INDEX IN K-THEORY FOR FAMILIES OF FIBRED CUSP OPERATORS

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Abstract. A families index theorem in K-theory is given for the setting of Atiyah, Patodi and Singer of a family of Dirac operators with spectral boundary condition. This result is deduced from such a K-theory index theorem for the calculus of cusp, or more generally fibred cusp, pseudodifferential operators on the fibres (with boundary) of a fibration; a version of Poincaré duality is also shown in this setting, identifying the stable Fredholm families with elements of a bivariant K-group.

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Introduction

The index theorem of Atiyah, Patodi and Singer gives a formula for the index of a Dirac operator on a compact manifold with boundary with boundary condition given by projection onto the range of the positive part of the boundary Dirac

2.0; Revised: 04-03-2006; Run: March 4, 2006.

operator. Versions of this result for families, with the formula being in cohomology for the Chern character of the (virtual) index bundle, were given by Bismut-Cheeger [7, 8] and the first author and Piazza [26, 27]. Here we formulate an index theorem in K-theory in the wider context of the algebras of pseudodifferential of fibred-cusp type, so generalizing the K-theory formulation of Atiyah and Singer in the boundaryless case. Our result specializes to give an index theorem in K-theory for the families of Dirac operators in the earlier contexts cited above. In a subsequent paper we will show how to derive a formula for the Chern character of the index class, reducing to the known formula in the Dirac case. There is also a relation, discussed below, with the 'direct' proof of the theorem of Atiyah, Patodi and Singer given by Dai and Zhang in [11].

We consider a smooth fibration of compact manifolds where the fibres are manifolds with boundary, with the boundary possibly carrying a finer fibration. With the addition of a normal trivialization of the boundary along the fibres we call this a fibration with fibred cusp structure. For such objects we formulate a notion of K-theory, denoted $\mathrm{K}_{\Phi\text{-}\mathrm{cu}}(\phi)$ (where ϕ is the overall fibration and Φ the boundary fibration) as the stable homotopy classes of the Fredholm families of corresponding 'fibred-cusp' pseudodifferential operators (introduced by Mazzeo and the first author in [21]) on the fibres. In the boundaryless case this reduces to the compactly supported K-theory of the fibre cotangent bundle as in [6]. The analytic index arises from the realization as Fredholm operators either through perturbation to make the null spaces have constant rank or through Kasparov's bivariant K-theory. In close analogy with the boundaryless case we define a topological index map by using an embedding of the fibration in the product of the base and a ball and then we show the equality of analytic and topological index homomorphisms

Theorem 1. *For a fibration with fibred cusp structure, the analytic and topological index maps, to K-theory of the base,*

$$\mathrm{K}_{\Phi\text{-}\mathrm{cu}}(\phi) \underset{\mathrm{ind}_t}{\overset{\mathrm{ind}_a}{\rightrightarrows}} \mathrm{K}(B) \tag{1}$$

are equal.

We also give an analogue of Atiyah's Poincaré duality in K-theory.

Theorem 2. *For a fibration with fibred cusp structure, there is a natural 'quantization' isomorphism*

$$\mathrm{quan} : \mathrm{K}_{\Phi\text{-}\mathrm{cu}}(\phi) \longrightarrow \mathrm{KK}^0_B(\mathcal{C}_\Phi(M), \mathcal{C}(B)) \tag{2}$$

where $\mathcal{C}_\Phi(M)$ is the C^* algebra of those continuous functions on the total space of the fibration which are constant on the fibres of Φ on the boundary.

A brief review of KK-theory and a definition of $\mathrm{KK}^0_B(\mathcal{C}_\Phi(M), \mathcal{C}(B))$ can be found in section 4 below. The 'cusp' case of these results (meaning $\Phi = \partial\phi = \phi\big|_{\partial M}$ and in which case we write the K-group as $\mathrm{K}_{\mathrm{cu}}(\phi)$) applies to give a families index theorem in K-theory for problems of Atiyah-Patodi-Singer type.

Theorem 3. *Let $\eth$ be a family of Dirac operators associated to a unitary Clifford module for a family of metrics of product type near the boundary of the fibres of a fibration and suppose P is a spectral section for the boundary Dirac operator, then $(\eth, P)$ defines a class $[(\eth, P)] \in \mathrm{K}_{\mathrm{cu}}(\phi)$ and*

$$\mathrm{ind}(\eth, P) = \mathrm{ind}_a([\eth, P]]).$$

This is a refinement at a K-theoretic level of the index theorem of [26] for families Dirac operators with Atiyah-Patodi-Singer boundary conditions. Note that *only* the cusp case is needed for this application, so the proof requires only a relatively small part of the discussion in the body of the paper, in particular Sections 6–8 are not required for this.

Since we generalize it here, let us briefly recall the families index in the boundaryless case. The index theorem in K-theory of Atiyah and Singer takes the form of the equality of an analytically defined and a topologically defined index for a family of elliptic pseudodifferential operators $P \in \Psi^m(M/B;\mathbb{E})$ on the fibres of a fibration

$$\begin{array}{ccc} Z & \text{---} & M \\ & & \big\downarrow \phi \\ & & B \end{array} \tag{3}$$

where $\mathbb{E} = (E_+, E_-)$ is a $\mathbb{Z}_2$-graded complex vector bundle over the total space, M, of the fibration and P maps sections of E_+ to sections of E_-. The analytic index is the element in $\mathrm{K}(B)$ which is the formal difference of vector bundles

$$\mathrm{ind}_{\mathrm{a}}(P) = [\mathrm{null}(P + A) \ominus \mathrm{null}(P^* + A^*)] \in \mathrm{K}(B) \tag{4}$$

for a perturbation $A \in \Psi^{-\infty}(M/B;\mathbb{E})$ such that the null spaces have constant rank (and for any choice of data leading to the adjoints). Such a perturbation always exists and $\mathrm{ind}_{\mathrm{a}}(P) \in \mathrm{K}(B)$ is independent of choices. The vanishing of the index is equivalent to the existence of such a perturbation with $P + A$ invertible. The symbol of the family P defines an element in the (compactly supported) K-theory of the fibre cotangent bundle, $[\sigma(P)] \in \mathrm{K_c}(T^*(M/B))$ and the analytic index of P depends only on $[\sigma(P)]$. Since all K-classes with compact support on $T^*(M/B)$ arise in this way, the analytic index gives a map

$$\mathrm{ind}_{\mathrm{a}} : \mathrm{K_c}(T^*(M/B)) \longrightarrow \mathrm{K}(B). \tag{5}$$

