Math 430 – Problem Set 4 Solutions

Due March 18, 2016

9.8. Prove that $\mathbb{Q}$ is not isomorphic to $\mathbb{Z}$.

Solution. Suppose that $\phi : \mathbb{Q} \to \mathbb{Z}$ is an isomorphism. Since ϕ is surjective, there is an $x \in \mathbb{Q}$ with $\phi(x) = 1$. Then $2\phi(x/2) = \phi(x) = 1$, but there is no integer n with $2n = 1$. Thus ϕ cannot exist.

9.12. Prove that S_4 is not isomorphic to D_{12}.

Solution. Note that D_{12} has an element of order 12 (rotation by 30 degrees), while S_4 has no element of order 12. Since orders of elements are preserved under isomorphisms, S_4 cannot be isomorphic to D_{12}.

9.23. Prove or disprove the following assertion. Let G, H, and K be groups. If $G \times K \cong H \times K$, then $G \cong H$.

Solution. Take $K = \prod_{i=1}^{\infty} \mathbb{Z}$ and $G = \mathbb{Z}$ and $H = \mathbb{Z} \times \mathbb{Z}$. Then

$$G \times K \cong K \cong H \times K$$

but $G \ncong H$. Thus the assertion is false.

Note that the assertion is true if K is finite, but it's difficult to show. Many people tried to used an isomorphism $\phi : G \times K \to H \times K$ to construct an isomorphism $G \to H$. The difficulty is that ϕ does not necessarily map $G \times \{1\}$ to $H \times \{1\}$ (and if it does, it may not be surjective).

9.29. Show that S_n is isomorphic to a subgroup of A_{n+2}.

Solution. Let $\tau = (n+1, n+2) \in S_{n+2}$. Identifying S_n with the subgroup of S_{n+2} that fix $n+1$ and $n+2$, we define

$$\begin{aligned} \phi : S_n &\to A_{n+2} \\ \sigma &\mapsto \begin{cases} \sigma & \text{if } \sigma \text{ even} \\ \sigma\tau & \text{if } \sigma \text{ odd} \end{cases} \end{aligned}$$

We check that ϕ is an injective homomorphism. Note that $\sigma\tau = \tau\sigma$ for all $\sigma \in S_n$. Then

$$\phi(\sigma_1\sigma_2) = \begin{cases} \sigma_1\sigma_2 = \phi(\sigma_1)\phi(\sigma_2) & \text{if } \sigma_1 \text{ even, } \sigma_2 \text{ even,} \\ \sigma_1\sigma_2\tau = \phi(\sigma_1)\phi(\sigma_2) & \text{if } \sigma_1 \text{ even, } \sigma_2 \text{ odd,} \\ \sigma_1\tau\sigma_2 = \phi(\sigma_1)\phi(\sigma_2) & \text{if } \sigma_1 \text{ odd, } \sigma_2 \text{ even,} \\ \sigma_1\sigma_2\tau^2 = \phi(\sigma_1)\phi(\sigma_2) & \text{if } \sigma_1 \text{ odd, } \sigma_2 \text{ odd.} \end{cases}$$

Thus ϕ is a homomorphism. Moreover, since $\sigma\tau$ is never 1 and ϕ is the identity on A_n, ϕ is injective. Thus it defines an isomorphism with its image, a subgroup of A_{n+2}.

9.41. Let G be a group and $g \in G$. Define a map $i_g : G \to G$ by $i_g(x) = gxg^{-1}$. Prove that i_g defines an automorphism of G.

Solution. Since $i_g(xy) = gxyg^{-1} = gxg^{-1}gyg^{-1} = i_g(x)i_g(y)$, we see that i_g is a homomorphism. It is injective: if $i_g(x) = 1$ then $gxg^{-1} = 1$ and thus $x = 1$. And it is surjective: if $y \in G$ then $i_g(g^{-1}yg) = y$. Thus it is an automorphism.

10.4. Let T be the group of nonsingular upper triangular 2×2 matrices with entries in $\mathbb{R}$; that is, matrices of the form

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix},$$

where $a, b, c \in \mathbb{R}$ and $ac \neq 0$. Let U consist of matrices of the form

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix},$$

where $x \in \mathbb{R}$.

(a) Show that U is a subgroup of T.

Solution. Taking $x = 0$, we see that the identity matrix is in U. The inverse of $\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$ is $\left(\begin{smallmatrix} 1 & -x \\ 0 & 1 \end{smallmatrix}\right)$, which is also in U. Finally,

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x+y \\ 0 & 1 \end{pmatrix},$$

which is in U.

