{t-s} \end{aligned}$$

and likewise for $v_q(j!)$ and $v_q((i+j)!)$. Therefore

$$\begin{aligned} v_q(\binom{i+j}{i}) &= v_q((i+j)!) - v_q(i!) - v_q(j!) \\ &= \sum_{s=1}^{r} \sum_{t=s}^{r} \tau_t q^{t-s} - \sum_{s=1}^{r} \sum_{t=s}^{r} \kappa_t q^{t-s} - \sum_{s=1}^{r} \sum_{t=s}^{r} \lambda_t q^{t-s} \\ &= \sum_{s=1}^{r} \sum_{t=s}^{r} (c(t-1) - qc(t)) q^{t-s} \\ &= \sum_{s=1}^{r} c(s-1) + \sum_{s=1}^{r} \sum_{t=s}^{r-1} c(t) q^{t-s+1} - \sum_{s=1}^{r} \sum_{t=s}^{r} c(t) q^{t-s+1} \\ &= \sum_{s=0}^{r-1} c(s), \end{aligned}$$

where this last equality holds since $c(r) = 0$. This last sum is exactly the number of carries occuring when adding i and j in base q. $\square$

We now proceed to the proof of Lemma 2.11.

Proof of Lemma 2.11.

(i) We compute the coefficient of $x_1^i x_2^j$ in the power series $\mu^R(x_1, x_2) = \exp(\log(x_1) + \log(x_2))$.

$$\begin{aligned}\exp(\log(x_1)+\log(x_2)) &= \sum_{k\geq 0} b_k\left(\sum_{l\geq 0} m_l(x_1^{l+1}+x_2^{l+1})\right)^{k+1}\\ &\equiv x_1+x_2-\sum_{l\geq 1} b_l(x_1^{l+1}+x_2^{l+1})+\sum_{k\geq 1} b_k\left(x_1+x_2-\sum_{l\geq 1} b_l(x_1^{l+1}+x_2^{l+1})\right)^{k+1}\\ &\equiv x_1+x_2-\sum_{l\geq 1} b_l(x_1^{l+1}+x_2^{l+1})+\sum_{k\geq 1} b_k(x_1+x_2)^{k+1}.\end{aligned}$$

Thus the coefficient of $x_1^i x_2^j$ is $\binom{i+j}{i} = \frac{(i+j)!}{i!j!}$ for $i, j \geq 1$.

(ii) We need to prove that the image of $Q_{2n}(\theta_R)$ consists of multiples of either $\overline{b_n}$ or of $p\overline{b_n}$ (if $n+1 = p^f$).

Since L is generated as a ring by the a_{ij}, which lie in degree $2(i+j-1)$, $Q_{2n}(L)$ is generated as an abelian group by a_{ij} for $i+j = n+1$. By the previous part, it suffices to show that the greatest common divisor of $\{\binom{n+1}{i}\colon 1 \leq i \leq n\}$ is $\gamma(n+1)$. If $n+1 = p^f$, then both i and j have f digits but their sum has $f+1$, so at least one carry occurs for every i in the range $1 \leq i \leq n$. Thus $v_p(\gamma(p^f-1)) \geq 1$ by Proposition 2.14. If $i = p^{f-1}$ and $j = (p-1)p^{f-1}$, then exactly one carry occurs, so in fact $v_p(\gamma(p^f-1)) = 1$. Now suppose q is any prime and $n+1 \neq q^f$ for any $f \geq 1$, and write the base q expansion of $n+1$ as $\kappa_0 + \kappa_1 q + \cdots \kappa_r q^r$. Suppose first that $\kappa_j \neq 0$ for some j in the range $0 \leq j < r$. If no such j exists, then we must have $n+1 = aq^r$ with $2 \leq a < q$; in this case set $j = r$. In either situation, we can write $n+1 = q^j + (n+1-q^j)$, and no carries occur in that sum. Therefore $v_q(\binom{n+1}{q^j}) = 0$, so $v_q(\gamma(n+1)) = 0$. Since this holds for an arbitrary prime q, we have the desired conclusion.

□

In order to prove Lemma 2.12, we first develop the machinery of symmetric 2-cocycles. Many of the arguments here are due to [5, p. 11-16].

Definition 2.15. Let $A[\![x,y]\!]$ be the abelian group consisting of formal sums $\sum_{i,j\geq 0} a_{ij}x^iy^j$ with $a_{ij} \in A$. A *symmetric 2-cocycle* with coefficients in A is a power series $f(x,y) \in A[\![x,y]\!]$ such that $f(x,y) = f(y,x)$ and $f(x,0) = 0$ and

$$f(y,z) - f(x+y,z) + f(x,y+z) - f(x,y) = 0.$$

Write $Z(A)$ for the set of such f's and $Z_d(A)$ for the subset consisting of homogeneous polynomials of degree d.

We have that $Z_0(A) = Z_1(A) = 0$ by the conditions that $f(x,0) = 0$ and $f(x,y) = f(y,x)$. It is also clear that $Z(A) = \prod_{d>1} Z_d(A)$. The following desciption of $Z(A)$, together with the isomorphism $Z_d(A) \cong A$ which we will prove in Lemma 2.17, form the key ingredients in the proof of Lemma 2.12.

Lemma 2.16. *There is a natural ismorphism of abelian groups* $Z(A) = \mathrm{Hom}(Q_*(L),A)$*, where the* Hom *is in the category of abelian groups. Under this isomorphism,* $Z_d(A)$ *is identified with* $\mathrm{Hom}(Q_{2(d-1)}(L),A)$.

Proof. Let $Y(A)$ be the set of formal group laws μ over $\mathbb{Z} \oplus A$. We have $\mu(x, y) = x + y + f(x, y)$ for some $f(x, y) \in A[\![x, y]\!]$. The conditions $\mu(x, 0) = x$ and $\mu(x, y)$ translate directly to $f(x, 0) = 0$ and $f(x, y) = f(y, x)$, and we have

$$\mu(\mu(x, y), z) = x + y + z + f(x, y) + f(x + y + f(x, y), z).$$

Since f has coefficients in A and $A^2 = 0$ in $\mathbb{Z} \oplus A$, the last term equals $f(x + y, z)$. Thus the associativity condition on μ translates to the cocycle condition on f. The map $\mu \mapsto f$ thus gives a bijection $Y(A) = Z(A)$.

However, by Theorem 2.8, $Y(A)$ bijects with ring homomorphisms $\phi\colon L \to \mathbb{Z} \oplus A$, and the condition that $(\pi_{\mathbb{Z}}\mu)(x, y) = x + y)$ translates to the condition that $\pi_{\mathbb{Z}}\phi(I_L)$, or equivalently that $\phi(I_L) \subset A$. If this is the case, then $\phi(I_L^2) \subset A^2 = 0$, so ϕ induces a homomorphism $Q_*(L) = I_L/I_L^2 \to A$.

Since $Q_*(L) = \bigoplus_{n\geq 1} Q_{2n}(L)$, by the duality of direct sums and products, it suffices to check that the map just defined restricts to an isomorphism of abelian groups $Z_d(A) \to \mathrm{Hom}(Q_{2(d-1)}(L), A)$ for $d > 1$. A cocycle $f = \sum_{i+j=d} \alpha_{ij} x^i y^j \in Z_d(A)$ corresponds to $x + y + \sum_{i+j=d} \alpha_{ij} x^i y^j \in Y(A)$, which corresponds to the homomorphism $\phi_f\colon Q_{2(d-1)}(L) \to A$ given by $\phi_f(a_{ij}) = \alpha_{ij}$. The product of these correspondences gives our map $Z(A) = \prod_{d>1} Z_d(A) \to \mathrm{Hom}(\oplus_{d>1} Q_{2(d-1)}(L), A) = \mathrm{Hom}(Q_*(L), A)$. Moreover, $f + g \mapsto \phi_{f+g} = \phi_f + \phi_g$, so our correspondence is a homomorphism. In addition, $\phi_f = 0$ implies $\alpha_{ij} = 0$ for all i, j which implies $f = 0$, and conversely the conditions on the a_{ij} in L are precisely those required to guarantee that for a homomorphism $\phi\colon Q_{2(d-1)}(L) \to A$ there exists a cocycle f with $\phi = \phi_f$. $\square$

Given any A we can define the map $\phi_A\colon A \to Z_d(A)$ by $\phi_A(a) = a \sum_{i=1}^{d-1} \frac{1}{\gamma(d)} \binom{d}{i} x^i y^{d-i}$. This makes sense since $\gamma(d)$ is the greatest common divisor of all of the $\binom{d}{i}$, and it's a symmetric 2-cocycle since $\phi_A(a)(x, y) = \frac{a}{\gamma(d)}((x + y)^d - x^d - y^d)$.

