Math 0430 Final Exam
April 20, 2015

Name: ______________________

Instructions: Attempt all problems. Show your work! No books or notes may be used.

1. (5 points each)

   (a) Define the term *cyclic group*.

   (b) Show that every group of prime order is cyclic.

2. (3 points each) Consider the following permutation in S_6:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 4 & 1 & 5 & 2 \end{pmatrix}$$

(a) Find σ^{-1}.

(b) Express σ as a product of disjoint cycles.

(c) Express σ as a product of transpositions.

(d) Is σ even or odd? Explain.

(e) Find the order of σ.

3. (10 points) The group S_3 consists of the following six permutations.

$$\rho_0 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = () \qquad \mu_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = (2,3)$$
$$\rho_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (1,2,3) \qquad \mu_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = (1,3)$$
$$\rho_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = (1,3,2) \qquad \mu_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = (1,2)$$

Find all subgroups of S_3, and say which ones are normal.

4. (4 points each)

   (a) Define the term *zero divisor*.

   (b) Define the term *unit*.

   (c) List the units and the zero divisors in the ring $\mathbb{Z}_{10}$.

5. (3 points each) Which of the following polynomials is irreducible? Justify your answers.

(a) $x^2 - 2$ in $\mathbb{Q}[x]$

(b) $x^2 - 2$ in $\mathbb{R}[x]$

(c) $x^2 - 2$ in $\mathbb{Z}_7[x]$

(d) $x^4 + x^2 + 1$ in $\mathbb{R}[x]$

(e) $x^4 + x^2 + 1$ in $\mathbb{Z}_2[x]$

(f) $x^{100} + x^{11} + x + 1$ in $\mathbb{Z}_2[x]$

6. (5 points each)

(a) Let $S = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$. Show that S is a subfield of the field $\mathbb{R}$ of real numbers.

(b) Show that $\mathcal{I} = \{f(x) \in \mathbb{Q}[x] : f(\sqrt{2}) = 0\}$ is the principal ideal in $\mathbb{Q}[x]$ generated by $x^2 - 2$.

(c) Show that S is isomorphic to the quotient ring $\mathbb{Q}[x]/(x^2 - 2)$.

7. (5 points each)

   (a) Define the term *principal ideal*.

   (b) Prove that every ideal in $\mathbb{Z}$ is principal.

8. (5 points each)

   (a) Define greatest common divisors in an integral domain.

   (b) Prove that if R is a principal ideal domain, and $a, b \in R$ are not both 0, then a and b have a greatest common divisor.