Math 430 – Problem Set 4 Solutions

Due March 18, 2016

9.8. Prove that \( \mathbb{Q} \) is not isomorphic to \( \mathbb{Z} \).

**Solution.** Suppose that \( \phi: \mathbb{Q} \to \mathbb{Z} \) is an isomorphism. Since \( \phi \) is surjective, there is an \( x \in \mathbb{Q} \) with \( \phi(x) = 1 \). Then \( 2\phi(x/2) = \phi(x) = 1 \), but there is no integer \( n \) with \( 2n = 1 \). Thus \( \phi \) cannot exist.

9.12. Prove that \( S_4 \) is not isomorphic to \( D_{12} \).

**Solution.** Note that \( D_{12} \) has an element of order 12 (rotation by 30 degrees), while \( S_4 \) has no element of order 12. Since orders of elements are preserved under isomorphisms, \( S_4 \) cannot be isomorphic to \( D_{12} \).

9.23. Prove or disprove the following assertion. Let \( G, H, \) and \( K \) be groups. If \( G \times K \cong H \times K \), then \( G \cong H \).

**Solution.** Take \( K = \prod_{i=1}^{\infty} \mathbb{Z} \) and \( G = \mathbb{Z} \) and \( H = \mathbb{Z} \times \mathbb{Z} \). Then

\[ G \times K \cong K \cong H \times K \]

but \( G \not\cong H \). Thus the assertion is false.

Note that the assertion is true if \( K \) is finite, but it’s difficult to show. Many people tried to use an isomorphism \( \phi: G \times K \to H \times K \) to construct an isomorphism \( G \to H \). The difficulty is that \( \phi \) does not necessarily map \( G \times \{1\} \) to \( H \times \{1\} \) (and if it does, it may not be surjective).

9.29. Show that \( S_n \) is isomorphic to a subgroup of \( A_{n+2} \).

**Solution.** Let \( \tau = (n+1, n+2) \in S_{n+2} \). Identifying \( S_n \) with the subgroup of \( S_{n+2} \) that fix \( n+1 \) and \( n+2 \), we define

\[ \phi: S_n \to A_{n+2} \]

\[ \sigma \mapsto \begin{cases} 
\sigma & \text{if } \sigma \text{ even} \\
\sigma \tau & \text{if } \sigma \text{ odd}
\end{cases} \]

We check that \( \phi \) is an injective homomorphism. Note that \( \sigma \tau = \tau \sigma \) for all \( \sigma \in S_n \). Then

\[ \phi(\sigma_1 \sigma_2) = \begin{cases} 
\sigma_1 \sigma_2 = \phi(\sigma_1)\phi(\sigma_2) & \text{if } \sigma_1 \text{ even, } \sigma_2 \text{ even,} \\
\sigma_1 \sigma_2 \tau = \phi(\sigma_1)\phi(\sigma_2) & \text{if } \sigma_1 \text{ even, } \sigma_2 \text{ odd,} \\
\sigma_1 \tau \sigma_2 = \phi(\sigma_1)\phi(\sigma_2) & \text{if } \sigma_1 \text{ odd, } \sigma_2 \text{ even,} \\
\sigma_1 \sigma_2 \tau^2 = \phi(\sigma_1)\phi(\sigma_2) & \text{if } \sigma_1 \text{ odd, } \sigma_2 \text{ odd.}
\end{cases} \]

Thus \( \phi \) is a homomorphism. Moreover, since \( \sigma \tau \) is never 1 and \( \phi \) is the identity on \( A_n \), \( \phi \) is injective. Thus it defines an isomorphism with its image, a subgroup of \( A_{n+2} \).

9.41. Let \( G \) be a group and \( g \in G \). Define a map \( i_g : G \to G \) by \( i_g(x) = gxg^{-1} \). Prove that \( i_g \) defines an automorphism of \( G \).
Solution. Since \( i_g(xy) = gxg^{-1} = gxg^{-1}gyg^{-1} = i_g(x)i_g(y) \), we see that \( i_g \) is a homomorphism. It is injective: if \( i_g(x) = 1 \) then \( gxg^{-1} = 1 \) and thus \( x = 1 \). And it is surjective: if \( y \in G \) then \( i_g(g^{-1}yg) = y \). Thus it is an automorphism.

10.4. Let \( T \) be the group of nonsingular upper triangular \( 2 \times 2 \) matrices with entries in \( \mathbb{R} \); that is, matrices of the form
\[
\begin{pmatrix}
a & b \\
0 & c
\end{pmatrix},
\]
where \( a, b, c \in \mathbb{R} \) and \( ac \neq 0 \). Let \( U \) consist of matrices of the form
\[
\begin{pmatrix}
1 & x \\
0 & 1
\end{pmatrix},
\]
where \( x \in \mathbb{R} \).