Fix a choice of integers $\eta_{i,d}$ for $d > 1$ and $0 < i < d$ such that

$$\gamma(d) = \sum_{i=1}^{d-1} \eta_{i,d} \binom{d}{i}.$$

Then define a map $\psi_A\colon Z_d(A) \to A$ by

$$\psi_A(\sum_{i=1} d - 1\alpha_i x^i y^{d-i}) = \sum_{i=1} d - 1\eta_{i,d}\alpha_i$$

By construction, $\psi_A \circ \phi_A = \mathrm{Id}_A$ for all A and all $d > 1$, so ϕ_A is always a split monomorphism.

Lemma 2.17. *ϕ_A is an isomorphism for every abelian group A.*

Proof. We proceed by analyzing various possibilities for A. The first case is when A is a vector space over $\mathbb{Q}$. This case then allows us to prove the result for any torsion free A. We use an argument involving the p-adic integers $\mathbb{Z}_p$ in order to show the result for $\mathbb{Z}/p^f\mathbb{Z}$. The classification of finitely generated abelian group then implies the conclusion for any finitely generated A, and a simple argument extends the result to an arbitrary abelian group.

(i) Suppose first that A is a vector space over $\mathbb{Q}$. Define $\psi\colon Z_d(A) \to A$ by $\psi(\sum_i a_i x^i y^{d-i}) = \gamma(d)a_1/d$. We have

$$\psi(\phi_A(a)) = \psi(a \sum_{i=1} d - 1 \frac{1}{\gamma(d)} \binom{d}{i} x^i y^{d-i}) = a,$$

so $\psi\phi_A = \mathrm{Id}_A$. Now we show that ψ is injective. Suppose $\psi(f) = 0$. Then $a_1 = 0$. Since $f(x,y) = \sum_{i=1}^{d-1} a_i x^i y^{d-i}$ this condition is equivalent to the condition that $f_1(0,y) = 0$, where f_1 is the derivative of f with respect to the first coordinate. But differentiating the cocycle condition with respect to x yields

$$\begin{aligned} 0 &= -f_1(y,z) + f_1(0, y+z) - f_1(0,y) \\ &= -f_1(y,z), \end{aligned}$$

and therefore $a_i = 0$ for all i.

Now we have $\psi\phi_A = \mathrm{Id}_A$, which implies that $\psi\phi_A\psi = \psi$. But ψ is injective, so $\phi_A\psi = \mathrm{Id}_{Z_d(A)}$. Thus ϕ_A is an isomorphism. In fact we have $\psi = \psi_A$.

(ii) Now suppose that A is torsion-free. Then define $A' = A \otimes \mathbb{Q}$. Since A is torsion-free, $a \mapsto a \otimes 1$ embeds A into A'. We have that $Z_d(A) = Z_d(A') \cap A[x,y]$, and we know that $\phi_{A'}$ is an isomorphism by the previous part. Since ϕ_A is just the restriction of $\phi_{A'}$ to A, it is bijective and thus an isomorphism.

As special cases, we note that we now have the result for $A = \mathbb{Z}$ and $A = \mathbb{Z}_p$.

(iii) We now want to show the result for $A = \mathbb{Z}/p^f\mathbb{Z}$. We already know that ϕ_A is injective, so it suffices to check that it is surjective. Consider the diagram

$$\begin{tikzcd} \mathbb{Z}_p \arrow[r, "\phi_{\mathbb{Z}_p}"] \arrow[d] & Z_d(\mathbb{Z}_p) \arrow[d] \\ \mathbb{Z}/p^f\mathbb{Z} \arrow[r, "\phi_{\mathbb{Z}/p^f\mathbb{Z}}"] & Z_d(\mathbb{Z}/p^f\mathbb{Z}) \end{tikzcd}$$

It commutes and by the previous case the top arrow is an isomorphism. Therefore the bottom arrow is a surjection.

(iv) If A is any finitely generated abelian group, it can be written as a direct sum of finitely many terms of the form $\mathbb{Z}$ or $\mathbb{Z}/p^f\mathbb{Z}$. Since $Z_d(A \oplus B) \cong Z_d(A) \oplus Z_d(B)$ we have the desired result.

(v) Now suppose that A is any abelian group. Let B be the subgroup of A generated by the coefficients of f. Then B is finitely generated and $f \in Z_d(B)$. But ϕ_B is surjective, so there is some $b \in B \subset A$ with $\phi_A(b) = \phi_B(b) = f$. So ϕ_A is surjective and thus an isomorphism.

$\square$

Proof of Lemma 2.12.

(i) We need to prove that there is a unique $\varphi_A \colon Q_{2n}(L) \to A$ that takes μ^L to μ^A.

By Theorem 2.8, there is a unique ring homomorphism $\theta_{\mathbb{Z}\oplus A} \colon L \to \mathbb{Z} \oplus A$ with $\mu^A = \theta_{\mathbb{Z}\oplus A *}\mu^L$. Since the grading on $\mathbb{Z} \oplus A$ is in dimensions 0 and $2n$, and since $\theta_{\mathbb{Z}\oplus A}$ preserves degree, $\theta_{\mathbb{Z}\oplus A}$ maps a_{ij} to 0 unless $i + j = n + 1$. Thus $\theta_{\mathbb{Z}\oplus A}$ factors as

$$\begin{tikzcd} L \arrow[rr, "\theta_{\mathbb{Z}\oplus A}"] \arrow[dr] & & \mathbb{Z} \oplus A \\ & \mathbb{Z} \oplus Q_{2n}(L) \arrow[ur, "\mathrm{Id}_{\mathbb{Z}} \oplus \varphi_A"'] & \end{tikzcd}$$

(ii) In order to prove that φ_A factors through the homomorphism $Q_{2n}(L) \to T_n$, we consider different possibilities for A. We first prove the claim for $A = \mathbb{Z}$ and $A = \mathbb{Z}/p\mathbb{Z}$, then for

$\mathbb{Z}/p^f\mathbb{Z}$. We then use the classification of finitely generated abelian groups to prove the claim for finitely generated A, and finally consider completely general A.

$\square$

In this section we sketch definitions and basic results in the transition from CW-complexes to spectra in algebraic topology. For the underlying theory of CW-complexes, see Hatcher [3]. For a more extensive exposition of spectra consult Adams [1, pp. 123-373] or Switzer [6].

.

3.1. CW **complexes and geometric operations.** We include the definition of a CW-complex for completeness. See Hatcher [3, p. 5] or Switzer [6, pp. 64-65] for examples and further details. Let $D^n \subset \mathbb{R}^n$ be the subspace $\{x \in \mathbb{R}^n : |x| \leq 1\}$ and $S^n \subset \mathbb{R}^{n+1}$ be the subspace $\{x \in \mathbb{R}^{n+1} : |x| = 1\}$.

