Math 430 – Problem Set 3 Solutions

Due February 19, 2016

4.14. Let $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ be elements in $\mathrm{GL}_2(\mathbb{R})$. Show that A and B have finite orders but AB does not.

Solution.

- $A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $A^3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and $A^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ so A has order 4.
- $B^2 = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$ and $B^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ so B has order 3.
- I claim that $(AB)^n = \begin{pmatrix} 1 & -n \\ 0 & 1 \end{pmatrix}$. $AB = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$, which is the base case, and $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -n \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -n-1 \\ 0 & 1 \end{pmatrix}$, which is the induction step. Thus $(AB)^n$ is never the identity matrix for $n > 0$ and AB has infinite order.

4.15(c). Evaluate $(5 - 4i)(7 + 2i)$.

Solution.

$$(5 - 4i)(7 + 2i) = 35 + 10i - 18i + 8 = 43 - 18i.$$

4.15(f). Evaluate $(1 + i) + \overline{(1 + i)}$.

Solution.

$$(1 + i) + \overline{(1 + i)} = 1 + i + 1 - i = 2$$

4.16(c). Convert $3\,\mathrm{cis}(\pi)$ to the form $a + bi$.

Solution.

$$3\,\mathrm{cis}(\pi) = 3(\cos(\pi) + i\sin(\pi) = -3$$

4.17(c). Change $2 + 2i$ to polar representation.

Solution. Using the formulas $r = \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1}(b/a)$ (which holds since $2 + 2i$ is in the first quadrant), we get $r = \sqrt{8}$ and $\theta = \tan^{-1}(1)$ so $2 + 2i = 2\sqrt{2}\,\mathrm{cis}(\pi/4)$.

4.27. If g and h have orders 15 and 16 respectively in a group G, what is the order of $\langle g \rangle \cap \langle h \rangle$?

Solution. The intersection $\langle g \rangle \cap \langle h \rangle$ is a subgroup of both $\langle g \rangle$ and $\langle h \rangle$. By Lagrange's theorem, its order must therefore divide both 15 and 16. Since $\gcd(15, 16) = 1$, we get that $|\langle g \rangle \cap \langle h \rangle| = 1$.

5.2(c). Compute $(143)(23)(24)$.

Solution.

$$(143)(23)(24) = (14)(23)$$

5.2(i). Compute $(123)(45)(1254)^{-2}$.

Solution. Since (1254) has order 4, $(1254)^{-2} = (1254)^2 = (15)(24)$. Thus

$$(123)(45)(1254)^{-2} = (123)(45)(15)(24) = (143)(25)$$

5.2(n). Compute $(12537)^{-1}$.

Solution. We reverse the order of the cycle, yielding

$$(12537)^{-1} = (73521) = (17352).$$

5.7. Find all possible orders of elements in S_7 and A_7.

Solution. Orders of permutations are determined by least common multiple of the lengths of the cycles in their decomposition into disjoint cycles, which correspond to partitions of 7.

Representative Cycle	Order	Sign
()	1	Even
(12)	2	Odd
(123)	3	Even
(1234)	4	Odd
(12345)	5	Even
(123456)	6	Odd
(1234567)	7	Even
(12)(34)	2	Even
(12)(345)	6	Odd
(12)(3456)	4	Even
(12)(34567)	10	Odd
(123)(456)	3	Even
(123)(4567)	12	Odd
(12)(34)(56)	2	Odd
(12)(34)(567)	6	Even

Therefore the orders of elements in S_7 are $1, 2, 3, 4, 5, 6, 7, 10, 12$ and the orders of elements in A_7 are $1, 2, 3, 4, 5, 6, 7$.

5.16. Find the group of rigid motions of a tetrahedron. Show that this is the same group as A_4.

Solution. Let G be the group of rigid motions. Label the vertices of the tetrahedron $1, 2, 3, 4$. A rotation is determined by where it sends vertex 1 (four possibilities) and the orientation of the edges emanating from that vertex (three possibilities). So there are 12 elements in G. Define a map ϕ from G to the symmetric group on the vertices by mapping a given rotation to the permutation it induces on the vertices. There are eight rotations of order 3 that fix a single vertex and rotate around the axis connecting that vertex to the center of the opposite face. The images of these rotations under ϕ are $\{(123), (132), (124), (142), (134), (143), (234), (243)\}$. There are three rotations of order 2 around the axis between midpoints of opposite edges. The images of these rotations under ϕ are $\{(12)(34), (13)(24), (14)(23)\}$. Together with the identity, this gives all twelve rotations. The image of ϕ is A_4, it is injective, and it preserves the group operation (since the operation is function composition in both cases), so ϕ gives an isomorphism between the group of rigid motions of the tetrahedron and A_4.