(a) Show that \( U \) is a subgroup of \( T \).

Solution. Taking \( x = 0 \), we see that the identity matrix is in \( U \). The inverse of \( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \) is \( \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \), which is also in \( U \). Finally,
\[
\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x + y \\ 0 & 1 \end{pmatrix},
\]
which is in \( U \).

(b) Prove that \( U \) is abelian.

Solution. This follows from the formula for multiplication of elements of \( U \) given above, together with the commutativity of addition in \( \mathbb{R} \).

(c) Prove that \( U \) is normal in \( T \).

Solution.
\[
\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}^{-1} = \begin{pmatrix} a & ax + b \\ 0 & c \end{pmatrix} \begin{pmatrix} 1/a & -b/(ac) \\ 0 & 1/c \end{pmatrix} = \begin{pmatrix} 1 & ax/c \\ 0 & 1 \end{pmatrix},
\]

(d) Show that \( T/U \) is abelian.

Solution. Note that
\[
\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} 1 & b/a \\ 0 & 1 \end{pmatrix},
\]
so every coset in \( T/U \) has a representative that is a diagonal matrices. Since diagonal matrices commute with each other, \( T/U \) is commutative.

Alternatively, note that
\[
\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}^{-1} \begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix}^{-1} = \begin{pmatrix} aa' & ab' + bc' \\ 0 & cc' \end{pmatrix} \begin{pmatrix} 1/(aa') & -(b'/c')/(aca'e') \\ 0 & 1/(cc') \end{pmatrix} = \begin{pmatrix} 1 & (ab' - b'/c')/(cc') \\ 0 & 1 \end{pmatrix}.
\]

Since \( U \) contains the commutator subgroup of \( T \), \( T/U \) is abelian by 10.14.

(e) Is \( T \) normal in \( GL_2(\mathbb{R}) \)?
Solution. No. For example,
\[
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.
\]

10.7. Prove or disprove: If \( H \) is a normal subgroup of \( G \) such that \( H \) and \( G/H \) are abelian, then \( G \) is abelian.

Solution. \( U \triangleleft T \) from the previous problem provides a counterexample, as does \( A_3 \triangleleft S_3 \).

10.9. Prove or disprove: If \( H \) and \( G/H \) are cyclic, then \( G \) is cyclic.

Solution. \( A_3 \triangleleft S_3 \) provides a counterexample, as does \( Z_2 \triangleleft Z_2 \times Z_2 \).

10.14. Let \( G \) be a group and let \( G' = \langle aba^{-1}b^{-1} \rangle \); that is, \( G' \) is the subgroup of all finite products of elements in \( G \) of the form \( aba^{-1}b^{-1} \). The subgroup \( G' \) is called the commutator subgroup of \( G \).

(a) Show that \( G' \) is a normal subgroup of \( G \).

Solution. Suppose \( \gamma = aba^{-1}b^{-1} \) is a generator of \( G' \). Since \( g\gamma g^{-1} = (gag^{-1})(gbg^{-1})(gag^{-1})^{-1}(gbg^{-1})^{-1} \), we have that \( g\gamma g^{-1} \in G' \). Since conjugation by \( g \) is a homomorphism, every product of such elements will also be an element of \( G' \). Thus \( G' \) is normal. Alternatively, note that \( g\gamma g^{-1} = g\gamma g^{-1}\gamma^{-1} \gamma \in G' \) since \( \gamma \in G' \) and \( g\gamma g^{-1}\gamma^{-1} \) is a commutator.

(b) Let \( N \) be a normal subgroup of \( G \). Prove that \( G/N \) is abelian if and only if \( N \) contains the commutator subgroup of \( G \).

Solution. Suppose \( a, b \in G \). Then
\[
(aN)(bN) = (bN)(aN) \iff Nab = Nba \\
\iff Naba^{-1}b^{-1} = N \\
\iff aba^{-1}b^{-1} \in N.
\]

So \( G/N \) is abelian \( \iff (aN)(bN) = (bN)(aN) \) for all \( a, b \in G \)
\( \iff aba^{-1}b^{-1} \in N \) for all \( a, b \in G \)
\( \iff G' \subseteq N \).

11.2. Which of the following maps are homomorphisms? If the map is a homomorphism, what is the kernel?

(a) \( \phi : \mathbb{R}^* \to \text{GL}_2(\mathbb{R}) \) defined by
\[
\phi(a) = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}
\]

Solution. This is a homomorphism since \( \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & ab \end{pmatrix} \). The kernel is \( \{1\} \subset \mathbb{R}^* \).