Definition 3.1. A CW-complex is a Hausdorff topological space that can be inductively constructed from discs of varying dimensions and attaching maps. More precisely, a *cell complex* on a Hausdorff space X is a collection $K = \{e^n_\alpha : n \in \mathbb{N}, \alpha \in J_n\}$ of subsets of X satisfying the properties given below. Set $K^n = \{e^r_\alpha : r \leq n, \alpha \in J_n\}$, the *n-skeleton* of K, and $X^{(n)} = \bigcup_{\substack{r \leq n \\ \alpha \in J_r}} e^r_\alpha$. Set

$$\dot{e}^n_\alpha = e^n_\alpha \cap X^{(n-1)} \qquad = \textit{boundary of } e^n_\alpha,$$
$$\mathring{e}^n_\alpha = e^n_\alpha - \dot{e}^n_\alpha \qquad = \textit{interior of } e^n_\alpha.$$

We say that a cell e^m_β is a face of e^n_α if there is a chain of faces $e^m_\beta = e^{m_0}_{\beta_0}, e^{m_1}_{\beta_1}, \ldots, e^{m_k}_{\beta_k} = e^n_\alpha$ with $\mathring{e}^{m_i}_{\beta_i} \cap e^{m_{i+1}}_{\beta_{i+1}} \neq \emptyset$. We require that

(i) $X = \bigcup_{n,\alpha} e^n_\alpha$,
(ii) $\mathring{e}^n_\alpha \cap \mathring{e}^m_\beta \neq \emptyset$ if and only if $\alpha = \beta$ and $m = n$,
(iii) for each cell e^n_α there is a map

$$\varphi^n_\alpha : (D^n, S^{n-1}) \to (e^n_\alpha, \dot{e}^n_\alpha)$$

that is surjective and maps $\mathring{D}^n$ homeomorphically onto $\mathring{e}^n_\alpha$.

If in addition K satisfies the following two properties then we say that K gives X the structure of a CW-complex, or just that X is a CW-complex.

(i) For each cell e^n_α in K, there are only finitely many other cells e^m_β with $\mathring{e}^m_\beta \cap e^n_\alpha \neq \emptyset$.
(ii) A subset $S \subset X$ is closed if and only if $S \cap e^n_\alpha$ is closed in e^n_α for each cell.

We say that $L \subset K$ is a *subcomplex* of K if for each cell $e^n_\alpha \in L$, every face of e^n_α is also an element of L.

CW-complexes provide the geometric framework for much of classical algebraic topology. We can form new CW-complexes out of previously constructed ones in a number of ways (see Hatcher [3, pp. 8-10]). Say X (with cells e^n_α) and Y (with cells d^m_β) are CW-complexes and A is a subcomplex of X. When they appear, let x_0 and y_0 be base points of X and Y (chosen 0-cells). Let $I = D^1$ be the unit interval $\{x \in \mathbb{R} : 0 \leq x \leq 1\}$ with its standard CW-structure.

(i) The disjoint union $X \amalg Y$ is made into a CW-complex in the obvious fashion.
(ii) We can make the set theoretic product $X \times Y$ a CW-complex by giving cells $e^n_\alpha \times d^m_\beta$ and maps $\varphi^{m+n}_{\alpha,\beta} : (D^{m+n}, S^{m+n-1}) \cong (D^n \times D^m, S^{n-1} \times D^m \cup D^n \times S^{m-1}) \to e^n_\alpha \times d^m_\beta$. Note that the topology on the CW-complex $X \times Y$ is not necessarily the same as the product topology.
(iii) The quotient X/A has a cell d^n_α for each cell e^n_α that is not an element of A (the image of that cell under the projection map $X \to X/A$), and one new 0-cell e^0_A corresponding to the

image of A in X/A. For each cell d^n_α, the map $\phi^n_\alpha\colon (D^n, S^{n-1}) \to d^n_\alpha$ is given by composing φ^n_α with the projection map $X \to X/A$.

(iv) The reduced suspension SX of a base pointed space X is defined as $X \times I/(X \times \{0\} \cup \{x_0\} \times I \cup X \times \{1\})$. If $f\colon X \to Y$ is continuous, define $Sf\colon SX \to SY$ to be the quotient of the map $f \times \mathrm{Id}\colon X \times I \to Y \times I$.

(v) The smash product $X \wedge Y$ is defined as $X \times Y/(X \times \{y_0\} \cup \{x_0\} \times Y)$. The projection maps $\pi_X\colon X \times Y \to X$ and $\pi_Y\colon X \times Y \to Y$ descend to maps on $X \wedge Y$. Moreover, given maps $f\colon X \to X'$ and $g\colon Y \to Y'$ of base-pointed topological spaces we get a map $f \wedge g\colon X \wedge Y \to X' \wedge Y'$ by taking the quotient of $f \times g$. Thus the smash product is the product in the category of base pointed topological spaces.

(vi) The wedge sum $X \vee Y$ is defined as $X \amalg Y/\{x_0, y_0\}$. The wedge sum is the coproduct in the category of base pointed topological spaces.

(vii) The path space PX is the topological space consisting of all maps $\gamma\colon I \to X$. If X has a base point, we require that $\gamma(0) = x_0$ and give PX the base point consisting of the constant map sending every t to x_0. It turns out that PX has the homotopy type of a CW-complex.

(viii) The loop space ΩX is the subspace of PX consisting of paths γ with $\gamma(1) = x_0$.

Note that $S^m \wedge S^n \cong S^{m+n}$, that $SX \cong S^1 \wedge X$ and that under this identification Sf corresponds to $\mathrm{Id} \wedge f$.

We generally consider arbitrary continuous maps between CW-complexes, but sometimes it is useful to restrict our attention to maps which satisfy an additional property.

Definition 3.2. We say a map $f\colon X \to Y$ is *cellular* if $f(X^{(n)}) \subset Y^{(n)}$ for every n.

The cellular approximation theorem [3, p. 349] states that every continuous map between CW-complexes is homotopic to a cellular map.

.

3.2. **Spectra.** The motivation to consider spectra comes from a number of sources. The suspension functor on CW-complexes induces isomorphisms in homology:

Proposition 3.3. *There is a natural isomorphism $H_n(X) \cong H_{n+1}(SX)$.*

See Hatcher [3, p. 448].

The Freudenthal suspension theorem provides an analogue in homotopy theory. Define $\Sigma_r\colon \pi_r(X, x_0) \to \pi_{r+1}(SX, *)$ by $\Sigma([f]) = [Sf] = \mathrm{Id}_{S^1} \wedge f$.

Theorem 3.4. *For every n-connected CW-complex X, Σ_r is an isomorphism for $1 \le r \le 2n$ and an epimorphism for $r = 2n + 1$.*

See Hatcher [3, p. 360] or Switzer [6, p. 85].

These results motivate us to ask for a category where an analogue of suspension had an inverse S^{-1}. The category of spectra will satisfy this objective.

One can also motivate the construction of spectra by attempting to generalize classical homology and cohomology theories. This generalization is based upon Brown's representation theorem (see Switzer [6, pp. 152-157] for more details).

We consider contravariant functors F from the category of pointed CW-complexes to the category of pointed sets that satisfy three properties. We require that

(i) $F(X)$ depends only on the homotopy type of (X, x_0) and $F(f) = F(g)$ if f and g are homotopic maps.

(ii) For an arbitrary wedge sum $\bigwedge_\alpha X_\alpha$ with inclusions $i_\alpha\colon X_\alpha \to \bigwedge_\alpha X_\alpha$, the morphism $F(\prod_\alpha i_\alpha)\colon F(\bigwedge_\alpha X_\alpha) \to \prod_\alpha F(X_\alpha)$ is an isomorphism.

(iii) If A is any subcomplex of X with inclusion $i\colon A \hookrightarrow X$ and $u \in F(X)$, write $u|A$ for $i^*(u)$. We assume that if X is a CW-complex with subcomplexes A_1 and A_2 are such that $X = A_1 \cup A_2$, and if $x_1 \in F(A_1)$, $x_2 \in F(A_2)$ with $x_1|A_1 \cap A_2 = x_2|A_1 \cap A_2$ then there is a $y \in F(X)$ with $y|A_1 = x_1$ and $y|A_2 = x_2$.

