Math 220 - Practice Final (Spring 2007) Solutions

1. (a) $2^{2\log_2 3+\log_2 5} = \left(2^{\log_2 3}\right)^2 \cdot \left(2^{\log_2 5}\right) = 3^2 \cdot 5 = 45.$

   (b) We let $y = \frac{x+1}{x-1}$ and solve for x:

$$\begin{aligned} y(x-1) &= x+1 \\ xy - x - y - 1 &= 0 \\ x(y-1) &= y+1 \\ x &= \frac{y+1}{y-1}. \end{aligned}$$

   So the inverse function is $f^{-1}(x) = \frac{x+1}{x-1}$, which happens to be equal to $f(x)$ (note that the graph of $f(x)$ is symmetric about the line $y = x$).

   (d) The derivative at $x = 1$ is $2x = 2$, so the tangent line has slope 2 and equation $y - 1 = 2(x - 1)$.

2. (a)

$$\begin{aligned} f'(1) &= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h\to 0} \frac{h2^{1+h}\sqrt{1+(1+h)^2} - 0 \cdot 2 \cdot \sqrt{2}}{h} \\ &= \lim_{h\to 0} 2^{1+h}\sqrt{1+(1+h)^2} \\ &= 2\sqrt{2}. \end{aligned}$$

   (b) The graph of this function is a semicircle around the point $(0, 2)$, so the area between it and the x-axis is the sum of a semicircle of radius 1 and a square of side length 2. This area is $4 + \frac{\pi}{2}$.

   (c) We integrate the velocity function to get $s(t) = t + t^2 + C$. Since $s(0) = 0$, we have $C = 0$ and $s(t) = t + t^2$.

   (d) We make the substitution $u = x - \frac{\pi}{2}$ and use the identities $\cos(x) = -\sin(x - \frac{\pi}{2})$ and $\sin(x) = \cos(x - \frac{\pi}{2})$:

$$\begin{aligned} \int_0^{\pi} \frac{\cos(x)}{1 + \sin^2(x) + \sin^4(x)}\,dx &= \int_{-pi/2}^{\pi/2} \frac{-\sin(u)}{1+\cos^2(u)+\cos^4(u)} \\ &= 0 \end{aligned}$$

   since the integrand is an odd function and the interval is of the form $[-a, a]$.

3. (a)

$$\begin{aligned} \lim_{x\to 2^-} \frac{|x-2|}{x^2-4} &= \lim_{x\to 2^-} \frac{2-x}{x^2-4} \\ &= \lim_{x\to 2^-} \frac{-1}{x+2} \\ &= -\frac{1}{4} \end{aligned}$$

(b) Since the numerator and denominator both evaluate to 0, L'Hospital's rule applies.

$$\begin{aligned}\lim_{x\to 0}\frac{e^x-1}{x} &= \lim_{x\to 0}\frac{e^x}{1}\\ &= 1.\end{aligned}$$

(c) Since

$$\frac{1}{x}-\frac{1}{\sin(x)}=\frac{\sin(x)-x}{x\sin(x)}$$

and both numerator and denominator evaluate to 0 when $x = 0$, we may use L'Hospital's rule. Differentiating top and bottom, we get

$$\frac{\cos(x)-1}{\sin(x)+x\cos(x)}.$$

Again, both numerator and denominator evaluate to 0 so we apply L'Hospital's rule again.

$$\frac{-\sin(x)}{2\cos(x)-x\sin(x)}$$

evaluates to $\frac{0}{2}$, so

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin(x)}\right)=0.$$

(d) This limit is of indeterminate form 1^∞, so we need to take the logarithm and apply L'Hospital's rule. The natural log is

$$\frac{\ln(1+x)}{x},$$

and differentiating numerator and denominator yields

$$\frac{1/(1+x)}{1}.$$

Evaluating at $x = 0$ gives 1, so the original limit is $e^1 = e$.

4. (a) Using the chain rule,

$$\begin{aligned}h'(1) &= f'(g(1))g'(1)\\ &= f'(3)g'(1)\\ &= 6\cdot 5\\ &= 30.\end{aligned}$$

Using the product rule,

$$\begin{aligned}k'(1) &= f'(1)g(1)+f(1)g'(1)\\ &= 4\cdot 3+2\cdot 4\\ &= 20.\end{aligned}$$

(b) When $x = 2$, we observe that $y = 1$ is a solution to $y + y^3 = 2$ (note that this is the only real solution, since $y^3+y-2 = (y-1)(y^2+y+2)$ and the discriminant of y^2+y+2 is $1^2-4\cdot 2\cdot 1 < 0$). Therefore $y(2) = 1$.

We differentiate twice, yielding

$$\begin{aligned}1 &= y' + 3y^2y' \\ 0 &= y'' + 6yy' + 3y^2y''.\end{aligned}$$

Substituting $y = 1$ into the first equation gives

$$4y' = 1,$$

and thus $y'(2) = \frac{1}{4}$. Substituting $y = 1$ and $y' = \frac{1}{4}$ into the second equation gives

$$4y'' + \frac{6}{4} = 0,$$

and thus $y''(2) = -\frac{3}{8}$.

5. (a) $f'(x) = e^x + \frac{2}{x} + 3\cos(x) + \frac{4}{1+x^2} + \frac{5}{\sqrt{1-x^2}}$

   (b) $g(x) = e^{x\ln(x)}$ so $g'(x) = (\ln(x) + 1)x^x$. You can also use logarithmic differentiation.

