Math 220 - Practice Exam 2 Solutions

1. Determine the derivatives of the following functions:

   (a) $f(x) = \ln(x + e^{-x})$,

   Solution. $f'(x) = \frac{1-e^{-x}}{x+e^{-x}}$

   (b) $f(x) = e^{\sin^{-1}(x)}$,

   Solution. $f'(x) = \frac{e^{\sin^{-1}(x)}}{\sqrt{1-x^2}}$

   (c) $f(x) = (x + \sin(x))^{\cos(x)}$,

   Solution. Since $f(x) = e^{\ln(x+\sin(x))\cos(x)}$, we have

   $$f'(x) = (x + \sin(x))^{\cos(x)} \left(\frac{\cos x + \cos^2(x)}{x + \sin(x)} - \ln(x + \sin(x))\sin(x) \right).$$

   You can also use logarithmic differentiation.

   (d) $f(x) = x^{(2^x)}$,

   Solution. Since $f(x) = e^{\ln(x)e^{\ln(2)x}}$, we have

   $$f'(x) = x^{(2^x)} 2^x \left(\frac{1}{x} + \ln(x)\ln(2) \right).$$

   You can also use logarithmic differentiation.

   (e) $f(x) = \sinh(2x) + \cosh^{-1}(3x)$.

   Solution. $f'(x) = 2\cosh(2x) + \frac{3}{\sqrt{9x^2-1}}$.

2. Evaluate so that your answer is a fraction.

   (a) $\tanh(\ln(2)) =$

   Solution. $\tanh(\ln(2)) = \frac{e^{\ln(2)} - e^{-\ln(2)}}{e^{\ln(2)} + e^{-\ln(2)}} = \frac{2-1/2}{2+1/2} = \frac{3}{5}$.

   (b) $\sin(\tan^{-1}(\frac{3}{4}) + \cos^{-1}(\frac{5}{13})) =$

   Solution. By the sum-angle formula for sine, we have

   $$\begin{aligned} \sin(\tan^{-1}(\frac{3}{4}) + \cos^{-1}(\frac{5}{13})) &= \sin(\tan^{-1}(\frac{3}{4}))\cos(\cos^{-1}(\frac{5}{13}))) + \cos(\tan^{-1}(\frac{3}{4}))\sin(\cos^{-1}(\frac{5}{13}))) \\ &= \frac{3}{5} \cdot \frac{5}{13} + \frac{4}{5} \cdot \frac{12}{13} \\ &= \frac{3}{13} + \frac{48}{65} \\ &= \frac{63}{65}. \end{aligned}$$

   Here we used the right triangles of side lengths 3,4,5 and 5,12,13.

3. Determine each limit. Show your work.

   (a) $\lim_{x\to 0} \frac{\cos(2x)-1}{x^2+x^3}$

   Solution. Evaluating numerator and denominator at 0 yields 0/0, so L'Hospital's Rule applies. Differentiating top and bottom, we compute

$$\lim_{x\to 0} \frac{-2\sin(2x)}{2x+3x^2}.$$

   Evaluating numerator and denominator at 0 again yields 0/0, so we apply L'Hospital's Rule a second time and consider

$$\lim_{x\to 0} \frac{-4\cos(2x)}{2+6x}.$$

   Now, evaluating at 0 gives -2, which is thus the value for the original limit:

$$\lim_{x\to 0} \frac{\cos(2x)-1}{x^2+x^3} = -2.$$

   (b) $\lim_{x\to 0} \left(x+e^{2x}\right)^{\cot(x)}$

   Solution. We take the natural log of the function and use the definition of $\cot(x) = \frac{\cos(x)}{\sin(x)}$, yielding

$$\frac{\cos(x)\ln(x+e^{2x})}{\sin(x)}.$$

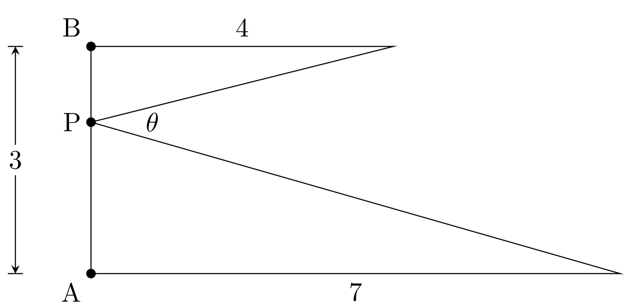
   Evaluating numerator and denominator at 0 gives 0/0, so L'Hospital's Rule applies. Differentiating top and bottom, we compute

$$\lim_{x\to 0} \frac{-\sin(x)\ln(x+e^{2x}) + \frac{\cos(x)\cdot(1+2e^{2x})}{x+e^{2x}}}{\cos(x)}.$$

   Evaluating numerator and denominator at 0 now gives 3. Since the limit of the natural log of the original function is 3,

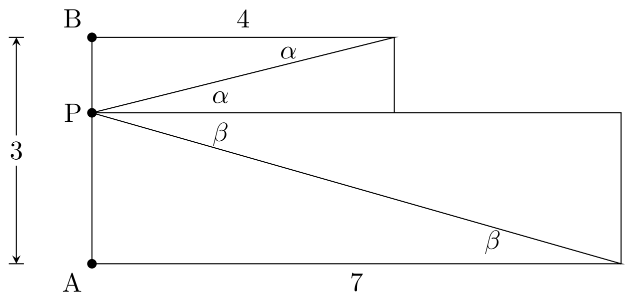
$$\lim_{x\to 0} \left(x+e^{2x}\right)^{\cot(x)} = e^3.$$

4. Consider the following diagram, where the point P is on the line segment AB.



   (a) Where should P be placed on AB in order to maximize the angle θ?

