Math 220 Final Exam (part 1) Solutions

1. Evaluate each of the following limits, **showing your work**. If a limit has value $\pm\infty$, give that rather than "does not exist." (3 points each)

   (a) $\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1}$

   Solution. This is a limit giving the derivative of $f(x) = \sqrt{x}$ at $x = 1$. Since $f'(x) = \frac{1}{2\sqrt{x}}$, the value of the limit is $f'(1) = \frac{1}{2}$. Alternatively, you can use L'Hospital's rule.

   (b) $\lim_{h\to 2^+} \frac{h^2-2}{h-2}$

   Solution. As $h \to 2^+$, the numerator is positive and the denominator approaches 0 from above. Therefore the limit is ∞. Note that L'Hospital's rule does not apply since the limit of the numerator is not 0.

   (c) $\lim_{x\to 0}(1 + \sin(2x))^{1/x}$

   Solution. Let $f(x) = (1 + \sin(2x))^{1/x}$. Taking logarithms,

   $$\ln(f) = \frac{\ln(1 + \sin(2x))}{x}.$$

   Both numerator and denominator tend to 0, so L'Hospital's rule applies. Differentiating top and bottom, we get

   $$\frac{2\cos(2x)/(1 + \sin(2x))}{1}.$$

   This tends to 2 as $x \to 0$. Since $\lim_{x\to 0} \ln(f(x)) = 2$, $\lim_{x\to 0} f(x) = e^2$.

   (d) $\lim_{x\to\infty} x^2e^{-x}\sin(x)$

   Solution. As $x \to \infty$, note that $x^2 \to \infty$, $e^{-x} \to 0$ and $\sin(x)$ oscillates. Consider x^2e^{-x}. Either using L'Hospital's rule twice, or what we learned about limits of products of exponentials and polynomials, we see that $x^2e^{-x} \to 0$. Therefore $\lim_{x\to\infty} x^2e^{-x}\sin(x) = 0$ as well. If you want to be more rigorous, you can use the Squeeze theorem.

2. Find the following derivatives. (3 points each)

   (a) Find the derivative of

   $$f(x) = \frac{1 + x^3e^x}{1 - x^2}.$$

   Solution. By the quotient rule and product rule,

   $$\begin{aligned} f'(x) &= \frac{(3x^2e^x + x^3e^x)(1 - x^2) - (1 + x^3e^x)(-2x)}{(1 - x^2)^2} \\ &= \frac{(-x^5 - x^4 + x^3 + 3x^2)e^x + 2x}{(1 - x^2)^2} \end{aligned}$$

   (b) Find the derivative of

   $$g(x) = \cosh(\cos(\ln|x|)).$$

Solution. Using the chain rule,

$$g'(x) = \sinh(\cos(\ln|x|))(-\sin(\ln|x|)) \cdot \frac{1}{x}.$$

(c) Find the derivative of

$$h(x) = (2 + \sin(x))^x.$$

Solution. Using logarithmic differentiation,

$$\begin{aligned} \ln(h(x)) &= x\ln(2+\sin(x)) \\ \frac{h'(x)}{h(x)} &= \ln(2+\sin(x)) + \frac{x\cos(x)}{2+\sin(x)} \\ h'(x) &= \left(\ln(2+\sin(x)) + \frac{x\cos(x)}{2+\sin(x)}\right)(2+\sin(x))^x. \end{aligned}$$

You can also rewrite $h(x) = e^{x\ln(2+\sin(x))}$ and use the chain rule.

(d) Suppose that $y(x)$ satisfies

$$3x^2y^3 - e^y = 3 - e$$

and $y(1) = 1$. Find $y'(1)$.

Solution. Using implicit differentiation,

$$\begin{aligned} 6xy^3 + 9x^2y^2y' - e^yy' &= 0 \\ 6 + (9-e)y'(1) &= 0 \\ y'(1) &= \frac{6}{e-9}. \end{aligned}$$

3. Let

$$f(x) = \frac{1}{2}x^2e^{1-x^2}$$

with derivatives

$$\begin{aligned} f'(x) &= (x - x^3)e^{1-x^2} \\ f''(x) &= (2x^4 - 5x^2 + 1)e^{1-x^2} \end{aligned}$$

(a) Where is $f(x)$ increasing and where is it decreasing? **Show your work**. (3 points)

Solution. Note that e^{1-x^2} is always positive.

$f(x)$ is increasing when $f'(x) > 0$, which occurs when $x - x^3 = x(1-x)(1+x) > 0$, which occurs for $x < -1$ and $0 < x < 1$.

$f(x)$ is decreasing when $f'(x) < 0$, which occurs when $x(1-x)(1+x) < 0$, or $-1 < x < 0$ and $x > 1$.

(b) Where is $f(x)$ concave up and where is it concave down? **Show your work**. (3 points)

Hint: The equation $2x^4 - 5x^2 + 1 = 0$ has solutions $\pm\sqrt{\frac{5\pm\sqrt{17}}{2}}$.