(b) Prove that U is abelian.

Solution. This follows from the formula for multiplication of elements of U given above, together with the commutativity of addition in $\mathbb{R}$.

(c) Prove that U is normal in T.

Solution.

$$\begin{aligned}
\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}^{-1} &= \begin{pmatrix} a & ax+b \\ 0 & c \end{pmatrix}\begin{pmatrix} 1/a & -b/(ac) \\ 0 & 1/c \end{pmatrix} \\
&= \begin{pmatrix} 1 & ax/c \\ 0 & 1 \end{pmatrix}
\end{aligned}$$

(d) Show that T/U is abelian.

Solution. Note that

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}\begin{pmatrix} 1 & b/a \\ 0 & 1 \end{pmatrix},$$

so every coset in T/U has a representative that is a diagonal matrices. Since diagonal matrices commute with each other, T/U is commutative.

Alternatively, note that

$$\begin{aligned}
\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix}\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}^{-1}\begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix}^{-1} &= \begin{pmatrix} aa' & ab'+bc' \\ 0 & cc' \end{pmatrix}\begin{pmatrix} 1/(aa') & -(b'c+bc')/(aca'c') \\ 0 & 1/(cc') \end{pmatrix} \\
&= \begin{pmatrix} 1 & (ab'-b'c)/(cc') \\ 0 & 1 \end{pmatrix}.
\end{aligned}$$

Since U contains the commutator subgroup of T, T/U is abelian by 10.14.

(e) Is T normal in $\mathrm{GL}_2(\mathbb{R})$?

Solution. No. For example,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

10.7. Prove or disprove: If H is a normal subgroup of G such that H and G/H are abelian, then G is abelian.

Solution. $U \lhd T$ from the previous problem provides a counterexample, as does $A_3 \lhd S_3$.

10.9. Prove or disprove: If H and G/H are cyclic, then G is cyclic.

Solution. $A_3 \lhd S_3$ provides a counterexample, as does $\mathbb{Z}_2 \lhd \mathbb{Z}_2 \times \mathbb{Z}_2$.

10.14. Let G be a group and let $G' = \langle aba^{-1}b^{-1} \rangle$; that is, G' is the subgroup of all finite products of elements in G of the form $aba^{-1}b^{-1}$. The subgroup G' is called the commutator subgroup of G.

(a) Show that G' is a normal subgroup of G.

Solution. Suppose $\gamma = aba^{-1}b^{-1}$ is a generator of G'. Since $g\gamma g^{-1} = (gag^{-1})(gbg^{-1})(gag^{-1})^{-1}(gbg^{-1})^{-1}$, we have that $g\gamma g^{-1} \in G'$. Since conjugation by g is a homomorphism, every product of such elements will also be an element of G'. Thus G' is normal.

Alternatively, note that $g\gamma g^{-1} = g\gamma g^{-1}\gamma^{-1}\gamma \in G'$ since $\gamma \in G'$ and $g\gamma g^{-1}\gamma^{-1}$ is a commutator.

(b) Let N be a normal subgroup of G. Prove that G/N is abelian if and only if N contains the commutator subgroup of G.

Solution. Suppose $a, b \in G$. Then

$$\begin{aligned} (aN)(bN) = (bN)(aN) &\Leftrightarrow Nab = Nba \\ &\Leftrightarrow Naba^{-1}b^{-1} = N \\ &\Leftrightarrow aba^{-1}b^{-1} \in N. \end{aligned}$$

So

$$\begin{aligned} G/N \text{ is abelian} &\Leftrightarrow (aN)(bN) = (bN)(aN) \text{ for all } a, b \in G \\ &\Leftrightarrow aba^{-1}b^{-1} \in N \text{ for all } a, b \in G \\ &\Leftrightarrow G' \subseteq N. \end{aligned}$$

11.2. Which of the following maps are homomorphisms? If the map is a homomorphism, what is the kernel?

(a) $\phi : \mathbb{R}^* \to \mathrm{GL}_2(\mathbb{R})$ defined by

$$\phi(a) = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$$

Solution. This is a homomorphism since $\left(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 0 & b \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & ab \end{smallmatrix}\right)$. The kernel is $\{1\} \subset \mathbb{R}^*$.

(b) $\phi : \mathbb{R} \to \mathrm{GL}_2(\mathbb{R})$ defined by

$$\phi(a) = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$$

Solution. This is a homomorphism since $\left(\begin{smallmatrix} 1 & 0 \\ a & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ b & a \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ a+b & 1 \end{smallmatrix}\right)$. The kernel is $\{0\} \subset \mathbb{R}$.