   Solution. Let x be the length of BP and y the length of AP. Then $x+y=3$ and $\theta = \tan^{-1}(x/4) + \tan^{-1}(y/7)$ (see the following diagram, where $\theta = \alpha+\beta$ and $\tan(\alpha) = x/4$ and $\tan(\beta) = y/7$).



Substituting for y we get

$$\theta = \tan^{-1}(x/4) + \tan^{-1}((3-x)/7).$$

Differentiating,

$$\frac{d\theta}{dx} = \frac{1/4}{1+x^2/16} + \frac{-1/7}{1+(3-x)^2/49}$$
$$= \frac{4}{16+x^2} - \frac{7}{49+(3-x)^2}$$

We set this equal to 0 and multiply by $(16+x^2)(49+(3-x)^2)$, giving

$$4(49+(3-x)^2) = 7(16+x^2)$$
$$4x^2 - 24x + 232 = 112 + 7x^2$$
$$3x^2 + 24x - 120 = 0$$
$$x^2 + 8x - 40 = 0$$
$$x = \frac{-8 \pm \sqrt{64+160}}{2}$$
$$x = \pm\sqrt{56} - 4$$

Since $7 < \sqrt{56} < 8$, these local extrema are actually not within the interval AB. Since there are no local extrema within the interval, the minimum and maximum will occur at the endpoints. So we need to determine which of $\tan^{-1}(3/4)$ and $\tan^{-1}(3/7)$ is larger. Since $\tan^{-1}(x)$ is an increasing function and $3/7 < 3/4$,

$$\tan^{-1}(3/7) < \tan^{-1}(3/4).$$

Thus the maximum value is $\tan^{-1}(3/4)$ when $P = A$.

(b) Where should P be placed on AB in order to minimize the angle θ?

Solution. The minimum value is $\tan^{-1}(3/7)$ when $P = B$.

5. Sketch a graph of the function $f(x) = \ln(\sin^2(x))$ on the interval $(-2\pi, 2\pi)$.

(a) Show all vertical asymptotes, if they exist.

(b) Show intercepts if they exist.

(c) Show the coordinates of local minima or maxima if they exist. For each, explain why it is a minimum or a maximum.

Solution. $\ln(y)$ has an asymptote when $y = 0$, and $\sin^2(x) = 0$ when x is a multiple of π. Similarly, $\ln(y) = 0$ when $y = 1$, and $\sin^2(x) = 1$ when x is an odd multiple of $\pi/2$.

$$f'(x) = \frac{2\cos(x)}{\sin(x)} = 2\cot(x)$$
$$f''(x) = -2\csc^2(x).$$

The critical points occur at odd multiples of $\pi/2$ as well, and they are all maxima by the second derivative test. Moreover, by the nature of the asymptote for $\ln(y)$ and the fact that $\sin^2(x)$ is never negative, $\lim_{x\to n\pi} f(x) = -\infty$ for any integer n. So the graph of $f(x)$ on $(-2\pi, 2\pi)$ has four concave down pieces, rising up from $-\infty$ at the multiples of π to touch the x-axis in a maximum value of 0 at the odd multiples of $\pi/2$.

6. Let $f(x) = xe^{-x^2/8}$.

   (a) Find the minimum and maximum values of $f(x)$ on the interval $[-5, 1]$.

   Solution. We differentiate to find the critical points.

$$f'(x) = e^{-x^2/8}(1 - x^2/4),$$

   so $f'(x) = 0$ when $x = \pm 2$. Differentiating again,

$$f''(x) = e^{-x^2/8}(-3x/4 + x^3/16).$$

   When $x = 2$ this is negative (maximum) and when $x = -2$ this is positive (minimum). Moreover,

$$\lim_{x\to\pm\infty} f(x) = 0,$$

   so these are a global maximum and minimum. But 2 is not in the interval, so the largest value for $f(x)$ will occur at $x = 1$. In summary, the minimum is $\frac{-2}{\sqrt{e}}$ and the maximum is $\frac{1}{\sqrt[8]{e}}$.

   (b) On what intervals is $f(x)$ concave up?

   Solution. $f(x)$ is concave up when $f''(x) > 0$. We factor

$$-3x/4 + x^3/16 = \frac{x}{16}(x - \sqrt{12})(x + \sqrt{12}).$$

   This is positive for $x > \sqrt{12}$ and for $-\sqrt{12} < x < 0$.

7. Suppose that $f(x) = e^{\sin(x)} + 2\cos(e^x - 1)$ and note that $f(0) = 3$. Find $(f^{-1})'(3)$.

   Solution. By the inverse function theorem, $(f^{-1})'(3) = \frac{1}{f'(0)}$. We now compute

$$f'(x) = \cos(x)e^{\sin(x)} - 2\sin(e^x - 1)e^x.$$

   Evaluating at $x = 0$ gives $f'(0) = 1$ and thus $(f^{-1})'(3) = 1$.

8. Let $f(x) = x^3 + x$. Find a point c in the interval $[2, 11]$ satisfying the conclusion of the Mean Value Theorem.

   Solution. The Mean Value Theorem states that there is some c with

$$\begin{aligned} f'(c) &= \frac{f(11) - f(2)}{11 - 2} \\ &= \frac{1331 + 11 - 8 - 2}{9} \\ &= \frac{1332}{9} \\ &= 148. \end{aligned}$$

   To find c, we set $f'(c) = 3c^2 + 1 = 148$ and solve, yielding

$$\begin{aligned} 3c^2 &= 147 \\ c^2 &= 49 \\ c &= 7. \end{aligned}$$