Alternatively, the analytic index may be defined via the bivariant K-theory of Kasparov, consisting of equivalence classes of almost-unitary Fredholm modules. Thus, the choice of a selfadjoint family $Q \in \Psi^{-m}(M/B;E_+)$, such that $Q^2P^*P - \mathrm{Id}$ is smoothing, fixes a class

$$\left[\begin{pmatrix} 0 & QP^* \\ PQ & 0 \end{pmatrix}\right] \in \mathrm{KK}^0_B(\mathcal{C}(M), \mathcal{C}(B)) \tag{6}$$

which also depends only on the class of the symbol, i.e. defines a homomorphism which combined with the natural push-forward map again gives the analytic index

$$\begin{array}{ccc} \mathrm{K_c}(T^*(M/B)) & \xrightarrow{\quad\mathrm{ind}_{\mathrm{a}}\quad} & \mathrm{K}(B) = \mathrm{KK}^0_B(\mathcal{C}(B), \mathcal{C}(B)) \;. \\ & \searrow \quad \nearrow & \\ & \mathrm{KK}^0_B(\mathcal{C}(M), \mathcal{C}(B)) & \end{array} \tag{7}$$

In fact the map on the left is an isomorphism which is a realization of Atiyah's map from elliptic pseudodifferential operators to K-homology, i.e. is a form of Poincaré duality.

The topological index is defined as a Gysin map, using Bott periodicity. By a standard generalization of Whitney's embedding theorem , the fibration (3) may be embedded as a subfibration of a real vector bundle V over B (indeed the bundle may be taken to be a product); the K-theory of $T^*(V/B)$ is then canonically isomorphic to the K-theory of the base. The composite map

$$\begin{array}{ccccc} & & \mathrm{K_c}(T^*(V/B)) & & \\ & \nearrow & & \searrow^{\text{Thom}} & \\ \mathrm{K_c}(T^*(M/B)) & & \xrightarrow{\quad \mathrm{ind_t} \quad} & & \mathrm{K}(B) \end{array} \tag{8}$$

is the ‘topological index’ and is again independent of choices. The index theorem of Atiyah and Singer [5], [6] in K-theory is the equality of these two maps; the one obtained by ‘quantizing’ symbols by use of pseudodifferential operators, the other by ‘trivializing’ the symbols using embeddings.

In this paper the corresponding problem is considered for fibred-cusp pseudodifferential operators. The full case is discussed below, initially the discussion is restricted to the cusp algebra; this indeed is the special case which is most closely related to the index theorem of Atiyah, Patodi and Singer. This relationship itself is made precise below following the discussion of the pseudodifferential index theorem.

The algebra of cusp pseudodifferential operators $\Psi^m_{\mathrm{cu}}(Z)$ on a compact manifold with boundary has properties similar to those of the usual algebra on a compact manifold without boundary Z; it is discussed extensively in [28]. The definition of cusp operators depends on the choice of a boundary defining function $x \in \mathcal{C}^\infty(Z)$, that is, a nonegative function which is zero on ∂Z, positive everywhere else and such that dx is non-zero on ∂Z. Given such a defining function, consider the Lie algebra of cusp vector fields. These are arbitrary smooth vector fields in the interior which, near the boundary are of the form

$$ax^2\partial_x + \sum_{j=1}^m a_j \partial_{z_j}, \; a, a_j \in \mathcal{C}^\infty(Z)$$

where (x, z) are coordinates near ∂Z. The cusp differential operatorsform the universal enveloping algebra of this Lie algebra (as a $\mathcal{C}^\infty(Z)$-module). By microlocalization this algebra is extended to the cusp pseudodifferential operators. A typical example of cusp differential operator is obtained by considering the Laplacian associated to a Riemannian metric g which in a collar neighborhood of the boundary $\partial Z \times [0, 1)_x \subset Z$ takes the form

$$g = \frac{dx^2}{x^4} + h, \; h \in \mathcal{C}^\infty(\partial Z \times [0, 1)_x; T^*(\partial Z) \otimes T^*(\partial Z)).$$

The algebra of cusp pseudodifferential operators is closely related to the b-pseudodifferential algebra but has the virtue of admitting a $\mathcal{C}^\infty$ functional calculus. The main difference, compared to the boundaryless case and as far as Fredholm properties are concerned, is that as well as a symbol map in the usual sense, taking values in functions homogeneous of degree m on the cusp cotangent bundle, $\sigma_m : \Psi^m_{\mathrm{cu}}(Z) \longrightarrow S^m_{\mathrm{cu}}(Z)$, there is a non-commutative ‘indicial’ (or normal) homomorphism taking values in suspended families of pseudodifferential operators on the boundary

$$N : \Psi^m_{\mathrm{cu}}(Z) \longrightarrow \Psi^m_{\mathrm{sus}}(\partial Z); \tag{9}$$

the suspended algebra consists of pseudodifferential operators with a symbolic parameter representing the dual to the normal bundle to the boundary. As already noted, the cusp algebra is not quite naturally associated to a compact manifold with boundary but is fixed by the choice of a trivialization of the normal bundle to the boundary. This may be thought of as a residue of the product-type structure in the theorem of Atiyah, Patodi and Singer. A cusp pseudodifferential operator is Fredholm on the (weighted) cusp Sobolev spaces if and only if *both* the symbol and the normal operator are invertible; we describe such an operator as 'fully elliptic.' The data consisting of (compatible) pairs $\sigma_m(P)$ and $N(P)$ constitutes the *joint symbol*, so an operator is fully elliptic when its joint symbol is invertible.

Following the model of the theorem of Atiyah and Singer described above we consider a fibration as in (3) where now the model fibre, Z, is a compact manifold with boundary. We define analogues of the objects described above in the boundaryless case including a 'symbolic' K-group $\mathrm{K}_{\mathrm{cu}}(\phi)$ as the set of stable homotopy classes of compatible and invertible joint symbols. The analytic and topological indexes are homomorphisms from this group into the topological K-group of the base.

The definition of the analytic index is a straightforward extension of the boundaryless case. That is, for a fully elliptic family of cusp pseudodifferential operators $P \in \Psi^m_{\mathrm{cu}}(M/B;\mathbb{E})$ on the fibres of the fibration there exists a perturbation by a family of smoothing operators supported in the interior, such that the null bundle has constant rank and then (5) again fixes an element $\mathrm{ind}_{\mathrm{a}}(P) \in \mathrm{K}(B)$ which is independent of the perturbation. In fact it only depends on the stable homotopy class of the joint symbol, within invertible symbols, and so defines the top map in (1) in this case.

