Final Exam Practice

1. Basics (20 points)

   (a) Simplify the expression $2^{2\log_2 3+\log_2 5}$.

   (b) Find the inverse function of $f(x) = \dfrac{x+1}{x-1}$.

   (c) Eliminate the parameter t to find a Cartesian equation of the curve

   $$x = 1 + 3t, \qquad y = 2 - t^2.$$

   (d) Find the equation of the line that is tangent to the curve $y = x^2$ at the point $(1, 1)$.

2. Basics (25 points)

   (a) Use $f'(a) = \lim_{h\to 0} \dfrac{f(a+h) - f(a)}{h}$ to find $f'(1)$ where $f(x) = (x-1)2^x\sqrt{1+x^2}$

   (b) Using geometry, find $\int_{-1}^{1} (2 + \sqrt{1-x^2})\, dx$.

   (c) Find the position function $s(t)$ of a particle that moves along a straight line with velocity function $v(t) = 1 + 2t$ and initial displacement $s(0) = 0$.

   (d) Evaluate the definite integral $\displaystyle\int_0^{\pi} \frac{\cos x}{1+\sin^2 x+\sin^4 x}\, dx$.

3. Find the following limits (25 points)

   (a) $\displaystyle\lim_{x\to 2^-} \frac{|x-2|}{x^2-4}$

   (b) $\displaystyle\lim_{x\to 0} \frac{e^x - 1}{x}$

   (c) $\displaystyle\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{\sin x}\right)$

   (d) $\displaystyle\lim_{x\to 0} (1+x)^{1/x}$

4. Differentiation Techniques

   (a) Suppose $f(1) = 2$, $g(1) = 3$, $f(3) = 4$, $g(2) = 5$, $f'(1) = 4$, $g'(1) = 5$, $f'(3) = 6$. Suppose $h(x) = f(g(x)$ and $k(x) = f(x)g(x)$. Find $h'(1)$ and $k'(1)$.

   (b) Let $y(x)$ be implicitly defined by $x = y + y^3$. Find $y(2)$, $y'(2)$ and $y''(2)$.

5. Find the derivative of the following functons: (20 points)

   (a) $f(x) = e^x + 2\ln x + 3\sin x + 4\arctan x + 5\arcsin x$

   (b) $g(x) = x^x$

   (c) $h(x) = \cos\sqrt{1 + x^2}$

   (d) $k(x) = \int_0^{2x} e^{t^2}\, dt$

6. Find the following integrals (20 points)

   (a) $\int \left(x^2 + \frac{2}{x} + 3\sin x + 4^x + \frac{5}{1+x^2}\right) dx$

   (b) $\int (2x+1)(x^2+x+1)^3\, dx$

   (c) $\int x\cos x\, dx$

   (d) $\int \frac{1}{(2-x)(x+3)}\, dx$

7. Application of Derivatives (25 points)

   (a) Find the maximum volume V of a circular cylindrical tin can that has a total surface area $A = 600\pi$ cm^2. (Hint: If a tin can has base radius r and height h, then volume is $V = \pi r^2 h$ and surface area $A = 2\pi r^2 + 2\pi rh$.)

   (b) A car is traveling east 100 mile/hour and a truck is traveling north 80 mile/hour. At a time moment when the car is 3 miles east of an intersection and the truck is 4 miles north of the same intersection, what is the relative speed of departing between the car and the truck?

8. Approximation (20 points)

   (a) Consider the problem of finding a root of $x^3 - x + 2 = 0$. Using Newton's method and starting from the initial guess $x = 0$, find the next two iterations.

   (b) Find the linear approximation $L(x)$ for the function $f(x) = x^{1/10}$ around the point $a = 1$ and use $L(x)$ to calculate approximately the numerical value of $1.1^{1/10}$.

   (c) Find the Riemann sum $R_4 = \sum_{i=1}^{4} f(c_i)\,(x_i - x_{i-1})$ for $\int_0^8 x^2\,dx$ with regular partition points $x_i = 2i$ for $i = 0, 1, 2, 3, 4$ and the middle point rule: $c_i = \frac{1}{2}(x_{i-1} - x_i)$.

9. Plot Curve (20 points)
   Let $f(x) = xe^{(1-x^2)/2}$. Differentiation gives that

$$f'(x) = (1 - x^2)e^{(1-x^2)/2}, \qquad f''(x) = x(x^2 - 3)e^{(1-x^2)/2}.$$

   (a) Find the intervals where f is increasing or decreasing. Also find points of local or global minimum or maximum.

   (b) Find intervals where f is concave up of concave down.

   (c) Find any horizontal asymptotes.

   (d) Sketch the curve $y = f(x)$ for $-\infty < x < \infty$.