Given any CW-complex (Y, y_0) and $u \in F(Y)$, we have a natural transformation $T_u\colon [-,(Y,y_0)] \to F$ given by $T_u([f]) = F(f)(u) \in F(X)$ for any $f\colon (X,x_0) \to (Y,y_0)$. We say that an element $u \in F(Y)$ is universal if $T_u\colon [(S^q, s_0);(Y,y_0)] \to F(S^q)$ is an isomorphism for all q.

Theorem 3.5. *For any such F, there is a CW-complex (Y,y_0) and universal element $u \in F(Y)$ such that $T_u\colon [-,(Y,y_0)] \to F$ is a natural equivalence.*

A version of this theorem holds if we consider functors taking values in the category of groups. As a special case, we have that Eilenberg-MacLane spaces classify ordinary cohomology.

Theorem 3.6. *For any abelian group G, there is a natural equivalence $T\colon [-,K(G,n)] \to H^n(-;G)$.*

Suppose we have a reduced cohomology theory h^*. Then by Theorem 3.5, for each n there is a CW-complex E_n so that $h^n(-) = [-,(E_n,*)]$. In addition, the E_n are related: one of the axioms for a reduced cohomology theory gives natural equivalences σ_r with

$$\begin{tikzcd}
h^r \arrow[r, "(\sigma_r)^{-1}"] \arrow[d, equal] & h^{r+1}\circ S \arrow[d, equal] & \\
{[-,E_r]} & {[S(-),E_{r+1}]} \arrow[r, equal] & {[-,\Omega E_{r+1}]}.
\end{tikzcd}$$

We can consider $\sigma_r^{-1}([\mathrm{Id}_{E_r}]) \in [E_r, \Omega E_{r+1}] = [SE_r, E_{r+1}]$. So, given any cohomology theory, we get a collection of spaces E_r and maps $\epsilon_r\colon SE_r \to E_{r+1}$, at least up to homotopy. Moreover, we can recover our functor from the given data.

Definition 3.7. (i) A *CW-spectrum* **E**is a collection $\{(E_n,*)\colon n \in \mathbb{Z}\}$ of CW-complexes such that SE_n is (or is homeomorphic to) a subcomplex of E_{n+1}. We will use the term spectrum for a CW-spectrum.

(ii) A *subspectrum* $\mathbf{F} \subset \mathbf{E}$ consists of subcomplexes $F_n \subset E_n$ such that $SF_n \subset F_{n+1}$.

(iii) We now define a *cell* of a spectrum **E**. The subspectrum $F_n = *$ for all n we denote by $*$ and call a cell of dimension $-\infty$. Suppose we have a d-dimensional cell e^d_n of E_n other than $*$ such that e^d_n is not the suspension of any $(d-1)$-dimensional cell. Then we call the sequence

$$e = (e^d_n, Se^d_n, S^2e^d_n, \ldots)$$

a cell of dimension $d - n$.

(iv) We say a subspectrum $\mathbf{F} \subset \mathbf{E}$ is *cofinal* if for any cell $e^d_n \subset E_n$ there is some m with $S^m e^d_n \subset F_{n+m}$.

(v) A *function* $f\colon \mathbf{E} \to \mathbf{F}$ *of degree* r between spectra is a collection $\{f_n\colon n \in \mathbb{Z}\}$ of cellular maps $f_n\colon E_n \to F_{n-r}$ such that $f_{n+1}|_{SE_n} = Sf_n$.

(vi) Let **E** and **F** be spectra. Consider the set of pairs $(\mathbf{E}', f')$ so that $\mathbf{E}'$ is a cofinal subspectrum of **E**and $f'\colon \mathbf{E}' \to \mathbf{F}$ is a function. Say that $(\mathbf{E}', f')$ $(\mathbf{E}'', f'')$ if there is a cofinal subspectrum $\mathbf{E}''' \subset \mathbf{E}' \cap \mathbf{E}''$ with $f'|_{\mathbf{E}'''} = f''|_{\mathbf{E}'''}$. It is easy to show that is an equivalence relation. We call an equivalence class a *map* of spectra.

(vii) We will define smash products of spectra in Section 3.4; for the moment we define the smash product of a spectrum $\mathbf{E}$with a CW-complex X to be the spectrum $\mathbf{E} \wedge X$ defined by $(\mathbf{E} \wedge X)_n = \mathbf{E}_n \wedge X$.

(viii) Let I_+ be the interval I with a disjoint base point added. Then a *homotopy* is a map $h \colon \mathbf{E} \wedge I_+ \to \mathbf{F}$. Let i_0 and i_1 be the two maps $\mathbf{E} \to \mathbf{E} \wedge I_+$ induced by the inclusions of 0 and 1 into I_+. We say that two maps $f_0, f_1 \colon \mathbf{E} \to \mathbf{F}$ are *homotopic* if there is a homotopy $h \colon \mathbf{E} \wedge I_+ \to \mathbf{F}$ with $h \circ i_0 = f_0$ and $h \circ i_1 = f_1$. We write $[\mathbf{E}, \mathbf{F}]$ for the set of homotopy classes of maps from $\mathbf{E}$to $\mathbf{F}$(as a graded set), and $[\mathbf{E}, \mathbf{F}]_r$ for those of degree r.

We will be working with the category whose objects are spectra and whose morphisms are maps of spectra.

.

3.3. **Examples of Spectra.** We now list a few examples of spectra.

(i) Given any CW-complex X, we can form the suspension spectrum of X, which we will write $\mathbf{X}$. This spectrum is defined by

$$\mathbf{X}_n = \begin{cases} S^n X & \text{if } n \geq 0 \\ * & \text{otherwise.} \end{cases}$$

As a special case we have the sphere spectrum, which we will write $\mathbf{S}$:

$$\mathbf{S}_n = \begin{cases} S^n & \text{if } n \geq 0 \\ * & \text{otherwise.} \end{cases}$$

(ii) Write $\mathbf{H}$ for the Eilenberg-MacLane spectrum. The justification for using $\mathbf{H}$ rather than $\mathbf{K}$ comes from our notation for the generalized homology theory associated to a spectrum: by choosing $\mathbf{H}$ we will be able to write $\mathbf{H}^*(X)$ and $\mathbf{H}_*(X)$ for ordinary cohomology and homology. See Hatcher [3, p. 365] for a construction of Eilenberg-Maclane spaces, and define

$$\mathbf{H}_n = \begin{cases} K(\mathbb{Z}, n) & \text{if } n \geq 0 \\ * & \text{otherwise.} \end{cases}$$

More generally we can form the Eilenberg-MacLane spectrum with coefficients, given by

$$\mathbf{HG} = \begin{cases} K(G, n) & \text{if } n \geq 0 \\ * & \text{otherwise.} \end{cases}$$

Even more generally, if we have any spectrum $\mathbf{E}$, we can add coefficients to it by smashing with an appropriate Moore spectrum. We will return to this construction in Section 3.4, after we have some additional tools for working with spectra.

(iii) An important collection of spectra is obtained by considering classifying spaces of topological groups. In order to define these spectra we need to first define classifying spaces.

Suppose now that G is a topological group. Define a contravariant functor k_G from CW-complexes to pointed sets: if X is a CW-complex, $k_G(X)$ is the set of all equivalence classes of principal G-bundles over X, and if $[f]$ is a homotopy class of maps $f \colon (X, x_0) \to (Y, y_0)$ then $k_G([f])(\{\xi\}) = \{f^*\xi\}$. Switzer proves [6, p. 201] that this functor satisfies the conditions in Theorem 3.5. Thus there is a CW-complex BG, determined up to homotopy type, and a universal principal G-bundle ξ_G over BG such that the natural transformation $T_G \colon [-, (BG, *)] \to k_G$ defined by $T_G([f]) = \{f^*\xi_G\}$ is a natural equivalence.