The symbol, as opposed to the joint symbol, homomorphism leads to a short exact sequence

$$\mathrm{K}(B) \longrightarrow \mathrm{K}_{\mathrm{cu}}(\phi) \xrightarrow{\sigma} \mathrm{K}^0(\overline{{}^{\mathrm{cu}}T^*(M/B)}; {}^{\mathrm{cu}}S^*(M/B)) = \mathrm{K}_{\mathrm{c}}(T^*(M/B)) \tag{10}$$

where $\overline{{}^{\mathrm{cu}}T^*(M/B)}$ is the radial compactification of the fibrewise cusp cotangent bundle ${}^{\mathrm{cu}}T^*(M/B)$(which is isomorphic to $T^*(M/B)$) . The image group can also be interpreted as the compactly supported 'absolute' K-theory of ${}^{\mathrm{cu}}T^*(M/B)$– the more intricate notation here emphasizes that it is 'relative' to fibre infinity but 'absolute' as regards the boundary of M. The first map in (10) represents the inclusion of the fully elliptic operators of the form $\mathrm{Id} + \Psi^{-\infty}_{\mathrm{cu}}(M/B;E)$. The exactness of (10) follows from proposition 2.2 and theorem 5.2 in [28]. In fact it is shown there that any family of elliptic operators $P \in \Psi^m_{\mathrm{cu}}(M/B;\mathbb{E})$, so only assuming the invertibility of the symbol family, $\sigma(P)$, can be perturbed by an element of $\Psi^{-\infty}_{\mathrm{cu}}(M/B;\mathbb{E})$ to be invertible (hence of course fully elliptic). This leads to a splitting of the sequence (10) which we can write

$$\mathrm{K}(B) \underset{\mathrm{ind_a}}{\overset{i_*}{\rightleftarrows}} \mathrm{K}_{\mathrm{cu}}(\phi) \underset{\mathrm{inv}}{\overset{\sigma}{\rightleftarrows}} \mathrm{K}_{\mathrm{c}}(T^*(M/B)). \tag{11}$$

In this sense the index element of $\mathrm{K}(B)$ is a 'difference class' measuring the twisting of the given fully elliptic family relative to the invertible perturbation; since the index map from homotopy classes of fully elliptic families with a fixed elliptic symbol to $\mathrm{K}(B)$ is an isomorphism, the K-theory of the base can be represented by the fully elliptic perturbations of any one elliptic family. Were it easy to determine the map inv, i.e. to find an invertible family corresponding to a given symbol, the index problem would be much simpler!

As in the boundaryless case, this analytic index map can also be defined via bivariant K-theory. Let $\mathcal{C}_{\mathrm{cu}}(M) \subset \mathcal{C}(M)$ be the C^* subalgebra of the continuous functions on M consisting of those which are constant on the boundary of each fibre of ϕ; thus there is a short exact sequence of C^* algebras

$$\mathcal{C}_0(M) \longrightarrow \mathcal{C}_{\mathrm{cu}}(M) \xrightarrow{R} \mathcal{C}(B) \tag{12}$$

where $\mathcal{C}_0(M)$ is the C^* algebra of continuous functions on M vanishing on the boundary and R is restriction to the boundary. The quantization of an invertible joint symbol to a Fredholm cusp pseudodifferential operator (and choice of parametrix) then gives an alternative definition of the analytic index as in (6), (7):

$$\begin{array}{ccc} \mathrm{K}_{\mathrm{cu}}(\phi) & \xrightarrow{\mathrm{ind}_{\mathrm{a}}} & \mathrm{KK}^0_B(\mathcal{C}(B), \mathcal{C}(B)) = \mathrm{K}(B). \\ & {}_{\mathrm{quan}}\searrow \quad \nearrow & \\ & \mathrm{KK}^0_B(\mathcal{C}_{\mathrm{cu}}(M), \mathcal{C}(B)) & \end{array} \tag{13}$$

The quantization map here is an isomorphism which we can interpret as Poincaré duality. In fact we show that there is a commutative diagram

$$\begin{array}{ccccc} \mathrm{K}(B) & \longrightarrow & \mathrm{K}_{\mathrm{cu}}(\phi) & \xrightarrow{\sigma} & \mathrm{K}_{\mathrm{c}}(T^*(M/B)) \\ \updownarrow & & \updownarrow & & \updownarrow \\ \mathrm{KK}^0_B(\mathcal{C}(B), \mathcal{C}(B)) & \longrightarrow & \mathrm{KK}^0_B(\mathcal{C}_{\mathrm{cu}}(M), \mathcal{C}(B)) & \longrightarrow & \mathrm{KK}^0_B(\mathcal{C}_0(M), \mathcal{C}(B)). \end{array} \tag{14}$$

Here the left isomorphism is the natural identification of the KK group with K-theory and the right isomorphism is an absolute version of Atiyah's isomorphism as discussed in [25]; it follows that the central map is also an isomorphism.

To complete the analogy with the Atiyah-Singer theorem we proceed to define a 'topological' index using embeddings of the fibration and then prove the equality of the two index maps. In this case, because of the non-commutative structure of the definition of elements of $\mathrm{K}_{\mathrm{cu}}(\phi)$, there is more of an analytic flavour to the definition of this 'topological index' than in the boundaryless case.

The main step in defining $\mathrm{ind}_{\mathrm{t}}$ is to show that for an embedding of $\phi : M \longrightarrow B$ as a subfibration of $\tilde{\phi} : \widetilde{M} \longrightarrow B$ there is a natural lifting map

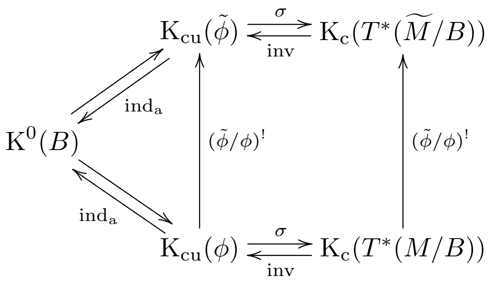
$$(\tilde{\phi}/\phi)^! : \mathrm{K}_{\mathrm{cu}}(\phi) \longrightarrow \mathrm{K}_{\mathrm{cu}}(\tilde{\phi}) \tag{15}$$

corresponding to tensoring with the 'Bott element' for the normal bundle of the smaller fibration. Our construction is closely related to that of Atiyah and Singer in [5] in the boundaryless case, but is of necessity more intricate since we need to preserve the invertibility of the indicial family, not just symbolic ellipticity. For this reason we replace the construction in [5] by a slightly different one involving pseudodifferential operators 'of product type.' In view of the identification above, this can also be considered as a lifting construction for KK theory, giving a map, which we believe but do not show, is an explicit realization of the map dual to the restriction $R : \mathcal{C}_{\mathrm{cu}}(\widetilde{M}) \longrightarrow \mathcal{C}_{\mathrm{cu}}(M)$

$$R^* : \mathrm{KK}^0_B(\mathcal{C}_{\mathrm{cu}}(M), \mathcal{C}(B)) \longrightarrow \mathrm{KK}^0_B(\mathcal{C}_{\mathrm{cu}}(\widetilde{M}), \mathcal{C}(B)). \tag{16}$$