   (c) $h'(x) = -\sin(\sqrt{1+x^2}) \cdot \frac{x}{\sqrt{1+x^2}}$

   (d) Let $K(u) = \int_0^u e^{t^2}\,dt$. Then $k(x) = K(u)$ with $u = 2x$, and $k'(x) = K'(u)u'(x) = e^{4x^2} \cdot 2$, using the fundamental theorem of calculus and the chain rule.

6. (a)

$$\int \left(x^2 + \frac{2}{x} + 3\sin(x) + 4^x + \frac{5}{1+x^2}\right) dx = \frac{x^3}{3} + 2\ln(x) - 3\cos(x) + \frac{4^x}{\ln(4)} + 5\tan^{-1}(x) + C.$$

   (b) With the substitution $u = x^2 + x + 1$,

$$\begin{aligned}\int (2x+1)(x^2+x+1)^3\,dx &= \int u^3\,du \\ &= \frac{u^4}{4} + C \\ &= \frac{1}{4}(x^2+x+1)^4 + C.\end{aligned}$$

   (c) Using integration by parts with $u = x$ and $dv = \cos(x)dx$,

$$\begin{aligned}\int x\cos(x)\,dx &= x\sin(x) - \int \sin(x)\,dx \\ &= x\sin(x) + \cos(x) + C.\end{aligned}$$

7. (a) We are given the equations

$$\begin{aligned}V &= \pi r^2 h \\ A &= 2\pi r^2 + 2\pi r h \\ A &= 600\pi.\end{aligned}$$

With four variables and three equations, we are ready to proceed, solving for V in terms of a single variable r. We solve the equation

$$2\pi r^2 + 2\pi r h = 600\pi$$

for h, giving $h = \frac{300-r^2}{r}$ and thus

$$V = \pi r(300 - r^2).$$

Differentiating and setting $V' = 0$, we get

$$300\pi - 3\pi r^2 = 0,$$

so $r = 10$. The volume is then $V = \pi \cdot 10 \cdot (300 - 100) = 2000\pi$.

(b) Suppose that the car is traveling along the positive x-axis (with coordinate x) and the truck along the positive y-axis (with coordinate y). Then the distance between them is

$$D = \sqrt{x^2 + y^2}.$$

Differentiating, we get that

$$\frac{dD}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{x^2 + y^2}.$$

We're given the following information in the problem:

$$\begin{aligned} x &= 3 \\ y &= 4 \\ \frac{dx}{dt} &= 100 \\ \frac{dy}{dt} &= 80. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{dD}{dt} &= \frac{3 \cdot 100 + 4 \cdot 80}{\sqrt{3^2 + 4^2}} \\ &= \frac{620}{5} \\ &= 124. \end{aligned}$$

So they are separating at a rate of 124 miles per hour.

8. Newton's method iterates with $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$. Here, $f(x) = x^3 - x + 2$ and $f'(x) = 3x^2 - 1$. So, starting from $x_0 = 0$,

$$\begin{aligned} x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\ &= 0 - \frac{2}{-1} \\ &= 2 \\ x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} \\ &= 2 - \frac{8}{11} \\ &= \frac{14}{11}. \end{aligned}$$

The linear approximation is given by the tangent line:

$$\begin{aligned} L(x) &= f(a) + f'(a)(x-a) \\ &= 1^{1/10} + \frac{1}{10}1^{-9/10}(x-1) \\ &= 1 + \frac{1}{10}(x-1) \\ 1.1^{1/10} &\approx L(1.1) \\ &= 1 + \frac{1}{10}(1.1-1) \\ &= 1.01. \end{aligned}$$

(b) There is a typo in this problem: it should be $c_i = \frac{1}{2}(x_{i-1} + x_i)$. With this change,

$$\begin{aligned} R_4 &= \sum_{i=1}^{4} f(c_i)(x_i - x_{i-1}) \\ &= \sum_{i=1}^{4} c_i^2 (2i - 2(i-1)) \\ &= 2\sum_{i=1}^{4} c_i^2 \\ &= 2(1+9+25+49) \\ &= 168. \end{aligned}$$

Note that this is quite close to the exact value of the integral, $[\frac{x^3}{3}]_0^8 = 170\frac{2}{3}$.

9. (a) $f(x)$ is increasing when $f'(x) > 0$, which occurs when $-1 < x < 1$. It is decreasing when $f'(x) < 0$, which occurs when $x < -1$ or $x > 1$. The only local minimum is therefore at $x = -1$, where $f(x) = -1$ and the only local maximum is at $x = 1$, where $f(x) = 1$. Here we use the first derivative test to determine whether each point is a minimum or maximum, and we will see in part (c) that these are also global extreme values.

(b) $f(x)$ is concave up when $f''(x) > 0$, which occurs when $x > \sqrt{3}$ or $-\sqrt{3} < x < 0$. Similarly, $f(x)$ is concave down when $f''(x) < 0$, which occurs when $x < -\sqrt{3}$ or $0 < x < \sqrt{3}$.

(c) As $x \to \pm\infty$, the exponent $(1 - x^2)/2 \to -\infty$ and thus $f(x) \to 0$ (either using L'Hospital's rule or the fact that exponentials dominate polynomials). Therefore $y = 0$ is a horizontal asymptote.

(d) Here's the graph for comparison with your sketch.