(b) \( \phi : \mathbb{R} \to \text{GL}_2(\mathbb{R}) \) defined by
\[
\phi(a) = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}
\]

Solution. This is a homomorphism since \( \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b & a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a+b & a \end{pmatrix} \). The kernel is \( \{0\} \subset \mathbb{R} \).

(c) \( \phi : \text{GL}_2(\mathbb{R}) \to \mathbb{R} \) defined by
\[
\phi \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = a + d
\]

Solution. This is not a homomorphism since it maps the identity to 2, which is not the identity in \( \mathbb{R} \).
(d) \( \phi : \text{GL}_2(\mathbb{R}) \rightarrow \mathbb{R}^* \) defined by
\[
\phi \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = ad - bc
\]

**Solution.** This is a homomorphism, since
\[
\phi \left( \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \right) = \phi \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \phi \left( \begin{pmatrix} a & b' \\ c & d' \end{pmatrix} \right) = (ad' + bc') - (a'd' + b'c') = \phi \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \phi \left( \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \right).
\]

The kernel is \( \text{SL}_2(\mathbb{R}) \), the subgroup of \( \text{GL}_2(\mathbb{R}) \) consisting of matrices of determinant 1.

(e) \( \phi : M_2(\mathbb{R}) \rightarrow \mathbb{R} \) defined by
\[
\phi \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = b,
\]
where \( M_2(\mathbb{R}) \) is the additive group of \( 2 \times 2 \) matrices with entries in \( \mathbb{R} \).

**Solution.** This is a homomorphism, since
\[
\phi \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \right) = b + b' = \phi \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) + \phi \left( \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \right).
\]

The kernel is the group (under addition) of lower triangular matrices:
\[
\left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} : a, b, c \in \mathbb{R} \right\}.
\]

11.9. If \( \phi : G \rightarrow H \) is a group homomorphism and \( G \) is abelian, prove that \( \phi(G) \) is abelian.

**Solution.** If \( x, y \in \phi(G) \) then there exist \( a, b \in G \) with \( x = \phi(a) \) and \( y = \phi(b) \). Then \( xy = \phi(a)\phi(b) = \phi(ab) = \phi(ba) = \phi(b)\phi(a) = yx \), so \( \phi(G) \) is abelian.

11.15. Let \( G_1 \) and \( G_2 \) be groups, and let \( H_1 \) and \( H_2 \) be normal subgroups of \( G_1 \) and \( G_2 \) respectively. Let \( \phi : G_1 \rightarrow G_2 \) be a homomorphism. Show that \( \phi \) induces a natural homomorphism \( \bar{\phi} : (G_1/H_1) \rightarrow (G_2/H_2) \) if \( \phi(H_1) \subseteq H_2 \).

**Solution.** We define \( \bar{\phi}(gH_1) = \phi(g)H_2 \) for \( g \in G_1 \). We show that this is well defined. If \( g'H_1 = gH_1 \) then \( g^{-1}g' \in H_1 \) so \( \phi(g^{-1}g') \in \phi(H_1) \subseteq H_2 \). Thus \( \phi(g')\phi(g)^{-1} \in H_2 \), so \( \bar{\phi}(g'H_1) = \phi(g')H_2 = \bar{\phi}(gH_1) \).

It is also a homomorphism, since
\[
\bar{\phi}((gH_1)(g'H_1)) = \bar{\phi}(gg'H_1) = \phi(gg'H_1) = \bar{\phi}(gH_1)\bar{\phi}(g'H_1).
\]

11.19. Given a homomorphism \( \phi : G \rightarrow H \) define a relation \( \sim \) on \( G \) by \( a \sim b \) if \( \phi(a) = \phi(b) \) for \( a, b \in G \). Show this relation is an equivalence relation and describe the equivalence classes.
Solution. Checking the conditions for an equivalence relation is straightforward: $a \sim a$ since $\phi(a) = \phi(a)$; if $a \sim b$ then $\phi(a) = \phi(b)$ and thus $b \sim a$; if $a \sim b$ and $b \sim c$ then $\phi(a) = \phi(b) = \phi(c)$ so $a \sim c$.

The equivalence classes are precisely the cosets of $K = \ker(\phi)$, since

\[
\begin{align*}
    a \sim b & \iff \phi(a) = \phi(b) \\
    & \iff \phi(ab^{-1}) = 1 \\
    & \iff ab^{-1} \in K \\
    & \iff aK = bK.
\end{align*}
\]