It is important for the subsequent computation of the Chern character that this map is given by an explicit smooth construction with the corresponding $\mathrm{K}_{\mathrm{cu}}(\phi)$. For the proof of the index theorem it is essential that (15) be consistent both with the index and with the symbolic lifting construction of [5], which is to say that it leads to two commutative diagrams (one for the left-directed and one for the right-directed arrows)

(17)



There is always an embedding of the fibration of compact manifolds with boundary as a subfibration of the product $\pi_N : B \times \mathbb{B}^N \longrightarrow B$ of the base with a ball $\mathbb{B}^N$ of sufficiently large dimension N and (15) allows $\mathrm{K}_{\mathrm{cu}}(\phi)$ to be mapped into the group for such a product

$$\mathrm{K}_{\mathrm{cu}}(\phi) \longrightarrow \mathrm{K}_{\mathrm{cu}}(\pi_N),\ N \text{ large.} \tag{18}$$

The K-groups for these products may be computed directly (see section 12) and for odd dimensions there is a 'Thom isomorphism'

$$\mathrm{Thom} : \mathrm{K}_{\mathrm{cu}}(\pi_{2N+1}) \longleftrightarrow \mathrm{K}(B).$$

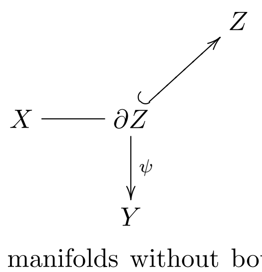
Hence there is a well-defined 'topological index map'

$$\mathrm{ind}_{\mathrm{t}} = \mathrm{Thom} \circ (\pi_{2N+1}/\phi)^! : \mathrm{K}_{\mathrm{cu}}(\phi) \longrightarrow \mathrm{K}(B) \tag{19}$$

and this completes the constructions of the ingredients in the statement of Theorem 1 in the cusp case.

The more general case of *fibred cusp* operators is similar to, but a little more complicated than, the cusp operators discussed above. These algebras of operators on a compact manifold correspond to the choice of a fibration of the boundary and the behaviour of the boundary becomes more 'commutative' as the fibres become smaller – that is, the cusp case is the most non-commutative. Thus Z, the typical fibre, is a compact manifold with boundary, ∂Z, which carries a distinguished fibration

(20)



where Y and X are compact manifolds without boundary. Associated with this structure, and a choice of trivialization, along the fibres of ψ, of the normal bundle to the boundary, is an algebra of 'fibred cusp pseudodifferential operators' $\Psi^*_{\psi\text{-}\mathrm{cu}}(Z)$ introduced in [21]. The cusp case corresponds to the extreme case of the one-fibre fibration $Y = \{\mathrm{pt}\}$. The other extreme case, choosing the point fibration of the boundary, $Y = \partial Z$, $\Phi = \mathrm{Id}$, is the 'scattering case' in which the index theorem is reducible directly to the usual Atiyah-Singer setting by a doubling construction (briefly discussed in [23] and below); this case is the simplest in most senses. It is discussed separately below, since a special case of the index theorem for scattering operators is used to handle the families index for perturbations of the identity in the general case. It is perhaps helpful to think of the fibred cusp operators as associated to the topological space Z/ψ in which the boundary is smashed to the base Y. Thus, for the cusp calculus, the boundary is smashed to a point whereas for the scattering calculus it is left unchanged.

Here we consider locally trivial families of such structures. Thus, suppose that M is a manifold with boundary which admits a fibration over a base B with typical fibre Z which can be identified with the manifold in (20). We suppose that the boundary of M has a finer fibration than over the base B giving a commutative diagram

$$
\begin{array}{ccccccc}
 & & & & Z & \text{———} & M \\
 & & & \nearrow & & \nearrow & \downarrow \phi \\
X & \text{———} & \partial Z & \text{———} & \partial M & \xrightarrow{\partial\phi} & B \\
 & & \downarrow \psi & & \downarrow \Phi & \nearrow & \\
 & & Y & \text{———} & D & &
\end{array}
\tag{21}
$$

in which X is the typical fibre of a fibration of ∂M over D and Y is the typical fibre of a fibration of D over B. We call such a pair of fibrations, together with a choice of normal trivialization, a fibration with fibred cusp structure. Associated to this geometry is an algebra of fibred cusp pseudodifferential operators acting on the fibres of M over B, which we shall denote $\Psi^*_{\Phi\text{-}\mathrm{cu}}(M/B)$. An element is a family parameterized by B with each operator related to the fibre (in M) above that point with its boundary smashed to the fibre of D above that point of B by the fibre of Φ.

In this more general setting we obtain similar results to those described above for the cusp calculus. First we define an abelian group, $\mathrm{K}_{\Phi\text{-}\mathrm{cu}}(\phi)$, with elements which are the stable homotopy classes of joint elliptic symbols, and a corresponding odd K-group, $\mathrm{K}^1_{\Phi\text{-}\mathrm{cu}}(\phi)$. The definition of analytic index maps, taking values respectively in $\mathrm{K}(B)$ and $\mathrm{K}^1(B)$ is essentially as above. We also give an analogue of (10) but now as a 6-term exact sequence

$$
\begin{array}{ccccc}
\mathrm{K_c}(T^*(D/B)) & \xrightarrow{i_0} & \mathrm{K}_{\Phi\text{-}\mathrm{cu}}(\phi) & \xrightarrow{\sigma_0} & \mathrm{K_c}(T^*(M/B)) \\
{\scriptstyle I_1}\uparrow & & & & \downarrow{\scriptstyle I_0} \\
\mathrm{K^1_c}(T^*(M/B)) & \xleftarrow[\sigma_1]{} & \mathrm{K}^1_{\Phi\text{-}\mathrm{cu}}(\phi) & \xleftarrow[i_1]{} & \mathrm{K^1_c}(T^*(D/B))
\end{array}
\tag{22}
$$

where σ_0, σ_1 are maps related to the symbol, I_0, I_1 are forms of the index map of Atiyah-Singer and i_0, i_1 correspond to inclusion of perturbations of the identity by operators of order $-\infty$. We show below that this is isomorphic to the corresponding sequence in KK-theory arising from the short exact sequence of C^* algebras (replacing (12))

$$\begin{gathered}\mathcal{C}_0(M) \longrightarrow \mathcal{C}_\Phi(M) \longrightarrow \mathcal{C}(D),\\ \mathcal{C}_\Phi(M) = \left\{ f \in \mathcal{C}(M); f\big|_{\partial M} = \Phi^* g,\ g \in \mathcal{C}(D) \right\},\end{gathered} \tag{23}$$