Solution. $f(x)$ is concave up when $f''(x) > 0$ and concave down when $f''(x) < 0$. The roots of $f''(x)$ are approximately $-3/2, -1/2, 1/2$, and $3/2$ (noting that $\sqrt{17} \approx \sqrt{16} = 4$). Thus we can compute

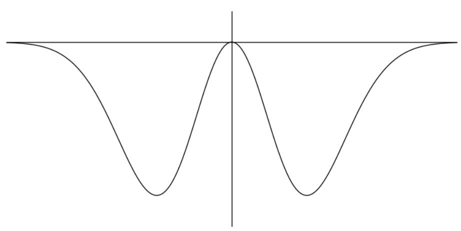
$$\begin{aligned} f''(-2) &= (32-20+1)e^{-3} > 0 \\ f''(-1) &= (2-5+1) < 0 \\ f''(0) &= (1)e^{1} > 0 \\ f''(1) &= (2-5+1) < 0 \\ f''(2) &= (32-20+1)e^{-3} > 0. \end{aligned}$$

Therefore $f(x)$ is concave up on $(-\infty, -\frac{\sqrt{5+\sqrt{17}}}{2}) \cup (-\frac{\sqrt{5-\sqrt{17}}}{2}, \frac{\sqrt{5-\sqrt{17}}}{2}) \cup (\frac{\sqrt{5+\sqrt{17}}}{2}, \infty)$ and concave down on $(-\frac{\sqrt{5+\sqrt{17}}}{2}, -\frac{\sqrt{5-\sqrt{17}}}{2}) \cup (\frac{\sqrt{5-\sqrt{17}}}{2}, \frac{\sqrt{5+\sqrt{17}}}{2})$.

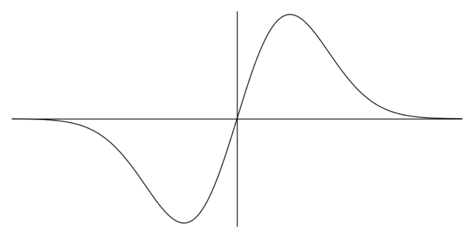
(c) Where does $f(x)$ have local maxima and local minima? **Show your work**. (3 points)

Solution. The zeros of $f'(x)$ are at $x = -1$, and $x = 0$ and $x = 1$. The first and third are maxima and the second is a minimum, all by the second derivative test and the computation in part (b).

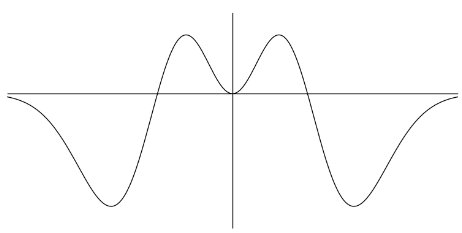
(d) Which of the following is the graph of $f(x)$? (3 points)



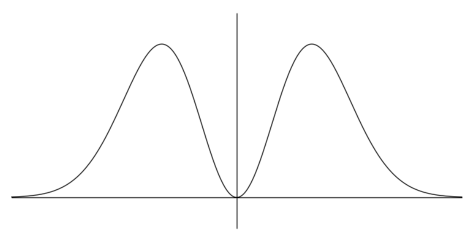
(i)



(ii)



(iii)



(iv)

Solution. The answer is (iv), since that is the only graph with the correct pattern of maximum, minimum, maximum.

4. Find the absolute minimum and maximum values of $f(x) = \frac{1}{4}x^4 - x^3 - 2x^2 + 1$ on the interval $[-1, 2]$. **Show your work**. (8 points)

Solution. The derivative is $f'(x) = x^3 - 3x^2 - 4x = x(x-4)(x+1)$. The only two critical points within the interval are at $x = -1$ and $x = 0$. Together with the endpoints, we need to compute

$$\begin{aligned} f(-1) &= \frac{1}{4} + 1 - 2 + 1 = \frac{1}{4} \\ f(0) &= 1 \\ f(2) &= 4 - 8 - 8 + 1 = -11. \end{aligned}$$

Thus the minimum value is -11 and the maximum is 1.

5. Let $f(x) = 2x - \sin(x)$.

(a) Find $f^{-1}(2\pi)$. **Show your work**. (4 points)

Solution. We need to solve $2x - \sin(x) = 2\pi$. There's not an easy way to solve equations like this generally (though you can use Newton's method to approximate a solution). In this case, without the sine term we would have $x = \pi$. Since $\sin(\pi) = 0$, $x = \pi$ is a solution (in fact, the only one). Thus $f^{-1}(2\pi) = \pi$.

(b) Find $(f^{-1})'(2\pi)$. **Show your work**. (4 points)

Solution. We have

$$\begin{aligned}(f^{-1})'(2\pi) &= \frac{1}{f'(f^{-1}(2\pi)} \\ &= \frac{1}{f'(\pi)} \\ &= \frac{1}{2 - \cos(\pi)} \\ &= \frac{1}{3}\end{aligned}$$

6. The volume of a cube is increasing at a constant rate of 30 cubic meters per second. When the cube has volume 1000 cubic meters, how fast is its surface area increasing? (8 points)

Solution. If x is the side length of the cube ($x = 10$ at the moment that $V = 1000$), then $V = x^3$ and $A = 6x^2$. Differentiating gives

$$\begin{aligned}V' &= 3x^2x' = 300x' \\ A' &= 12xx' = 120x'\end{aligned}$$

Since $V' = 30$, we have $x' = 0.1$ and $A' = 12$. So the surface area is increasing at a rate of 12 square meters per second.