For G of particular interest, namely various matrix groups including $U(n)$, $O(n)$, $SO(n)$ and $Sp(n)$ one can be more explicit. Since we will be using mostly $U(n)$ bundles, I will give the construction for $BU(n)$; the others can be found in Switzer [6, p. 203], and a more verbose description of the $O(n)$ case can be found in Milnor-Stasheff [4, pp. 55-68].

We can define the complex Stiefel manifold of all k-frames in $\mathbb{C}^n$ by

$$V_{k,n}(\mathbb{C}) = U(n)/U(n-k),$$

and the Grassmanian of all complex k-dimensional linear subspaces of $\mathbb{C}^n$ by

$$G_{k,n}(\mathbb{C}) = U(n)/(U(n-k) \times U(k)).$$

We have a fibration $U(n) \to U(n+1) \to U(n+1)/U(n) \cong S^{2n+1}$, and the long exact sequence of homotopy groups gives us that $\pi_q(U(n), *) \to \pi_q(U(n+1), *)$ is surjective for $q = 2n$ and an isomorphism for $q < 2n$. Therefore $\pi_q(U(n), *) \to \pi_q(U(n+k), *)$ is surjective for $q \leq 2n$ and an isomorphism for $q < 2n$, which shows that $\pi_q(V_{k,k+n}, *) = 0$ for $q \leq 2n$. So the principal $U(k)$-bundle with base space $G_{k,k+n}$ and total space $V_{k,k+n}$ is universal for $U(k)$ bundles over CW-complexes of dimension at most $2n$. The inclusions

$$\mathbb{C}^k \subset \mathbb{C}^{k+1} \subset \cdots$$

given by $(x_1, \ldots, x_k) \to (0, x_1, \ldots, x_k)$ induce inclusions

$$G_{k,k} \xrightarrow{j} G_{k,k+1} \xrightarrow{j} \cdots .$$

We set $BU(k) = \bigcup_{n\geq 0} G_{k,k+n}$ and $EU(k) = \bigcup_{n\geq 0} V_{k,k+n}$, both with the weak topology. This gives a universal complex vector bundle of dimension k. Moreover, the inclusions $: U(k) \subset U(k+1)$ induce inclusions $U(n+k)/(U(n)\times U(k)) \xrightarrow{i} U(n+k+1)/(U(n)\times U(k+1))$ which commute with the j and thus induce an inclusion $Bi\colon BU(k) \to BU(k+1)$. These Bi allow us to define a space $BU = \bigcup_{k\geq 1} BU(k)$.

One can show that the two functors Ω and B are almost inverses of each other. More precisely, we have the following proposition [6, p. 206]:

Proposition 3.8. *If G has the homotopy type of a CW-complex, there is a homotopy equivalence* $(G, 1) \simeq (\Omega BG, \omega_0)$*.*

Moreover, we have the following result, which will allow us to define spectra out of these BG.

Theorem 3.9 (Bott Periodicity Theorem)**.** *There are homotopy equivalences*

$$\Omega^2 BU \simeq \mathbb{Z} \times BU$$
$$\Omega^4 BSp \simeq \mathbb{Z} \times BO$$
$$\Omega^4 BO \simeq \mathbb{Z} \times BSp$$

This result allows us to define spectra **K**and **KO**by:

$$\mathbf{K}_n = \begin{cases} \mathbb{Z} \times BU & \text{if } n \text{ is even,} \\ \Omega BU & \text{if } n \text{ is odd,} \end{cases}$$

and

$$\mathbf{KO}_n = \begin{cases} \mathbb{Z} \times BO & \text{if } n \equiv 0 \pmod 8 \\ \Omega^{4-n'} BSp & \text{if } n' = 1, 2, 3 \text{ and } n \equiv n' \pmod 8 \\ \mathbb{Z} \times BSp & \text{if } n \equiv 4 \pmod 8 \\ \Omega^{8-n'} BO & \text{if } n' = 5, 6, 7 \text{ and } n \equiv n' \pmod 8 \end{cases}$$

The notation is justified because the associated cohomology theories are complex and real K-theory respectively.

(iv) Suppose $G \subset O(n)$ for some n. Given any principal G-bundle ξ with total space $E(\xi)$ and base $B(\xi)$, one can form the Thom space $M(\xi)$ as follows (see Milnor-Stasheff [4, pp. 205-208] for more details). Each fiber has an inner product that is preserved by the action of $G \subset O(n)$. Define a subspace $A(\xi) \subset E(\xi)$ by $A(\xi) = \{(b, v) \in E(\xi) \colon |v| \geq 1\}$.

Definition 3.10. The *Thom space* $M(\xi)$ is given by

$$M(\xi) = E(\xi)/A(\xi).$$

Write $MU(n)$ for $M(\xi)$ when ξ is the universal bundle over $BU(n)$.

The spectra **MU**is defined from the $MU(n)$. More precisely,

$$\mathbf{MU}_n = \begin{cases} MU(k) & \text{if } n = 2k, \\ MU(k) \wedge S^1 & \text{if } n = 2k+1. \end{cases}$$

The map $MU(k) \wedge S^2 \to MU(k+1)$ is given by noting that if ξ_k denotes the universal bundle over $BU(k)$ then the Whitney sum $\xi_k \oplus 1$

One can similarly define $MO(n)$, $MSO(n)$, $MSU(n)$ and $MSp(n)$, but we will not need any of these spaces or the associated spectra.

.

3.4. **Smash Products of Spectra.** Most of the geometric constructions that one can perform on spaces, such as those listed in Section 3.1, carry over to analogous constructions on spectra that satisfy similar properties. Unforetunately, the constructions become significantly more complicated. The most important of these constructions for us will be that of the smash product. The category of CW-spectra admits a smash product, which is a functor of two variables, with arguments and values in the homotopy category of CW-spectra. Note that the smash product is constant on spectra that are homotopy equivalent, but it is also only defined up to homotopy. This flaw may be remedied by working in other categories of spectra, but for us the problems caused by having the product only defined up to homotopy are not serious enough to warrant the extra complication of working with symmetric spectra or some other option.

The smash product is associative, commutative and has the sphere spectrum as a unit. I sketch the basic construction below: see Adams [1, pp. 158-189] for details.

Let A be a totally ordered set, isomorphic to $\mathbb{N}$, and let $A = B \amalg C$ be a partition of A. We define $\beta \colon A \to \mathbb{N}$ by setting $\beta(a) = \#\{b \in B \colon b < a\}$, and α and γ analogously with A and C replacing B. Then we set

$$(\mathbf{E} \wedge_{B,C} \mathbf{F})_{\alpha(a)} = \mathbf{E}_{\beta(a)} \mathbf{F}_{\gamma(a)}$$

and prove that under suitable conditions on B and C, the resulting spectra are all homotopy equivalent and satisfy associativity, commutativity and identity up to homotopy.

With smash products available, we can now make the following definitions, continuing into the next section.

Definition 3.11. (i) We call a spectrum $\mathbf{E}$ a *ring spectrum* if it is equipped with a map $m_{\mathbf{E}}\colon \mathbf{E} \wedge \mathbf{E} \to \mathbf{E}$ and a map $i_{\mathbf{E}}\colon \mathbf{S} \to \mathbf{E}$ so that $m_{\mathbf{E}}$ is associative and commutative up to homotopy and $i_{\mathbf{E}}$ acts as a unit up to homotopy.

(ii) We say that a spectrum $\mathbf{M}$ is a *module spectrum* over a ring spectrum $\mathbf{E}$ if it is equipped with a map $m_{\mathbf{M}}\colon \mathbf{E} \wedge \mathbf{M} \to \mathbf{M}$. Note that if $\mathbf{E}$ is a ring spectrum and $\mathbf{F}$ is any spectrum then $\mathbf{E} \wedge \mathbf{F}$ is a module spectrum via the map $m_{\mathbf{E}} \wedge \mathrm{Id}_{\mathbf{F}}\colon \mathbf{E} \wedge \mathbf{E} \wedge \mathbf{F} \to \mathbf{E} \wedge \mathbf{F}$.