namely

$$\begin{array}{ccccc} \mathrm{KK}^0_B(\mathcal{C}(D),\mathcal{C}(B)) & \longrightarrow & \mathrm{KK}^0_B(\mathcal{C}_\Phi(M),\mathcal{C}(B)) & \longrightarrow & \mathrm{KK}^0_B(\mathcal{C}_0(M),\mathcal{C}(B)) \\ \uparrow & & & & \downarrow \\ \mathrm{KK}^1_B(\mathcal{C}_0(M),\mathcal{C}(B)) & \longleftarrow & \mathrm{KK}^1_B(\mathcal{C}_\Phi(M),\mathcal{C}(B)) & \longleftarrow & \mathrm{KK}^1_B(\mathcal{C}(D),\mathcal{C}(B)) \end{array} \tag{24}$$

where the four isomorphisms between the corresponding spaces on the left and right in (24) and (22) are the Poincaré duality maps of Atiyah, as realized in Kasparov's KK theory. The remaining two isomorphisms are given by quantization maps in the fibred-cusp calculus amounting as before to Poincaré duality; this is Theorem 2 in the general case. Note that the fact that the symbol map in (22) is not (in general) surjective shows that in some sense the 'universal case' is that of the cusp calculus since only through it can *every* elliptic symbol be quantized to a Fredholm family.

This universality appears explicitly in the form of a natural map in which the finer fibration of the boundary is 'forgotten'

$$\begin{array}{ccc} \mathrm{K}_{\Phi\text{-}\mathrm{cu}}(\phi) & \xrightarrow{\ q_{\mathrm{ad}}\ } & \mathrm{K}_{\mathrm{cu}}(\phi) \\ & {\scriptstyle \mathrm{ind_a}}\searrow \quad \swarrow{\scriptstyle \mathrm{ind_a}} & \\ & \mathrm{K}(B) & \end{array} \tag{25}$$

through which the index factors; it is defined through an adiabatic limit. We then define the topological index as the composite with the topological index in the cusp case. It is also possible to proceed more directly, through an appropriate embedding of the fibration.

Since we make substantial use below of various classes ('calculi') of pseudodifferential operators we have tried to use a uniform notation. In the families index theorem of Atiyah and Singer the quantization map giving the analytic index is in terms of families of pseudodifferential operators on the fibres of a fibration. We use what is the standard notation for these, $\Psi^m(M/B;\mathbb{E})$, except that $\mathbb{E} = (E_+, E_-)$ is a $\mathbb{Z}_2$-graded bundle and the operators act from E_+ to E_-. We will sometime need to consider tensor products of the form

$$\mathbb{E} \otimes_+ F = (E_+ \otimes F, E_-)$$

where F is a bundle which is not $\mathbb{Z}_2$-graded. We also use $\mathbb{Z}_2$-graded bundle notation for the operators in the quantization map (2), on a fibration with fibred cusp structure

$$\Psi^m_{\Phi\text{-}\mathrm{cu}}(M/B;\mathbb{E})$$

where the suffix alone indicates the 'uniformity type' at the boundary. Several more such classes arise below. First the indicial operator for families as in (9) takes values in the (singly) suspended algebra for which we use the suffix 'sus.' These are the pseudodifferential operators on $\mathbb{R} \times Z$, or for families on the fibres of $\mathbb{R} \times \partial M$, which are translation-invariant and rapidly decreasing at infinity in the real factor – taking the Fourier transform therefore gives a parameterized family but the parameter enters as a symbolic variable. So more generally we use notation such as

$$\Psi^{m}_{\psi-\text{sus}}(X),\ \Psi^{m',m}_{\psi-\text{p}}(X),\ \Psi^{m}_{\psi\text{-ad}}(X)$$

to denote, respectively, pseudodifferential operators which are suspended with respect to the cotangent variables of a fibration, pseudodifferential operators which are of product type with respect to a fibration and pseudodifferential operators which depend adiabatically on a parameter (and pass from operators on the total space to fibre operators in the limit in the parameter). The general approach to pseudodifferential operators, by defining them in terms of classical conormal distributions on some blown up version of the product space, allows these types to be combined where required. Thus we use adiabatic families of fibred cusp operators to pass from one boundary fibration to another in the definition of (25) and product-type cusp pseudodifferential operators in the lifting construction – these have product-type suspended operators as normal operators with corresponding notation.

We do not put the 0's at the end of short exact sequences.

In §1 the analytic index, in K-theory, of fully elliptic fibred cusp pseudodifferential operators is described as a map from the K-group associated to invertible full symbols. In the special case of the scattering structure, the index theorem is derived from the Atiyah-Singer index theorem in §2 and used to discuss families which are perturbations of the identity for general fibred cusp structures in §3. The quantization map, to KK-theory, is introduced in §4 and shown to be an isomorphism for the cusp structure in §5. The 6-term symbol sequences in K-theory and KK-theory are described in §6 and related in §7, resulting in the proof of Theorem 2. The map (25) is constructed in §8 and a result on the extension of fibred cusp structures is contained in §9 and the related multiplicativity and lifting properties are examined in §§10,11. In §12 the model cases are analyzed and used to define the topological index map and to prove Theorem 1 in §13. The application of the cusp case to families of Atiyah-Patodi-Singer type is made in §14 and Theorem 3 is proved there. In the appendices the various classes of pseudodifferential operators appearing in the body of the paper are described, including product-type, fibred cusp and adiabatic algebras and their combinations.

1. Analytic index

As already briefly described above, the general setting of this paper is a compact manifold with boundary, M, with a fibration $\phi : M \longrightarrow B$ over a compact manifold usually without boundary; we also assume, without loss of generality, that the base is connected. In fact it is convenient at various points to allow the base (and hence also the total space) to be a manifold with corners, to allow especially products with intervals. We still treat M as a manifold with boundary, in that 'the' boundary is then the union of the boundary faces of the fibres. Thus, ϕ is a smooth surjective map with surjective differential at every point. It follows that the fibres, $\phi^{-1}(b)$ for $b \in B$, are compact manifolds with boundary, all diffeomorphic to a fixed manifold Z for which we use the notation (3). It is often notationally convenient to assume that the fibres are also connected, but it is by no means necessary and we believe that the paper is written so that this assumption is not used. Such a fibration is always locally trivial.

In addition we assume that the boundary of the total space of the fibration, ∂M, carries a second, finer, fibration, giving a commutative diagram of fibrations

$$\begin{array}{ccc} \partial M & \xrightarrow{\partial\phi} & B. \\ & {\scriptstyle\Phi}\searrow \quad \nearrow{\scriptstyle\partial\phi/\Phi} & \\ & D & \end{array} \tag{1.1}$$

Thus the boundary of each fibre carries a fibration and the overall fibre, with this fibration of its boundary, is always diffeomorphic to the model fibre with its model boundary fibration, as in (20). The maps fit together as in (21) and it is always the case that there is a local trivialization of the overall fibration ϕ in which the fibration of the boundary of the fibres is reduced to this normal form; in this sense the structure is locally trivial.

