Homework 23 Solutions

Problems

1. **Compute the following roots:**

   (a) **The 15th root of 2** (mod 29)**?**

   Since 29 is prime and 15 and 28 are relatively prime, our method applies. The Euclidean algorithm gives

$$\begin{aligned} 28 &= 15 + 13, \\ 15 &= 13 + 2, \\ 13 &= 6 \cdot 2 + 1. \end{aligned}$$

   Running it backwards, we have

$$\begin{aligned} 1 &= 13 - 6 \cdot 2, \\ 1 &= 13 - 6(15 - 13) = -6 \cdot 15 + 7 \cdot 13, \\ 1 &= -6 \cdot 15 + 7(28 - 15) = 7 \cdot 28 - 13 \cdot 15. \end{aligned}$$

   Thus $2^{1/15} \equiv 2^{-13} \equiv 2^{15} \pmod{29}$. To compute $2^{15} \pmod{29}$, we first compute

$$\begin{aligned} 2^2 &\equiv 4, \\ 2^4 &\equiv 16 \equiv -13, \\ 2^8 &\equiv 169 \equiv 24 \equiv -5, \\ 2^{16} &\equiv 25 \equiv -4. \end{aligned}$$

   Here we've gone up to $2^{16} \pmod{29}$ because we know that $1/2 \equiv 15 \pmod{29}$, and thus

$$2^{15} \equiv 1/2 \cdot 2^{16} \equiv 15 \cdot (-4) \equiv -60 \equiv \boxed{27} \pmod{29}.$$

   (b) **The 33rd root of** 8 (mod 17)

   $1/33 = 1/1 = 1 \pmod{16}$. So $8^{1/33} = 8^1 = \boxed{8} \pmod{17}$

2. **Find a number congruent to** 4 (mod 81) **and** 53 (mod 100)**.**

   We use the method from the handout. First, we find x and y with $81x + 100y = 1$ using the Euclidean algorithm. This is possible since 81 and 100 are relatively prime.

$$\begin{aligned} 100 &= 81 + 19 \\ 81 &= 4 \cdot 19 + 5 \\ 19 &= 3 \cdot 5 + 4 \\ 5 &= 1 \cdot 4 + 1. \end{aligned}$$

   Now

$$\begin{aligned} 1 &= 5 - 4 \\ &= 5 - (19 - 3 \cdot 5) = 4 \cdot 5 - 19 \\ &= 4 \cdot (81 - 4 \cdot 19) - 19 = 4 \cdot 81 - 17 \cdot 19 \\ &= 4 \cdot 81 - 17 \cdot (100 - 81) = 21 \cdot 81 - 17 \cdot 100. \end{aligned}$$

   So $21 \cdot 81 \cdot 53 - 17 \cdot 100 \cdot 4 = \boxed{83353}$ does the trick.

3. **Find the smallest positive number congruent to** $15 \pmod{91}$ **and** $88 \pmod{120}$**.**

   As above, we begin by expressing 1 as a combination of 91 and 120.

$$120 = 91 + 2991 \qquad\qquad = 3 \cdot 29 + 429 = 7 \cdot 4 + 1.$$

   Now

$$\begin{aligned} 1 &= 29 - 7 \cdot 4 \\ &= 29 - 7 \cdot (91 - 3 \cdot 29) = 22 \cdot 29 - 7 \cdot 91 \\ &= 22 \cdot (120 - 91) - 7 \cdot 91 = 22 \cdot 120 - 29 \cdot 91. \end{aligned}$$

   So $22 \cdot 120 \cdot 15 - 29 \cdot 91 \cdot 88 = -192532$ satisfies the congruences, and in order to find the smallest positive number also satsifying the congruences, we reduce modulo $91 \cdot 120 = 10920$, yielding $\boxed{3928}$.

4. **Suppose you work on the top floor of a skyscraper, and for exercise each day, you climb the stairs. Each day that you climb, you count the steps modulo a different number. One day you notice that when you count the steps modulo 47, there is only one left over, and when you count the steps modulo 54, there are 5 left over. Suppose that you're certain that you've been climbing at least 2000 stairs each day, but surely not more than 6000. How many steps have you been climbing each day?**

   Same idea.

$$\begin{aligned} 54 &= 47 + 7 \\ 47 &= 6 \cdot 7 + 5 \\ 7 &= 5 + 2 \\ 5 &= 2 \cdot 2 + 1. \end{aligned}$$

   Now

$$\begin{aligned} 1 &= 5 - 2 \cdot 2 \\ &= 5 - 2 \cdot (7 - 5) = 3 \cdot 5 - 2 \cdot 7 \\ &= 3 \cdot (47 - 6 \cdot 7) - 2 \cdot 7 = 3 \cdot 47 - 20 \cdot 7 \\ &= 3 \cdot 47 - 20 \cdot (54 - 47) = 23 \cdot 47 - 20 \cdot 54. \end{aligned}$$

   So the answer must be congruent to $23 \cdot 47 \cdot 5 - 20 \cdot 54 \cdot 1 = 4325 \equiv 1787 \pmod{54 \cdot 47}$. But we also know that it must lie between 2000 and 6000, so in fact you've been climbing $\boxed{4325}$ stairs each day.