Given an abelian group G, we can construct a Moore spectrum $\mathbf{M}(G)$ (see Adams [1, p. 200]) with

$$\begin{aligned} \pi_r(\mathbf{M}(G)) &= 0 && \text{for } r < 0, \\ \pi_0(\mathbf{M}(G)) &\cong G, && \\ \mathbf{H}_r(\mathbf{M}(G)) &= 0 && \text{for } r > 0. \end{aligned}$$

This spectrum allows the addition of “coefficients” to any spectrum $\mathbf{E}$: define

$$\mathbf{EG} = \mathbf{E} \wedge \mathbf{M}(G).$$

An analogoue of the universal coefficient theorem (see Adams [1, p. 200-202]) shows that the definitions of $\mathbf{HG}$ as $\mathbf{H} \wedge \mathbf{M}(G)$ and as the Eilenberg-MacLane spectrum with terms $K(G, n)$ agree.

.

3.5. **Homology, Cohomology and Homotopy.** We define the homotopy groups of a spectrum, which are really stable homotopy groups. Given a spectrum $\mathbf{E}$, we have homomorphisms

$$\pi_{n+r}(\mathbf{E}_n) \to \pi_{n+r+1}(S\mathbf{E}_n) \to \pi_{n+r+1}(\mathbf{E}_{n+1}),$$

so for fixed r the groups $\pi_{n+r}(\mathbf{E}_n)$ fit into a directed system.

Definition 3.12. The n^{th} homotopy group of a spectrum $\mathbf{E}$ is defined by

$$\pi_r(\mathbf{E}) = \lim_{n\to\infty} \pi_{n+r}(\mathbf{E}_n).$$

The following proposition [1, pp. 145-146] relates $\pi_r(\mathbf{E})$ to homotopy classes of maps into $\mathbf{E}$.

Proposition 3.13. *If X is a finite CW-complex and $\mathbf{X}$ the associated suspension spectrum, then for any spectrum $\mathbf{E}$, $[\mathbf{X}, \mathbf{E}]_r = \lim_{n\to\infty}[S^{n+r}X, \mathbf{E}_n]$. In the case $X = S^0$ this yields $\pi_r(\mathbf{E}) = [\mathbf{S}, \mathbf{E}]_r$.*

We now define generalized homology and cohomology in a fashion consistent with the comments at the beginning of Section 3.2.

Definition 3.14. Let $\mathbf{E}$ be a spectrum. Then the $\mathbf{E}$-homology of a spectrum $\mathbf{X}$ is defined by

$$\mathbf{E}_n(\mathbf{X}) = [\mathbf{S}, \mathbf{E} \wedge \mathbf{X}]_n,$$

and the $\mathbf{E}$-cohomology of $\mathbf{X}$ is defined by

$$\mathbf{E}^n(\mathbf{X}) = [\mathbf{X}, \mathbf{E}]_{-n}.$$

If X is a CW-complex with suspension spectrum $\mathbf{X}$ then we define $\mathbf{E}_n(X) = \mathbf{E}_n(\mathbf{X})$ and $\mathbf{E}^n(X) = \mathbf{E}^n(\mathbf{X})$.

The definitions immediately yield a number of relationships between homology, cohomology and homotopy groups.

Proposition 3.15. (i) *For any spectrum $\mathbf{E}$, $\mathbf{E}^*(pt) = \mathbf{E}_*(pt) = \pi_*(\mathbf{E}) = [\mathbf{S}, \mathbf{E}]$.*

(ii) *For any commutative ring spectrum* $\mathbf{E}$, $\pi_*(\mathbf{E})$ *is a commutative ring.*

(iii) *For any commutative ring spectrum* $\mathbf{E}$*and any spectrum* $\mathbf{Y}$*the map* $\mathbf{Y} \simeq \mathbf{S} \wedge \mathbf{Y} \xrightarrow{i_{\mathbf{E}} \wedge \mathrm{Id}_{\mathbf{Y}}} \mathbf{E} \wedge \mathbf{Y}$ *induces a homomorphism*

$$[\mathbf{X}, \mathbf{Y}]_* \xrightarrow{B} [\mathbf{X}, \mathbf{E} \wedge \mathbf{Y}]_*,$$

called the Boardman homomorphism. Specializing to $\mathbf{X} = \mathbf{S}$ *and* $\mathbf{E} = \mathbf{H}$ *yields the Hurewicz homomorphism* $\pi_*(\mathbf{Y}) \xrightarrow{h} \mathbf{H}_*(\mathbf{Y})$.

In addition to these basic properties, generalized homology and cohomology groups satisfy many of the same axioms as ordinary homology and cohomology. For further exposition see [1, pp. 196-214].

.

3.6. **Orientations.** Giving a spectrum an orientation allows us to choose generators coherently for many of the groups and rings that we need to consider. The presence of an orientation is also crucial in the process that produces a formal group law from a spectrum.

Definition 3.16. Let i be the inclusion $i\colon \mathbb{C}P^1 \to \mathbb{C}P^\infty$. An *orientation* of a spectrum $\mathbf{E}$is an element $x^{\mathbf{E}} \in \mathbf{E}^*(\mathbb{C}P^\infty)$ such that $\mathbf{E}^*(\mathbb{C}P^1)$ is a free module over $\pi_*(\mathbf{E})$ on the generator $i^*(x)$.

Since $\mathbb{C}P^1$ may be identified with S^2, the unit map $i_{\mathbf{E}}\colon \mathbf{S} \to \mathbf{E}$ that is part of the ring spectrum structure of $\mathbf{E}$ yields a generator $\gamma \in \mathbf{E}^2(\mathbb{C}P^1)$. We define a unit $u^{\mathbf{E}} \in \pi_*(\mathbf{E})$ by $i^*(x^{\mathbf{E}}) = u^{\mathbf{E}}\gamma$.

From now on all spectra come equipped with an orientation.

.

3.7. **Atiyah-Hirzebruch spectral sequence.** The Atiyah-Hirzebruch spectral sequence provides a powerful method for computing $\mathbf{E}$-homology and $\mathbf{E}$-cohomology.

Theorem 3.17. *Suppose that* $\mathbf{F}$ *and* $\mathbf{X}$ *are spectra so that* $\mathbf{X}$ *is bounded below: there is some* ν *such that* $\pi_r(\mathbf{X}) = 0$ *for* $r < \nu$. *Then there are spectral sequences*

$$E^2_{p,q} = \mathbf{H}_p(\mathbf{X}, \pi_q(\mathbf{F})) \Rightarrow \mathbf{F}_{p+q}(\mathbf{X})$$
$$E_2^{p,q} = \mathbf{H}^p(\mathbf{X}, \pi_q(\mathbf{F})) \Rightarrow \mathbf{F}^{p+q}(\mathbf{X})$$

This spectral sequence comes from considering the skeletal filtration on $\mathbf{X}$and gluing together the long exact sequences in homology or cohomology for each inclusion of the r-skeleton into the $r+1$-skeleton. In particular, we get exact couples

$$\begin{tikzcd}
\bigoplus_p \mathbf{F}_*(\mathbf{X}^{(p-1)}) \arrow[rr, "i_*"] & & \bigoplus_p \mathbf{F}_*(\mathbf{X}^{(p)}) \arrow[dl, "j_*"] \\
& \bigoplus_p \mathbf{F}_*(\mathbf{X}^{(p)}, \mathbf{X}^{(p-1)}) \arrow[ul, "\partial"] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
\bigoplus_p \mathbf{F}^*(\mathbf{X}^{(p-1)}) \arrow[dr, "\partial"'] & & \bigoplus_p \mathbf{F}^*(\mathbf{X}^{(p)}) \arrow[ll, "i^*"'] \\
& \bigoplus_p \mathbf{F}^*(\mathbf{X}^{(p)}, \mathbf{X}^{(p-1)}) \arrow[ur, "j^*"'] &
\end{tikzcd} .$$

One notable feature of this spectral sequence is that if we a priori have an element $x \in \mathbf{F}^*(\mathbf{X})$, all differentials vanish on its image in any $\mathbf{F}^*(\mathbf{X}^{(p)})$. This will allow us to argue that the sequence degenerates at the second page in a variety of cases using the existence of an orientation.