(c) $\phi : \mathrm{GL}_2(\mathbb{R}) \to \mathbb{R}$ defined by

$$\phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = a + d$$

Solution. This is not a homomorphism since it maps the identity to 2, which is not the identity in $\mathbb{R}$.

(d) $\phi : \mathrm{GL}_2(\mathbb{R}) \to \mathbb{R}^*$ defined by

$$\phi\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) = ad - bc$$

Solution. This is a homomorphism, since

$$\begin{aligned}
\phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\right) &= \phi\left(\begin{pmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{pmatrix}\right) \\
&= (aa' + bc')(cb' + dd') - (ab' + bd')(ca' + dc') \\
&= ada'd' + bcb'c' - adb'c' - bca'd' \\
&= (ad - bc)(a'd' - b'c') \\
&= \phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)\phi\left(\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\right).
\end{aligned}$$

The kernel is $\mathrm{SL}_2(\mathbb{R})$, the subgroup of $\mathrm{GL}_2(\mathbb{R})$ consisting of matrices of determinant 1.

(e) $\phi : \mathbb{M}_2(\mathbb{R}) \to \mathbb{R}$ defined by

$$\phi\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) = b,$$

where $\mathbb{M}_2(\mathbb{R})$ is the additive group of 2×2 matrices with entries in $\mathbb{R}$.

Solution. This is a homomorphism, since

$$\phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\right) = b + b' = \phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) + \phi\left(\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\right).$$

The kernel is the group (under addition) of lower triangular matrices:

$$\left\{\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} : a, b, c \in \mathbb{R}\right\}.$$

11.9. If $\phi : G \to H$ is a group homomorphism and G is abelian, prove that $\phi(G)$ is abelian.

Solution. If $x, y \in \phi(G)$ then there exist $a, b \in G$ with $x = \phi(a)$ and $y = \phi(b)$. Then $xy = \phi(a)\phi(b) = \phi(ab) = \phi(ba) = \phi(b)\phi(a) = yx$, so $\phi(G)$ is abelian.

11.15. Let G_1 and G_2 be groups, and let H_1 and H_2 be normal subgroups of G_1 and G_2 respectively. Let $\phi : G_1 \to G_2$ be a homomorphism. Show that ϕ induces a natural homomorphism $\bar{\phi} : (G_1/H_1) \to (G_2/H_2)$ if $\phi(H_1) \subseteq H_2$.

Solution. We define $\bar{\phi}(gH_1) = \phi(g)H_2$ for $g \in G_1$. We show that this is well defined. If $g'H_1 = gH_1$ then $g'g^{-1} \in H_1$ so $\phi(g'g^{-1}) \in \phi(H_1) \subseteq H_2$. Thus $\phi(g')\phi(g)^{-1} \in H_2$, so $\bar{\phi}(g'H_1) = \phi(g')H_2 = \phi(g)H_2 = \bar{\phi}(gH_1)$.

It is also a homomorphism, since

$$\begin{aligned}
\bar{\phi}((gH_1)(g'H_1)) &= \bar{\phi}(gg'H_1) \\
&= \phi(gg')H_2 \\
&= \phi(g)\phi(g')H_2 \\
&= (\phi(g)H_2)(\phi(g')H_2) \\
&= \bar{\phi}(gH_1)\bar{\phi}(g'H_1).
\end{aligned}$$

11.19. Given a homomorphism $\phi : G \to H$ define a relation $\sim$ on G by $a \sim b$ if $\phi(a) = \phi(b)$ for $a, b \in G$. Show this relation is an equivalence relation and describe the equivalence classes.

Solution. Checking the conditions for an equivalence relation is straightforward: $a \sim a$ since $\phi(a) = \phi(a)$; if $a \sim b$ then $\phi(a) = \phi(b)$ and thus $b \sim a$; if $a \sim b$ and $b \sim c$ then $\phi(a) = \phi(b) = \phi(c)$ so $a \sim c$.

The equivalence classes are precisely the cosets of $K = \ker(\phi)$, since

$$\begin{aligned} a \sim b &\Leftrightarrow \phi(a) = \phi(b) \\ &\Leftrightarrow \phi(ab^{-1}) = 1 \\ &\Leftrightarrow ab^{-1} \in K \\ &\Leftrightarrow aK = bK. \end{aligned}$$