Homework 12 Solutions

Problems

1. **Describe all solutions to $122x+84y=4$, if we require that x and y be whole numbers.**

   We break down the solution into steps.

   (a) Run the Euclidean algorithm forward, finding the gcd.

   (b) Check to see if the gcd divides 4. If it does not, then there are no solutions.

   (c) Assuming the gcd divides 4, express the gcd in terms of 122 and 84 by repeatedly substituting and simplifying using the equations derived in the Euclidean algorithm.

   (d) Multiply the equation obtained by $\frac{4}{\gcd(122,84)}$ in order to obtain an expression for 4 in terms of 122 and 84. Suppose that the solution obtained is

   $$4 = 122x_0 + 84y_0$$

   (e) Write down the smallest nontrivial expression for 0 in terms of 122 and 84, namely

   $$\begin{aligned} 0 &= \operatorname{lcm}(122,84) - \operatorname{lcm}(122,84) \\ &= \frac{122 \cdot 84}{\gcd(122,84)} - \frac{84 \cdot 122}{\gcd(122,84)} \\ &= 122 \cdot \frac{84}{\gcd(122,84)} - 84 \cdot \frac{122}{\gcd(122,84)} \end{aligned}$$

   (f) The general solution is found by multiplying this expression for zero by a variable n (which can range over all integers), and adding the result to the specific solution found in part 1d. Namely,

   $$4 = 122(x_0 + \frac{84}{\gcd(122,84)}n) + 84(y_0 + \frac{122}{\gcd(122,84)}n).$$

   Now we implement this strategy.

   (a)

   $$\begin{aligned} 122 &= 1 \cdot 84 + 38 \\ 84 &= 2 \cdot 38 + 8 \\ 38 &= 4 \cdot 8 + 6 \\ 8 &= 1 \cdot 6 + 2 \\ 6 &= 3 \cdot 2 + 0 \end{aligned}$$

   So $\gcd(122,84) = 2$.

   (b) The greatest common divisor does divide 4, so there are solutions.

   (c) First we rewrite the above equations to isolate the remainders.

   $$\begin{aligned} 38 &= 122 - 1 \cdot 84 \\ 8 &= 84 - 2 \cdot 38 \\ 6 &= 38 - 4 \cdot 8 \\ 2 &= 8 - 1 \cdot 6 \end{aligned}$$

   We then repeatedly substitute and simplify.

   $$\begin{aligned} 2 = \quad 8 - 1 \cdot 6 = \quad & 1 \cdot 8 - 1 \cdot (38 - 4 \cdot 8) = & 8 - 1 \cdot 38 + 4 \cdot 8 = & \quad 5 \cdot 8 - 1 \cdot 38 \\ = \quad & 5 \cdot (84 - 2 \cdot 38) - 1 \cdot 38 = & 5 \cdot 84 - 10 \cdot 38 - 1 \cdot 38 = & \quad 5 \cdot 84 - 11 \cdot 38 \\ = \quad & 5 \cdot 84 - 11 \cdot (122 - 1 \cdot 84) = & 5 \cdot 84 - 11 \cdot 122 + 11 \cdot 84 = & \quad 16 \cdot 84 - 11 \cdot 122 \end{aligned}$$

(d) Multiplying by $\frac{4}{2} = 2$ yields

$$4 = 32 \cdot 84 - 22 \cdot 122.$$

(e) We have

$$\begin{aligned} 0 &= \frac{122}{2} \cdot 84 - \frac{84}{2} \cdot 122 \\ &= 61 \cdot 84 - 42 \cdot 122 \end{aligned}$$

is the minimal nontrivial expression for 0 in terms of 84 and 122.

(f) So the general solution is

$$4 = (32 + 61n) \cdot 84 + (-22 - 42n) \cdot 122,$$

where n is allowed to range over all integers, positive or negative.
One can also write this solution as

$$\begin{aligned} x &= -22 - 42n \\ y &= 32 + 61n, \end{aligned}$$

where again n is allowed to range over all integers, positive or negative.

2. **Using the Sieve of Eratosthenes to compute prime numbers, find which of the numbers between 400 and 440 are prime.**

The square root of 440 is less than 21 since $21^2 = 441$. So the highest prime number we need to check is 19.

Kill off numbers divisible by 2:

$$401, 403, 405, 407, 409, 411, 413, 415, 417, 419, 421, 423, 425, 427, 429, 431, 433, 435, 437, 439$$

Kill off numbers divisible by 3:

$$401, 403, 407, 409, 413, 415, 419, 421, 425, 427, 431, 433, 437, 439$$

Kill off numbers divisible by 5:

$$401, 403, 407, 409, 413, 419, 421, 427, 431, 433, 437, 439$$

Kill off numbers divisible by 7:

$$401, 403, 407, 409, 419, 421, 431, 433, 437, 439$$

Kill off numbers divisible by 11:

$$401, 403, 409, 419, 421, 431, 433, 437, 439$$

Kill off numbers divisible by 13:

$$401, 409, 419, 421, 431, 433, 437, 439$$

Kill off numbers divisible by 17:

$$401, 409, 419, 421, 431, 433, 437, 439$$

Kill off numbers divisible by 19:

$$401, 409, 419, 421, 431, 433, 439$$

3. **Explain why none of the nine consecutive numbers $10! + 2$, $10! + 3$, . . . , $10! + 10$ can be a prime number. (Hint: each of the numbers has a small factor.) This is called a "prime desert" of length 9.**

   10! is divisible by all the numbers between 2 and 10. So $10! + 2$ is divisible by 2, and $10! + 3$ is divisible by 3, and so on...

4. **Can you find a prime desert of [at least] length 99?**

   Yes: $100! + 2$, $100! + 3$, $100! + 4$, ..., $100! + 100$.

   Each term $100! + i$ is divisible by i, hence not prime.