For more details about this spectral sequence see Adams [1, pp. 214-220].

In this section we will prove the following theorem, due to Quillen.

Theorem 4.1. *There is a natural formal group law $\mu^{\mathbf{MU}}$ on $\mathbf{MU}$ so that the homomorphism $\theta_{\mathbf{MU}}\colon L \to \pi_*\mathbf{MU}$ given by Theorem 2.8 is an isomorphism.*

Proof. We first define the formal group law on $\mathbf{MU}$using the computation of $\mathbf{E}^*(\mathbb{C}P^\infty)$ and $\mathbf{E}^(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$, and the H-space structure on $\mathbb{C}P^\infty$. The universal property of L then produces a homomorphism $L \to \pi_*(\mathbf{MU})$, which we proceed to show is an isomorphism. The central part of the argument is the commutative diagram in Lemma 4.5. We get injectivity from the injectivity of the homomorphism θ_R of Theorem 2.9, and surjectivity from an analysis of the indecomposible quotient of the Hurewicz homomorphism.

Lemma 4.2. *Suppose that* $\mathbf{E}$ *is a ring spectrum. Then*

(i) $\mathbf{E}^*(\mathbb{C}P^n) \cong \pi_*(\mathbf{E})[x]/(x^{n+1})$,
(ii) $\mathbf{E}^*(\mathbb{C}P^\infty) \cong \pi_*(\mathbf{E})[\![x]\!]$,
(iii) $\mathbf{E}^*(\mathbb{C}P^n \times \mathbb{C}P^m) \cong \pi_*(\mathbf{E})[x_1, x_2]/(x_1^{n+1}, x_2^{m+1})$,
(iv) $\mathbf{E}^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong \pi_*(\mathbf{E})[\![x_1, x_2]\!]$.

We know that $\mathbb{C}P^\infty$ is an H-space (by [3, p. 282] for example), and its product map $m\colon \mathbb{C}P^\infty \times CPi \to \mathbb{C}P^\infty$ is associative, commutative and with an identity element.

Definition 4.3. Define $\mu^{\mathbf{E}}(x_1, x_2) = m^*(x) \in \mathbf{E}^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong \pi_*(\mathbf{E})[\![x_1, x_2]\!]$.

Lemma 4.4. *The power series $\mu^{\mathbf{E}}$ defined above gives a commutative formal group law.*

Proof. The required associativity, identity and commutativity axioms follow from the corresponding properties of m. □

This formal group law allows us to define a homomorphism $\theta_{\mathbf{MU}}\colon L \to \pi_*(\mathbf{MU})$. Recall from Section 2.4 that $R = \mathbb{Z}[b_1, b_2, \ldots]$ and that we have a formal group law μ^R and thus a homomorphism $\theta_R\colon L \to R$. In order to connect these maps, we need the following result.

Lemma 4.5. (i) *The* $\mathbf{E}$*-homology of* $\mathbf{MU}$ *is given by*

$$\mathbf{E}_*(\mathbf{MU}) \cong \pi_*(E)[b_1, b_2, \ldots].$$

(ii) *Moreover, the following diagram commutes:*

$$\begin{tikzcd} L \arrow[r, "\theta_{\mathbf{MU}}"] \arrow[d, "\theta_R"'] & \pi_*(\mathbf{MU}) \arrow[d, "h"] \\ R \arrow[r, equal] & \mathbf{H}_*(\mathbf{MU}) \end{tikzcd}$$

This commutative diagram constitutes the central part of this proof. We know from Theorem 2.9 that θ_R is injective, and thus we have that $\theta_{\mathbf{MU}}$ is also injective. In order to prove that $\theta_{\mathbf{MU}}$ is surjective we use the following results.

Lemma 4.6.

$$Q_m(\pi_*(\mathbf{MU})) = \begin{cases} \mathbb{Z} & \text{\textit{for m positive and even,}} \\ 0 & \text{\textit{otherwise.}} \end{cases}$$

Lemma 4.7. *The image of $Q_{2n}(h)\colon Q_{2n}(\pi_*(\mathbf{MU})) \to Q_{2n}(\mathbf{H}_*(\mathbf{MU}))$ is the same as the image of $Q_{2n}(\theta_R)$.*

These two lemmas allow us to complete the proof of Theorem 4.1. Using Lemma 2 we have that $Q_{2n}(h)$ is an isomorphism onto its image. Since $Q_{2n}(\theta_R)$ is also an isomorphism onto its image by Lemma 2, by the commutativity of the diagram in Lemma 4.5 we know that $Q_{2n}(\theta_{\mathbf{MU}})$ is actually an isomorphism. Therefore $\theta_{\mathbf{MU}}$ is surjective, by the same argument that showed that θ_R was surjective given that $Q_{2n}(\theta_R)$ was an isomorphism. We have already seen that $\theta_{\mathbf{MU}}$ is injective, so it is an isomorphism. This concludes the proof of Theorem 4.1. ◻

Once again there are many details to fill in. We begin with the proof of Lemma 4.2, computing the cohomology of $\mathbb{C}P^n$, $\mathbb{C}P^\infty$and their products.

Proof of Lemma 4.2. We have the following spectral sequences from Theorem 3.17:

(i) $\mathbf{H}^*(\mathbb{C}P^n; \pi_*(E)) \Rightarrow E^*(\mathbb{C}P^n)$,
(ii) $\mathbf{H}^*(\mathbb{C}P^\infty; \pi_*(E)) \Rightarrow E^*(\mathbb{C}P^\infty)$,
(iii) $\mathbf{H}^*(\mathbb{C}P^n \times \mathbb{C}P^m; \pi_*(E)) \Rightarrow \mathbf{E}^*(\mathbb{C}P^n \times \mathbb{C}P^m)$,
(iv) $\mathbf{H}^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty; \pi_*(E)) \Rightarrow \mathbf{E}^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$.

The appropriate powers x^i and $x_1^i x_2^j$ form a basis for the E_2 page as a $\pi_*(\mathbf{E})$ module. In fact, x is the image of the orientation $x^{\mathbf{E}} \in \mathbf{E}^*(\mathbb{C}P^\infty)$, and thus by the construction of the Atiyah-Hirzebruch spectral sequence must survive to the E_∞ page. Therefore all differentials d_r vanish on x, and since these differentials are $\pi_*(E)$-algebra homomorphisms, all differentials vanish and thus $E^*(\mathbb{C}P^\infty) \cong \pi_*(E)[\![x]\!]$.

The arguments for the other spectral sequences are analogous. ◻

In order to prove Lemma 4.5, we need to pursue some preliminaries. In order to compute $\mathbf{E}_*(\mathbf{MU})$, we first compute $\mathbf{E}_*(\mathbb{C}P^\infty) = \mathbf{E}_*(BU(1))$, then proceed to $\mathbf{E}_*(BU)$ and finally $\mathbf{E}_*(\mathbf{MU})$.

The computation of the homology of $\mathbb{C}P^\infty$is dual to the computation of its cohomology. Indeed, we have the following.

Lemma 4.8. (i) *$\mathbf{E}_*(\mathbb{C}P^n)$ and $\mathbf{E}_*(\mathbb{C}P^\infty)$ are finitely generated free modules over $\pi_*(\mathbf{E})$, dual to $\mathbf{E}^*(\mathbb{C}P^n)$ and $\mathbf{E}^*(\mathbb{C}P^\infty)$.*
(ii) *Let β_i be the element of $\mathbf{E}^*(\mathbb{C}P^n)$ dual to x^i, and also write β_i for the image in $\mathbf{E}^*(\mathbb{C}P^\infty)$ of β_i. Then $\{\beta_i\colon 0 \le i \le n\}$ forms a $\pi_*(\mathbf{E})$ basis for $\mathbf{E}_*(\mathbb{C}P^n)$ and $\{\beta_i\colon i \ge 0\}$ forms a $\pi_*(\mathbf{E})$ basis for $\mathbf{E}_*(\mathbb{C}P^\infty)$.*

In order to compute the $\mathbf{E}$-homology of BU, we need to know the ordinary homology of BU, which is given in the following lemma (see [4, p. 161] for the proof of the dual statement).