To associate with ϕ and Φ a class of pseudodifferential operators on the fibres of ϕ, of the type introduced in [21], we need one more piece of information. Namely we need to choose a trivialization of the normal bundle to the boundary of the fibres of ϕ along the fibres of Φ. This simply amounts to the choice of a boundary defining function $x \in \mathcal{C}^\infty(M)$ (so $x \geq 0$, $\partial M = \{x = 0\}$ and $dx \neq 0$ on ∂M). Such a choice gives a trivialization of the normal bundle to M everywhere (simply choose the inward pointing normal vector v_p at $p \in \partial M$ to satisfy $v_p x = 1$). Two such choices x, x', are equivalent, in the sense that they give the same trivialization along the fibres of Φ if and only if

$$x' = ax + bx^2,\ a, b \in \mathcal{C}^\infty(M),\ a\big|_{\partial M} = \Phi^* a',\ a' \in \mathcal{C}^\infty(D). \tag{1.2}$$

Equivalent choices turn out to lead to the same algebra of pseudodifferential operators. Since even inequivalent choices are homotopic and lead to isomorphic structures the dependence on this choice of normal trivialization will not be emphasized; it should be fixed throughout but none of the results depend on which choice is made.

Definition 1.1. A *fibration with fibred cusp structure* is a fibration of compact manifolds (3) with fibres modelled on a fixed compact manifold with boundary, a finer fibration as in (1.1) of the boundary of the total space and a choice of trivialization of the normal bundle to the boundary along the fibres as discussed above.

Notice that the extreme cases in which $\Phi = \partial\phi$ is simply the restriction of the fibration to the boundary of M is a particularly interesting case, the 'cusp' case, which includes the setting of the index theorem of Atiyah, Patodi and Singer. The other extreme, in which Φ is the identity, is the essentially commutative 'scattering' case. In terms of [3], this corresponds to operators for which the obstruction in K-theory to the existence of local elliptic boundary conditions vanishes (in contrast with the global nature of the Atiyah-Patodi-Singer boundary condition). The general fibred cusp case is intermediate between these two extremes.

In general, even if we were assuming that M is connected, it would be artificial to assume that the boundary ∂M is connected, since many interesting examples involve a disconnected boundary. We therefore avoid any such assumption. When the boundary is not connected, the cusp case is still interpreted as $\Phi = \partial\phi$. As discussed in [28] and [32], this means one should allow terms of order $-\infty$ acting between different components of the boundary and correspondingly in the definition of the indicial family.

Under these conditions we may associate to a fibration with fibred cusp structure an algebra of 'fibred-cusp' pseudodifferential operators, $\Psi^{\mathbb{Z}}_{\Phi\text{-cu}}(M/B)$. The reader is referred to [21] for the original definition of the algebra and to the discussion in [32]. A generalization to product-type operators, used in the lifting construction below, is given in Appendix B. For our purposes here the main interest lies in the boundedness, compactness, Fredholm and related symbolic properties of these operators. Thus if $\mathbb{E} = (E_+, E_-)$ is a $\mathbb{Z}_2$-graded complex vector bundle we will denote by $\Psi^{\mathbb{Z}}_{\Phi\text{-cu}}(M/B; \mathbb{E})$ the space of these operators acting from sections of E_+ to sections of E_-; they always give continuous linear operators on weighted spaces of smooth sections

$$P \in \Psi^{\mathbb{Z}}_{\Phi\text{-cu}}(M/B; \mathbb{E}) : x^s\mathcal{C}^\infty(M; E_+) \longrightarrow x^s\mathcal{C}^\infty(M; E_-),\ \forall\ s \in \mathbb{R}. \tag{1.3}$$

Both the symbol and the normal operator can be defined by appropriate 'oscillatory testing' of these maps. The symbol map gives a short exact sequence

$$\Psi^{m-1}_{\Phi\text{-cu}}(M/B; \mathbb{E}) \longrightarrow \Psi^{m}_{\Phi\text{-cu}}(M/B; \mathbb{E}) \xrightarrow{\sigma} \mathcal{S}^{m}_{\Phi\text{-cu}}(M/B; \mathbb{E}) \tag{1.4}$$

with

$$\mathcal{S}^{m}_{\Phi\text{-cu}}(M/B; \mathbb{E}) = \mathcal{C}^\infty({}^{\Phi\text{-cu}}S^*(M/B); \hom(\mathbb{E}) \otimes N_{-m}) \tag{1.5}$$

where ${}^{\Phi\text{-cu}}S^*(M/B)$ is the sphere bundle of the fibred cusp cotangent bundle of the fibres (isomorphic, but not naturally so, to the 'usual' bundle $T^*(M/B)$) and $N_{-m} = N^{-m}$ is the bundle with sections which are the functions homogeneous of degree m on the fibres (then N can be identified with the normal bundle to the boundary of the radial compactification) and $\hom(\mathbb{E}) = \hom(E_+, E_-)$. The normal (or indicial) operator gives a short exact sequence

$$x\Psi^{m}_{\Phi\text{-cu}}(M/B; \mathbb{E}) \longrightarrow \Psi^{m}_{\Phi\text{-cu}}(M/B; \mathbb{E}) \xrightarrow{N} \Psi^{m}_{\Phi-\mathrm{sus}}(\partial M/D; \mathbb{E}). \tag{1.6}$$

Here the image is the space of suspended pseudodifferential operators on the fibres of the fibration Φ with symbolic parameters in

$${}^{\Phi\text{-cu}}T^*_{\partial M}(M/B)/T^*(\partial M/B) \tag{1.7}$$

which is to say the duals to the fibre variables of Φ together with a dual to the normal variable. The symbol and normal operators are connected by the identity

$$\sigma\big|_{\partial M} = \sigma \circ N \text{ on } \Psi^{m}_{\Phi\text{-cu}}(M/B; \mathbb{E}). \tag{1.8}$$

This is the only constraint, so if we set

$$A^m_\Phi(\phi; \mathbb{E}) = \big\{(\sigma, N) \in \mathcal{S}^{m}_{\Phi\text{-cu}}(M/B; \mathbb{E}) \oplus \Psi^{m}_{\Phi-\mathrm{sus}}(\partial M/D; \mathbb{E}); \sigma\big|_{\partial M} = \sigma_m(N)\big\} \tag{1.9}$$

the joint symbol sequence becomes

$$x\Psi^{m-1}_{\Phi\text{-cu}}(M/B; \mathbb{E}) \longrightarrow \Psi^{m}_{\Phi\text{-cu}}(M/B; \mathbb{E}) \xrightarrow{(j,N)} A^m_\Phi(\phi; \mathbb{E}). \tag{1.10}$$

Such a family extends by continuity to an operator between the natural continuous families of weighted fibred-cusp Sobolev spaces

$$P : \mathcal{C}(B; x^s H^M_{\Phi\text{-cu}}(M/B; E_+)) \longrightarrow \mathcal{C}(B; x^s H^{M-m}_{\Phi\text{-cu}}(M/B; E_-)),\ M, s \in \mathbb{R}. \tag{1.11}$$

It is a Fredholm family on these spaces if and only if its image in $A^m_\Phi(\phi;\mathbb{E})$ is invertible, in which case we say that the family is fully elliptic.