5.26. Prove that any element can be written as a finite product of the following permutations.

(a) $(12), (13), \ldots, (1n)$

Solution. Every element of S_n can be written as a product of transpositions, and any transposition (ab) can be written as $(1a)(1b)(1a)$. Thus $(12), (13), \ldots, (1n)$ generate S_n.

(b) $(12), (23), \ldots, (n-1, n)$

Solution. We prove by induction that $(1k)$ can be written in terms of $(12), (23), \ldots, (n-1, n)$ for $k = 2, 3, \ldots, n$. The base case is clear: $(12) = (12)$. The induction step follows from the identity $(1, k+1) = (1k)(k, k+1)(1k)$. By part (a), the set $(12), (13), \ldots, (1n)$ generates S_n, and thus $(12), (23), \ldots, (n-1, n)$ does as well.

(c) $(12), (12 \ldots n)$

Solution. We prove by induction that $(k-1, k)$ can be written in terms of $(12), (12 \ldots n)$ for $k = 2, 3, \ldots, n$. The base case is again clear: $(12) = (12)$. The induction step follows from the identity $(k, k+1) = (12 \ldots n)(k-1, k)(n \ldots 21)$. By part (b), the set $(12), (23), \ldots, (n-1, n)$ generates S_n, and thus $(12), (12 \ldots n)$ does as well.

5.30. Let $\tau = (a_1, a_2, \ldots, a_k)$ be a cycle of length k.

(a) Prove that if σ is any permutation, then

$$\sigma\tau\sigma^{-1} = (\sigma(a_1), \sigma(a_2), \ldots, \sigma(a_k))$$

is a cycle of length k.

Solution. Let $L = \sigma \cdot \tau$ and $R = (\sigma(a_1), \sigma(a_2), \ldots, \sigma(a_k)) \cdot \sigma$. We show that $L = R$ by proving that $L(x) = R(x)$ for $x = 1, 2, \ldots, n$. There are two cases: $x = a_i$ for some i and $x \neq a_i$ for any i. If $x = a_i$ then

$$L(x) = \sigma\tau(a_i) = \sigma(a_{i+1}),$$

where we set $a_{k+1} = a_1$ by convention. Since

$$R(x) = (\sigma(a_1), \sigma(a_2), \ldots, \sigma(a_k))(\sigma(a_i)) = \sigma(a_{i+1}),$$

L and R have the same value on x.

If $x \neq a_i$ then x is fixed by τ and thus $L(x) = \sigma(x)$. Similarly, $\sigma(x)$ is fixed by the cycle $(\sigma(a_1), \sigma(a_2), \ldots, \sigma(a_k))$ so $R(x) = \sigma(x)$.

Since $L = R$, we also have $L\sigma^{-1} = R\sigma^{-1}$.

(b) Let μ be a cycle of length k. Prove that there is a permutation σ such that $\sigma\tau\sigma^{-1} = \mu$.

Solution. Let $\mu = (b_1, b_2, \ldots, b_k)$. For $i = 1, \ldots, k$ define $\sigma(a_i) = b_i$. Since the sets $X = \{1, \ldots, n\} - \{a_1, \ldots, a_k\}$ and $Y = \{1, \ldots, n\} - \{b_1, \ldots, b_k\}$ both have cardinality $n - k$, there exists a bijection ϕ between them. Set $\sigma(x) = \phi(x)$ for $x \neq a_i$. Then $\sigma \in S_n$ and, by part (a), $\sigma\tau\sigma^{-1} = \mu$.

6.5(f). List the left and right cosets of D_4 in S_4.

Solution. Label the vertices of the square $1, 2, 3, 4$ in clockwise order. Then the elements of D_4, as a subgroup of S_4, are

$$\{(), (1234), (13)(24), (1432), (12)(34), (14)(23), (13), (24)\},$$

and this set is both a left and right coset.

Since $(12) \notin D_4$,

$$(12)D_4 = \{(12), (234), (1324), (143), (34), (1423), (132), (124)\}$$

is another left coset of D_4. Moreover, since $g_1H = g_2H \Leftrightarrow Hg_1^{-1} = Hg_2^{-1}$, the set consisting of the inverses of these elements is a right coset of D_4:

$$D_4(12) = \{(12), (243), (1423), (134), (34), (1324), (123), (142)\}$$

Finally, we can construct the remaining left coset by collecting the remaining elements,

$$(14)D_4 = \{(14), (23), (123), (142), (134), (243), (1243), (1342)\},$$

and the remaining right coset likewise:

$$D_4(14) = \{(14), (23), (132), (124), (143), (234), (1342), (1243)\}.$$

6.8. Use Fermat's Little Theorem to show that if $p = 4n+3$ is prime, there is no solution to the equation $x^2 \equiv -1 \pmod{p}$.

