Math 220 - Practice Final (Spring 2005) Solutions

3. (a) We find the derivative,

$$f'(x) = (x-1)^2 + 2x(x-1)$$
$$= (x-1)(3x-1).$$

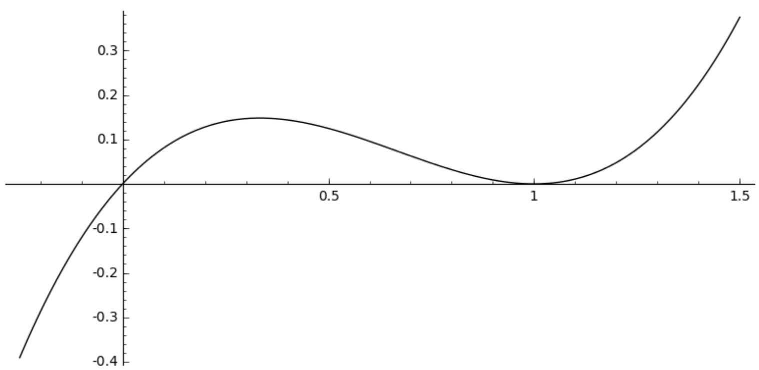
The critical points are at $x = 1$ and $x = 1/3$. Since $f'(x) < 0$ for $1/3 < x < 1$ and $f'(x) > 0$ elsewhere, $x = 1/3$ is a maximum and $x = 1$ is a minimum (by the first derivative test).

(b) The second derivative is

$$f''(x) = (3x-1) + 3(x-1)$$
$$= 6x - 4.$$

There is one inflection point at $x = 2/3$, where $f''(x)$ changes sign.

(c)



4. The equation to obtain the next approximation in Newton's method is

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Applying this to the function $f(x) = x^4 - 10100$ with $x_1 = -10$ gives

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$
$$= -10 - \frac{10000 - 10100}{4 \cdot (-1000)}$$
$$= -10 - \frac{1}{40}$$
$$= -10.025.$$

5. (a) The slope is given by $y' = \frac{1}{4}x^{-3/4} = \frac{1}{4000}$, and the equation by $y - 10 = \frac{1}{4000}(x - 10000)$.

(b) We find y on the tangent line when $x = 10100$:

$$\begin{aligned} y &= 10 + \frac{1}{4000}(100) \\ &= 10.025. \end{aligned}$$

1. (a) This has indeterminate form $0/0$, so L'Hospital's rule applies. Differentiating top and bottom, we get $\frac{1}{\frac{1}{2\sqrt{x+4}}} = 2\sqrt{x+4}$. Evaluating at $x = 0$ yields 4, so

$$\lim_{x\to 0} \frac{x}{\sqrt{x+4}-2} = 4.$$

(b) When $x < 3$, $|x-3| = 3 - x$, so

$$\lim_{x\to 3^-} \frac{|x-3|}{x-3} = lim_{x\to 3^-} \frac{3-x}{x-3} = -1.$$

(c) Near $x = 1$, $-\pi/2 < 2x - 5 < \pi/2$ so $\arctan(\tan(2x-5)) = 2x - 5$. Therefore

$$\lim_{x\to 1} \frac{\arctan(\tan(2x-5))}{2x-5} = \lim_{x\to 1} \frac{2x-5}{2x-5} = 1.$$

(d) As $x \to -\infty$, numerator and denominator both tend to 0, so L'Hospital's rule applies. Differentiating numerator and denominator yields

$$\frac{1/\left(1+\frac{3}{x^3}\right)\cdot\frac{-9}{x^4}}{\cos\left(\frac{4}{x^2}\right)\cdot\frac{-8}{x^3}} = \frac{9}{8x\left(1+\frac{3}{x^3}\right)\cos\left(\frac{4}{x^2}\right)}.$$

As $x \to -\infty$, this ratio tends to 0. Therefore

$$\lim_{x\to -\infty} \frac{\ln\left(1+\frac{3}{x^3}\right)}{\sin\left(\frac{4}{x^2}\right)} = 0.$$

(e) This limit is of the form $0 \cdot (-\infty)$, so we rewrite it as $\lim_{x\to 0} \frac{\ln(x^2)}{1/x^2}$. L'Hospital's rule now applies, and we differentiate numerator and denominator, yielding

$$\frac{2x/x^2}{-2/x^3} = -x^2.$$

Therefore

$$\begin{aligned} \lim_{x\to 0} x^2 \ln(x^2) &= \lim_{x\to 0} -x^2 \\ &= 0. \end{aligned}$$

7. (a) Using the substitution $u = 5x/2$,

$$\begin{aligned} \int \frac{dx}{4+25x^2} &= \frac{1}{4}\int \frac{dx}{1+(5x/2)^2} \\ &= \frac{1}{10}\int \frac{du}{1+u^2} \\ &= \frac{1}{10}\tan^{-1}(5x/2) + C. \end{aligned}$$

(b)

$$\int \left(12^x + x^{1/2}\right) dx = \frac{12^x}{\ln(12)} + \frac{2}{3}x^{3/2} + C.$$

(c) Let $g(u) = \int_0^u \frac{dt}{\sqrt{1+t^2}}$ and $u = 2x$. Then $f(x) = g(u(x))$ and $\frac{df}{dx} = \frac{dg}{du}\frac{du}{dx} = \frac{1}{\sqrt{1+4x^2}} \cdot 2$.

8. (a) Differentiating, we have

$$2yy' + e^{y^2} + 2xyy'e^{y^2} = 0.$$

Substituting $(x, y) = (0, 1)$ and solving gives $2y' + e = 0$, so $\frac{dy}{dx} = \frac{e}{2}$.

(b) Differentiating, we have

$$\begin{aligned} y' &= \frac{6\sin(x)\cos(x)}{1 + 9\sin^4(x)} \\ y'(\pi/4) &= \frac{6\sin(\pi/4)\cos(\pi/4)}{1 + 9\sin^4(\pi/4)} \\ &= \frac{6/2}{1 + 9/4} \\ &= \frac{12}{13}. \end{aligned}$$

(c) Writing $y = e^{2x\ln(x)}$ and differentiating, we get

$$\frac{dy}{dx} = 2(\ln(x) + 1)x^{2x}.$$

You can also use logarithmic differentiation.

9. Implicitly differentiating with respect to time,

$$4xx' - x'y - xy' + 6yy' = 0.$$

Substituting $x = -3$, $y = 1$ and $x' = 5$, we solve for y':

$$\begin{aligned} 4(-3)(5) - (5)(1) - (-3)y' + 6(1)y' &= 0 \\ 9y' &= 65 \\ y' &= \frac{65}{9}. \end{aligned}$$