Lemma 1.1. *If $P \in \Psi^{\mathbb{Z}}_{\Phi\text{-cu}}(M/B;\mathbb{E})$ is a fully elliptic family then there exists a smoothing perturbation in $x^\infty\Psi^{-\infty}_{\Phi\text{-cu}}(M/B;\mathbb{E})$ such that $P + A$ has null space in* (1.11) *forming a trivial smooth vector bundle, over B, contained in $\dot{\mathcal{C}}^\infty(M;E_+)$.*

Proof. The algebra is invariant under conjugation by x^s and such conjugation affects neither the symbol nor the normal operator; thus we can take $s = 0$. For simplicity of notation we will also suppose that $M = m$ in (1.11); this is all that is used below and the general case follows easily. The full ellipticity and properties of the calculus allow us to construct a parametrix, which is to say a family $Q \in \Psi^{-m}_{\Phi\text{-cu}}(M/B;\mathbb{E}^-)$, where $\mathbb{E}^- = (E_-,E_+)$, such that

$$PQ = \mathrm{Id}_+ - R_+,\ QP = \mathrm{Id}_- - R_-,\ R_\pm \in x^\infty\Psi^{-\infty}_{\Phi\text{-cu}}(M/B;E_\pm). \tag{1.12}$$

As a bundle of Hilbert spaces, $\mathcal{C}(B;H^m_{\Phi\text{-cu}}(M/B;E_+))$ is necessarily trivial. Thus there is a sequence of orthogonal projections Π_N, tending strongly to the identity, corresponding to projection onto the first N terms of an orthonormal basis in some trivialization. In particular for a compact operator such as R_+, $R_+\Pi_N \longrightarrow R_+$ in the topology of norm continuous families of bounded operators.

Furthermore, Π_N may be approximated by smooth families of projections of the same rank. To see this, fixing N, we may certainly find a sequence of smoothing operators with supports disjoint from the boundary such that $W_j \longrightarrow \Pi_N$ in the norm topology; it suffices to use a partition of unity and work locally. Then $W'_j = W_j\Pi_N W_j$ is a continuous (in B) family of smoothing operators, supported in the interior, which approximates Π_N and for large j has rank N. Taking j sufficiently large the range of W'_j is a trivial subbundle which has a basis $e_{l,j} \in \mathcal{C}(B;\mathcal{C}^\infty(M/B;E_+))$ with supports always in the interior. These sections can themselves be uniformly approximated by sections of the same form but smooth over B. Replacing W'_j by the orthogonal projection onto the span of these sections, for j large, we have succeeded in approximating Π_N by smooth families of projections with kernels supported in the interior. Thus we may suppose that the Π_N are smooth families of smoothing operators with kernels vanishing to infinite order at both boundaries of the product.

Returning to (1.12) we may compose on the right with $\mathrm{Id} - \Pi_N$ and, using the fact that it is also a projection, deduce that

$$QP(\mathrm{Id} - \Pi_N) = (\mathrm{Id} - B)(\mathrm{Id} - \Pi_N),\ B = R_+(\mathrm{Id} - \Pi_N). \tag{1.13}$$

For large N it follows that B has uniformly small norm. The inverse of $\mathrm{Id} - B$ is then also of the form $\mathrm{Id} + B'$ with B' a smoothing operator with kernel vanishing to infinite order at the boundary, and so is an element of $x^\infty\Psi^{-\infty}_{\Phi\text{-cu}}(M/B;E_+)$. Replacing Q by the new parametrix $(\mathrm{Id} + B')Q$ we have replaced the first identity in (1.12) by

$$Q(P + A) = \mathrm{Id} - \Pi_N,\ A = -P\Pi_N \in x^\infty\Psi^{-\infty}_{\Phi\text{-cu}}(M/B;\mathbb{E}). \tag{1.14}$$

This perturbs P as desired. □

Now once the null space of $P+A$ is arranged to be a smooth bundle it follows that its range has a complement of the same type (but in general not trivial), namely the null bundle of $P^* + A^*$ for some choice of inner products and smooth density.

Proposition 1.2. *The element*

$$\operatorname{null}(P+A)\ominus\operatorname{null}(P^*+A^*)\in \mathrm{K}(B)$$

defined by any fully elliptic element of $\Psi^m_{\Phi\text{-}\mathrm{cu}}(M/B;E)$ *using Lemma 1.1 is independent of the choice of perturbation* A, *is constant under smooth homotopy and is additive under direct sums.*

Proof. First suppose P has been perturbed so that (1.14) holds for some parametrix Q and some family of smooth projections Π_N. Changing to another family Π'_k we can choose k so large that $B'=\Pi_N(\operatorname{Id}-\Pi'_k)$, has small norm and then

$$(\operatorname{Id}-B')^{-1}Q(P+A')=\operatorname{Id}-\Pi'_k,\ A'=-P\Pi'_k+A(\operatorname{Id}-\Pi'_k). \tag{1.15}$$

The null space of the new family is the range of Π'_k into which the range of Π_N is mapped isomorphically by $\operatorname{Id}-\Pi'_k$. The range of $P+A'$ is then the direct sum of an isomorphic image of the complement of this image of Π_N plus the previous range. Thus the element in $\mathrm{K}(B)$ is unchanged.

To see the independence of the choice of stabilizing perturbation, suppose A and A' are two such perturbations. Consider the family depending on an additional parameter $P+(\cos\theta A+\sin\theta A')$, $\theta\in\mathbb{S}$. The circle can be included in the base of the product fibration and then the argument above can be applied to stabilize the new family. This shows that the pairs of bundles resulting from different stabilizations are homotopic and so define the same element in $\mathrm{K}(B)$. Indeed the same argument applies to a homotopy of the operator itself, through fully elliptic operators.

It is then immediate that the index class is additive under direct sums. □

Remark 1.2. Various 'stabilization' constructions like this are used below. For instance, if P_t is a family of totally elliptic operators depending smoothly on an additional parameter $t\in[0,1]$ and it is invertible for $t=0$ then there is a finite rank smoothing perturbation, vanishing at $t=0$, which makes the family invertible for all $t\in[0,1]$.

To formulate the index as a map we now consider the K-group which arises from the symbol algebra.

