Math 430 – Group Theory Extra Credit Project

Due April 25, 2016

If you choose to work on this project, I will e-mail you an integer N. Your goal is to classify all groups of order N up to isomorphism. Your main tools will be

- Sylow's Theorems (15.4, 15.7 and 15.8),
- Semidirect Products (see the introduction below),
- the following proposition, which you should prove.

  Proposition.

  1. *Let H and K be subgroups of G. Then HK is a subgroup of G if and only if $HK = KH$.*
  2. *If K is a normal subgroup of G and H is any subgroup of G, then HK is a subgroup of G.*

The book *Abstract Algebra* by Dummit and Foote may be useful (especially chapters 4 and 5). E-mail me if you're not able to get a copy from the library.

Hint. As a warmup, you may want to try to classify groups of order M for some divisors M of N. Also, try to list as many groups of order N as you can before attempting to prove that you have a complete list.

Introduction to Semidirect Products

Given two groups H and K, together with a left action of K on H (which we'll denote by $k \cdot h$), one can form a new group called the *semidirect product* of H and K. It will be denoted $H \rtimes K$.

As a *set*, its elements are just ordered pairs (h, k), like the direct product $H \times K$. The operation is defined by

$$(h_1, k_1) \times (h_2, k_2) = (h_1(k_1 \cdot h_2), k_1k_2).$$

There are natural injective homomorphisms $H \to H \rtimes K$ and $K \to H \rtimes K$ given by $h \mapsto (h, 1)$ and $k \mapsto (1, k)$. However, the image of H is a normal subgroup, while the image of K is not (unless the action of K on H is trivial, in which case the semidirect product is the same as the direct product).

Just as with direct products, there is a notion of internal semidirect product. Given a pair of subgroups H and K in G with H normal, multiplication induces an isomorphism between $H \rtimes K$ (with the action defined by $k \cdot h = khk^{-1}$) and G if the following two conditions are satisfied:

1. $H \cap K = \{1\}$,
2. $G = HK$ (or that $\#G = \#H\#K$ if all are finite).

Finally, there is a relationship between semidirect products and quotient groups. Given a normal subgroup H of G, let $K_q = G/H$. Then $G \cong H \rtimes K_q$ if and only if there is a subgroup $K \subseteq G$ that is mapped isomorphically onto K_q by the natural quotient map $G \to G/H$.

Example. If $G = D_n$ and H is the subgroup of rotations, then letting K be any subgroup of order 2 generated by a reflection we get $G \cong H \rtimes K \cong \mathbb{Z}_n \rtimes \mathbb{Z}_2$.

Example. If $G = S_n$ and $H = A_n$, letting K be any subgroup of order 2 generated by a transposition we get $G \cong H \rtimes K \cong A_n \rtimes \mathbb{Z}_2$.

Example. There is no way to express Q_8 as a semidirect product of smaller groups, since every subgroup of order 4 contains the unique subgroup of order 2.

Example. The subgroup of order 12 that we didn't define in class is a semidirect product $\mathbb{Z}_3 \rtimes \mathbb{Z}_4$. The action of $\mathbb{Z}_4$ on $\mathbb{Z}_3$ is: a acts trivially if it is even and acts by $x \mapsto -x$ if it is odd.