Solution. Suppose that $x^2 \equiv -1 \pmod{p}$ for some x. Then $x \not\equiv 0 \pmod{p}$, so Fermat's Little Theorem implies that $x^{p-1} \equiv 1 \pmod{p}$. Thus

$$\begin{aligned} 1 &\equiv x^{p-1} \pmod{p} \\ &\equiv x^{4n+2} \pmod{p} \\ &\equiv (x^4)^n \cdot x^2 \pmod{p} \\ &\equiv 1^n \cdot (-1) \pmod{p} \\ &\equiv -1 \pmod{p}. \end{aligned}$$

This is a contradiction since $p \neq 2$.

6.11. Let H be a subgroup of a group G and suppose that $g_1, g_2 \in G$. Prove that the following conditions are equivalent.

(a) $g_1H = g_2H$
(b) $Hg_1^{-1} = Hg_2^{-1}$
(c) $g_1H \subseteq g_2H$
(d) $g_2 \in g_1H$
(e) $g_1^{-1}g_2 \in H$

Solution.

(a) $\Rightarrow$ (c) Since $g_1H = g_2H$, certainly $g_1H \subseteq g_2H$.

(c) $\Rightarrow$ (d) Since $g_1 \in g_1H \subseteq g_2H$, there is an $h \in H$ with $g_1 = g_2h$. Then $g_2 = g_1h^{-1} \in g_1H$ since $h^{-1} \in H$.

(d) $\Rightarrow$ (e) Since $g_2 = g_1h$ for some $h \in H$, $g_1^{-1}g_2 = h \in H$.

(e) $\Rightarrow$ (b) Let $g_1^{-1}g_2 = h \in H$. We show first that $Hg_1^{-1} \subseteq Hg_2^{-1}$. Suppose $h'g_1^{-1} \in Hg_1^{-1}$. Since $g_1^{-1} = hg_2^{-1}$, we have $h'g_1^{-1} = h'hg_2^{-1} \in Hg_2^{-1}$. Conversely,$h''g_2^{-1} = h''h^{-1}g_1^{-1} \in Hg_1^{-1}$, so $Hg_2^{-1} \subseteq Hg_1^{-1}$.

(b) $\Rightarrow$ (a) Since $Hg_1^{-1} = Hg_2^{-1}$, there is an $h \in H$ with $g_1^{-1} = hg_2^{-1}$, so $g_1^{-1}g_2 = h$ and $g_2 = g_1h$. Then for any $h' \in H$, $g_2h' = g_1hh'$, so $g_2H \subseteq g_1H$. Similarly, $g_1h' = g_2h^{-1}h'$, so $g_1H \subseteq g_2H$.

6.15. Show that any two permutations $\alpha, \beta \in S_n$ have the same cycle structure if and only if there exists a permutation γ such that $\beta = \gamma\alpha\gamma^{-1}$.

Solution. Suppose first that $\beta = \gamma\alpha\gamma^{-1}$, and let $\alpha = \alpha_1\alpha_2 \ldots \alpha_k$ be a decomposition of α into disjoint cycles. Then $\beta = (\gamma\alpha_1\gamma^{-1})(\gamma\alpha_2\gamma^{-1}) \ldots (\gamma\alpha_k\gamma^{-1})$. By 5.30(a), $(\gamma\alpha_i\gamma^{-1})$ is a cycle of the same length as α_i, and if $i \neq j$ then $(\gamma\alpha_i\gamma^{-1})$ is disjoint from $(\gamma\alpha_j\gamma^{-1})$. Thus the cycle structures of α and β are the same.

Conversely, suppose that α and β have the same cycle structure. Then we get write $\alpha = \alpha_1\alpha_2 \ldots \alpha_k$ and $\beta = \beta_1\beta_2 \ldots \beta_k$, with $\alpha_i = (a_1, \ldots, a_{n_i})$ and $\beta_i = (b_1, \ldots, b_{n_i})$. Let X be the complement of the $a_{i,j}$ in $\{1, \ldots, n\}$ and let Y be the complement of the $b_{i,j}$. Then the cardinality of X is the same as the cardinality of Y, and we may choose a bijection γ between them. Extending γ to all of $\{1, \ldots, n\}$ by setting $\gamma(a_{i,j}) = b_{i,j}$ yields a permutation, and by 5.30(a), $\beta = \gamma\alpha\gamma^{-1}$.

6.18. If $[G : H] = 2$, prove that $gH = Hg$.

Solution. Since there are only two left cosets of H, which are disjoint, and one of them is H itself, the left cosets are H and $G - H$. The same holds for the right cosets. Moreover, $gH = H$ iff $g \in H$ iff $Hg = H$, and $gH = G - H$ iff $g \notin H$ iff $Hg = G - H$. Thus $Hg = gH$ for all $g \in G$.