Lemma 4.9. *The homology $H_*(BU)$ is given by*

$$H_*(BU) = \mathbb{Z}[\beta_1, \beta_2, \ldots],$$

where β_i is the image in $H_*(BU)$ of the corresponding element of $H_*(\mathbb{C}P^\infty)$ under the inclusion $\mathbb{C}P^\infty = BU(1) \hookrightarrow BU$.

The Atiyah-Hirzebruch spectral sequence is trivial since all differentials vanish on the monomials in the β_i. This gives us the following:

Lemma 4.10.

$$\mathbf{E}_*(BU) = \pi_*(\mathbf{E})[\beta_1, \beta_2, \ldots].$$

Proof of Lemma 4.5. Cupping with the Thom class gives an isomorphism $\mathbf{E}_*(BU) \cong \mathbf{E}_*(\mathbf{MU})$ (see [4, p. 206]). We want to use slightly different generators for $\mathbf{E}_*(\mathbf{MU})$ though, in order to make the diagram in Lemma 4.5 commute. In particular, we have that $BU(1) = MU(1)$ and the "inclusion" of $MU(1)$ in $\mathbf{MU}$ induces a homomorphism

$$\mathbf{E}_p(MU(1)) \to \mathbf{E}_{p-2}(\mathbf{MU}).$$

Write b_i for the image of $u^{\mathbf{E}}\beta_{i+1}$ in $\mathbf{E}_*(\mathbf{MU})$. Since $u^{\mathbf{E}}$ is a unit in $\pi_*(\mathbf{E})$ and $b_0 = 1$, this gives Lemma 1.

We now need to prove the commutativity of the diagram in Lemma 4.5. Recall from Section 3.5 that the map $h\colon \pi_*(\mathbf{MU}) = [\mathbf{S}, \mathbf{MU}] \to [\mathbf{S}, \mathbf{H} \wedge \mathbf{MU}] = \mathbf{H}_*(\mathbf{MU})$ is induced from the map $\mathbf{MU} \to \mathbf{H} \wedge \mathbf{MU}$. We need to understand the pushforward of $\mu^{\mathbf{MU}}$, which has coefficients in $\pi_*(\mathbf{MU})$, under this map.

The orientations $x^{\mathbf{MU}} \in \mathbf{MU}^*(\mathbb{C}P^\infty)$ and $x^{\mathbf{H}} \in \mathbf{H}^*(\mathbb{C}P^\infty)$ map to orientations $y^{\mathbf{MU}}$ and $y^{\mathbf{H}}$ in $(\mathbf{H} \wedge \mathbf{MU})^*(\mathbb{C}P^\infty)$. I claim that $y^{\mathbf{MU}} = \sum_{i\geq 0} b_i^{\mathbf{H}}(y^{\mathbf{H}})^{i+1}$. Note that $b_i^{\mathbf{H}} \in \mathbf{H}_*(\mathbf{MU}) = [\mathbf{S}, \mathbf{H} \wedge \mathbf{MU}] \to [\mathbf{S}, \mathbf{H} \wedge \mathbf{MU} \wedge \mathbb{C}P^\infty]$. The proof of this claim is given in [1, p. 60] and consists mostly of chasing down definitions.

Recall that the power series $\exp_{\mathbf{H}}(x) \in \pi_*(\mathbf{H} \wedge \mathbf{MU})[\![x]\!]$ is defined by

$$\exp_{\mathbf{H}}(x) = \sum_{i\geq 0} b_i^{\mathbf{H}} x^{i+1}.$$

Then we've shown that, tracing around the top and right of the commutative diagram, we have

$$h_*\mu^{\mathbf{MU}}(x_1, x_2) = \exp_{\mathbf{H}}(\log_{\mathbf{H}}(x_1) + \log_{\mathbf{H}}(x_2)).$$

But this is exactly the the definition of μ^R. □

The only remaining task is to control $Q_m(\pi_*(\mathbf{MU}))$ and $Q_{2n}(h)$. We sketch the proofs of Lemmas 4.6 and 4.7; for more details see [1, pp. 75-79].

Sketch for Lemma 4.6. The proof that

$$Q_m(\pi_*(\mathbf{MU})) = \begin{cases} \mathbb{Z} & \text{for } m \text{ positive and even,} \\ 0 & \text{otherwise} \end{cases}$$

rests on the Adams spectral sequence. If A is the mod-p Steenrod algebra, then

$$\mathrm{Ext}_A^{s,t}(\mathbf{H}^*(\mathbf{MU}; \mathbb{Z}/p\mathbb{Z}), \mathbb{Z}/p\mathbb{Z}) \Rightarrow \pi_{t-s}(\mathbf{MU}).$$

We prove that, for any p,

$$Q_m(\pi_*(\mathbf{MU})) \otimes \mathbb{Z}/p\mathbb{Z} = \begin{cases} \mathbb{Z}/p\mathbb{Z} & \text{for } m \text{ positive and even,} \\ 0 & \text{otherwise.} \end{cases}$$

Since $\mathbf{H}^*(\mathbf{MU})$ is concentrated in even degrees, this result is easy for odd or non-positive m. In order to prove that the dimension of $Q_m(\pi_*(\mathbf{MU})) \otimes \mathbb{Z}/p\mathbb{Z}$ is at least one, we can use the fact that the Hurewicz map is an isomorphism after tensoring with $\mathbb{Q}$ and note that

$$Q_{2n}(\pi_*(\mathbf{MU})) \otimes \mathbb{Q} \cong Q_{2n}\mathbf{H}_*(\mathbf{MU})) \otimes \mathbb{Q} \cong \mathbb{Q}.$$

So it remains to prove that $Q_{2n}(\pi_*(\mathbf{MU})) \otimes \mathbb{Z}/p\mathbb{Z}$ has dimension at most 1 over $\mathbb{Z}/p\mathbb{Z}$. At this point I will defer to the treatment in Adams [1] and the papers that he references. □

Sketch for Lemma 4.7. We need to prove that the image of $Q_{2n}(h)$ is the same as the image of $Q_{2n}(\theta_R)$. The inclusion $\mathrm{im}(Q_{2n}(\theta_R)) \subset \mathrm{im}(Q_{2n}(h))$ is clear from the commutativity of the diagram in Lemma 4.5. Since $\mathrm{im}(Q_{2n}(\theta_R)) = Q_{2n}(R)$ for $n+1 \neq p^f$ by Lemma 2, we need to prove that for $n+1 = p^f$, the inclusion $\mathrm{im}(Q_{2n}(h)) \subset p\langle\overline{b_n}\rangle$ holds.

Set $G = \mathbb{Z}/p\mathbb{Z}$. There is a canonical map $\mathbf{MU} \to \mathbf{H}$ (see Adams [1, p. 52]) and another $\mathbf{H} \to \mathbf{HG}$. Call the composition g, and the induced map

$$q_* \colon \mathbf{H}_*(\mathbf{MU}) = [\mathbf{S}, \mathbf{H} \wedge \mathbf{MU}] \to [\mathbf{S}, \mathbf{HG} \wedge \mathbf{HG}] = \mathbf{HG}_*(\mathbf{HG}).$$

Adams argues that the image of $q_*(b_n)$ in $Q_{2n}(\mathbf{HG}_*(\mathbf{HG})) \cong \mathbb{Z}/p\mathbb{Z}$ is nonzero, but that q_* annihilates $h(\pi_*(\mathbf{MU}))$ [1, p. 79]. This implies the inclusion $\mathrm{im}(Q_{2n}(h)) \subset p\langle\overline{b_n}\rangle$. □

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