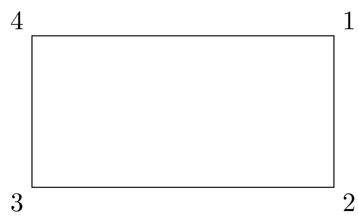
Math 430 Midterm Exam Solutions

1. Consider the symmetry group G of a rectangle with side lengths 1 and 2.



   (a) (4 points) List the elements of G.

   Solution. You can either give geometric descriptions — $\{\text{id}, \mu_h, \mu_v, \rho\}$ where μ_h and μ_v are horizontal and vertical reflections and ρ is rotation by 180 degrees — or descriptions in terms of the action on vertices — $\{(), (14)(23), (12)(34), (13)(24)\}$.

   (b) (4 points) Show that G is abelian.

   Solution. Every element has order 2, so G is abelian by a homework problem. Alternately, you can give the Cayley table of G and observe that it is symmetric.

   (c) (5 points) List the subgroups of G.

   Solution.

   - $\{\text{id}\}$,
   - $\langle \mu_h \rangle = \{\text{id}, \mu_h\}$,
   - $\langle \mu_v \rangle = \{\text{id}, \mu_v\}$,
   - $\langle \rho \rangle = \{\text{id}, \rho\}$,
   - $G = \{\text{id}, \mu_h, \mu_v, \rho\}$.

   (d) (5 points) Do there exist nontrivial subgroups H and K so that G is an internal direct product of H and K? Explain.

   Solution. Yes. Any two of the subgroups of order 2 will work. For example, if we set $H = \langle \mu_h \rangle$ and $K = \langle \mu_v \rangle$ then

   - $H \cap K = \{\text{id}\}$,
   - $\#G = \#H \cdot \#K$,
   - G is abelian, so $hk = kh$ for all $h \in H$ and $k \in K$.

2. Let $\sigma = (123)(4567) \in S_7$.

   (a) (5 points) What is the order of σ? Explain.

   Solution. Since σ is a product of disjoint cycles of length 3 and 4, it has order $\text{lcm}(3, 4) = 12$.

   (b) (5 points) Find σ^{-1}.

Solution. We reverse the order of the cycles, yielding $(7654)(321) = (132)(4765)$.

(c) (5 points) Is σ even or odd? Why?

Solution. Since σ is the product of a 3-cycle (even) and a 4-cycle (odd), it is odd. Alternately, you could give a decomposition of σ into transpositions and note that there are an odd number of them (the shortest decomposition will have 5 transpositions).

(d) (5 points) Give an isomorphism between the subgroup $\langle\sigma\rangle$ generated by σ and $\mathbb{Z}_n$ for some n.

Solution. The map

$$\begin{aligned} f : \mathbb{Z}_{12} &\to \langle\sigma\rangle \\ a &\mapsto \sigma^a \end{aligned}$$

is an isomorphism. It is well defined since $\sigma^{12} = ()$, bijective since the order of σ is 12 and a homomorphism since $\sigma^{a+b} = \sigma^a\sigma^b$ for all $a, b \in \mathbb{Z}$.

(e) (6 points) Find all $\tau \in S_7$ that also generate $\langle\sigma\rangle$, i.e. all τ with $\langle\tau\rangle = \langle\sigma\rangle$.

Solution. The other generators of $\langle\sigma\rangle$ will be the powers of σ that also have order 12: the σ^a for $\gcd(a, 12) = 1$. These are

$$\begin{aligned} \sigma^1 &= (123)(4567) \\ \sigma^5 &= (132)(4567) \\ \sigma^7 &= (123)(4765) \\ \sigma^{11} &= (132)(4765) \end{aligned}$$

3. Suppose H and K are normal subgroups of a group G.

(a) (8 points) Prove that $H \cap K$ is a subgroup of G.

Solution. We check that it contains the identity and is closed under multiplication and inversion.

- Since H and K are subgroups, $1 \in H$ and $1 \in K$ so $1 \in H \cap K$.
- Let $g, g' \in H \cap K$. Since H and K are subgroups, $gg' \in H$ and $gg' \in K$. Thus $gg' \in H \cap K$.
- Let $g \in H \cap K$. Since H and K are subgroups, $g^{-1} \in H$ and $g^{-1} \in K$. Thus $g^{-1} \in K$.

(b) (8 points) Prove that $H \cap K$ is a normal subgroup of G.

Solution. Suppose that $g \in G$ and $h \in H \cap K$. We show that $ghg^{-1} \in H \cap K$, and thus that $g(H \cap K)g^{-1} \subseteq H \cap K$. This proves that $H \cap K$ is normal by a theorem from the book.
Since $h \in H$ and H is normal, $ghg^{-1} \in H$. Similarly, since $h \in K$ and K is normal, $ghg^{-1} \in K$. Thus $ghg^{-1} \in H \cap K$ and $H \cap K$ is normal.

4. (8 points) Consider the subgroup $H = D_5$ of $G = S_5$. How many cosets does H have in G? Justify your answer.

Solution. By Lagrange's theorem, $[G : H] = \frac{\#G}{\#H} = \frac{5!}{2\cdot 5} = 12$.

5. (8 points) Suppose G is a group and $g, h \in G$. Prove that the order of hgh^{-1} is the same as the order of g.

Solution. We showed that $(hgh^{-1})^m = hg^mh^{-1}$ as a homework problem (for all $m \in \mathbb{Z}$). Note that $hg^mh^{-1} = 1$ if and only if $g^m = 1$. Thus $(hgh^{-1})^m = 1$ if and only if $g^m = 1$. Since the order of an element is the smallest positive integer m with the mth power equal to 1, the order of g will be the same as the order of hgh^{-1}.

6. (Bonus) Prove that there is no cyclic group G that has 34 different generators (i.e. $G = \langle g \rangle$ for 34 different $g \in G$).

Solution. The number of generators of a cyclic group of order n is $\phi(n)$. So we need to show that there is no integer n with $\phi(n) = 34 = 2 \cdot 17$. Recall that if $n = \prod_{i=1}^{k} p_i^{e_i}$ then $\phi(n) = \prod_{i=1}^{k} (p_i - 1) p_i^{e_i - 1}$.

- If n were prime then $\phi(n) = n - 1$, so we would need $n = 35$. But 35 is not prime.
- More generally, suppose p divides n. Then $p - 1$ divides $\phi(n)$. The divisors of 34 are $d = 1, 2, 17, 34$. Of these, only $d = 1$ and $d = 2$ have $d + 1$ prime, so the only primes that can divide n are 2 and 3.. If 9 divides n then 3 would divide $\phi(n)$, which it doesn't. If 8 divides n then 4 would divide $\phi(n)$, which it doesn't. So the only possible n are $1, 2, 4, 3, 6, 12$, none of which are large enough to have $\phi(n) = 34$.