Definition 1.3. For a fibration with fibred cusp structure, $\mathrm{K}_{\Phi\text{-}\mathrm{cu}}(\phi)$ denotes the set of equivalence classes of the collection of the invertible elements of the $A^0_\Phi(\phi,\mathbb{E})$ (with inverse in $A^0_\Phi(\phi,\mathbb{E}^-)$, $\mathbb{E}^-=(E^-,E^+)$) where the equivalence relation is a finite chain consisting of the following

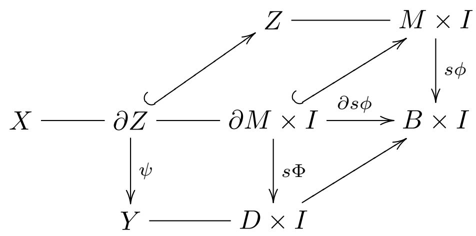
$$\begin{gathered}(\sigma,N)\in A^0_\Phi(\phi;\mathbb{E})\sim(\sigma_1,N_1)\in A^0_\Phi(\phi;\mathbb{F})\\ \text{if there exist bundle isomorphisms over } M,\ a_\pm:E_\pm\longrightarrow F_\pm\\ \text{such that } \sigma=a_-\circ\sigma_1\circ a_+ \text{ and } N=(a_-\big|_{\partial M})\circ N_1\circ(a_+\big|_{\partial M})\end{gathered} \tag{1.16}$$

$$\begin{gathered}(\sigma,N)\in A^0_\Phi(\phi;\mathbb{E})\sim(\sigma_2,N_2)\in A^0_\Phi(\phi;\mathbb{E})\\ \text{if there exists a homotopy } (\sigma(t),N(t)),\ t\in[0,1],\ (\sigma(t)^{-1},N(t)^{-1})\in A^0_\Phi(\phi;\mathbb{E}^-)\\ \text{such that } \sigma(0)=\sigma,\ \sigma(1)=\sigma_2,\ N(0)=N,\ N(1)=N_2,\end{gathered} \tag{1.17}$$

$$\begin{gathered}(\sigma,N)\in A^0_\Phi(\phi;\mathbb{E})\sim(\sigma_3,N_3)\in A^0_\Phi(\phi;\mathbb{E}\oplus F)\\ \text{if } F \text{ is ungraded, } \sigma_3=\sigma\oplus\operatorname{Id}_F \text{ and } N_3=N\oplus\operatorname{Id}_F\,.\end{gathered} \tag{1.18}$$

There is also a corresponding odd K-group. Given the diagram (21), let $I = [0,1]$ be the unit interval and consider the suspended version

(1.19)



where $s\phi = \phi \times \mathrm{Id}$ and $s\Phi = \Phi \times \mathrm{Id}$. Let E be an ungraded complex vector bundle. To the fibration (1.19) corresponds the space of joint symbols $A^m_{s\Phi}(s\phi; E)$.

Definition 1.4. For a fibration with fibred cusp structure, $\mathrm{K}^1_{\Phi\text{-}\mathrm{cu}}(\phi)$ denotes the set of equivalence classes of the collection of the invertible elements of the $A^0_{s\Phi}(s\phi, E)$ (with E ungraded and with inverse in $A^0_{s\Phi}(s\phi, E)$) which are the identity when restricted to $B \times \{0,1\}$ under, the equivalence relation corresponding to a finite chain as in (1.16), (1.17) and (1.18) with bundle transformations and homotopies required to be the identity on $B \times \{0,1\}$.

Proposition 1.3. *For any fibration with fibred cusp structure, both* $\mathrm{K}_{\Phi\text{-}\mathrm{cu}}(\phi)$ *and* $\mathrm{K}^1_{\Phi\text{-}\mathrm{cu}}(\phi)$ *are abelian groups under direct sum (or equivalently stabilized product) which are naturally independent of the choice of boundary trivialization and the index construction of Lemma 1.1 defines group homomorphisms*

$$\mathrm{ind}_a : \mathrm{K}_{\Phi\text{-}\mathrm{cu}}(\phi) \longrightarrow \mathrm{K}(B),\ \mathrm{ind}_a : \mathrm{K}^1_{\Phi\text{-}\mathrm{cu}}(\phi) \longrightarrow \mathrm{K}^1(B). \tag{1.20}$$

Proof. The abelian group structure follows as in the boundaryless case. Since changing the boundary defining function only affects the calculus through a change of the trivialization of the normal bundle and all such trivializations are homotopic the resulting abelian groups are independent of this choice and Proposition 1.2 shows that the analytic index map, (1.20), is then well defined and additive. □

2. The scattering case

The scattering case, in which the fibres of the boundary fibration are reduced to points is effectively 'commutative' compared to the others. In particular it is very close to the setting of the original Atiyah-Singer index theorem and we show here that it is reducible to it. The resulting identification of the analytic and topological indexes allows us to derive, in the next section, an index theorem in the setting of a general boundary fibration but for perturbations of the identity of order $-\infty$.

Lemma 2.1. *For the scattering structure on any compact fibration,* (3)*,*

$$\mathrm{K}_{\mathrm{sc}}(\phi) = \mathrm{K}_{\mathrm{Id}\text{-}\mathrm{cu}}(\phi) \equiv \mathrm{K}_c(T^*(M/B); T^*_{\partial M}(M/B)), \tag{2.1}$$

is identified with the compactly supported K-theory of the fibre cotangent bundle of the interior of M.

Proof. In this case an element of $A^0_{\mathrm{Id}}(\phi; \mathbb{E})$ is a pair (σ, b) each of which is a bundle isomorphism, taking values in the lift of $\hom(\mathbb{E})$. The 'symbolic part' σ is defined on the sphere bundle at infinity of the (radial compactification of the) appropriately rescaled fibre cotangent bundle ${}^{\mathrm{sc}}T^*(M/B)$ and the boundary part is smooth on the radial compactification of the restriction, ${}^{\mathrm{sc}}T^*_{\partial M}(M/B)$ of this bundle to the boundary. Since they are compatible at the intersection, which is to say the corner of $\overline{^{\mathrm{sc}}T^*}(M/B)$, together this gives *precisely* a section of $\hom(\mathbb{E})$ lifted to the boundary of $\overline{^{\mathrm{sc}}T^*}(M/B)$. This is the data needed for the standard definition of a compactly supported K-class in the interior of a manifold with boundary (to which such a manifold with corners is homeomorphic) and all classes arise this way. This gives (2.1). □

Poincaré duality reduces to one of the cases discussed (for $B = \{\mathrm{pt}\}$) in [25]

$$\mathrm{K_c}(T^*(M/B); T^*_{\partial M}(M/B)) \longleftrightarrow \mathrm{KK}^0_B(\mathcal{C}(M), \mathcal{C}(B)). \tag{2.2}$$

The index theorem in the scattering case has a 'simple' formulation and proof in the sense that it reduces directly to the Atiyah-Singer theorem through the following observation from [23].

Lemma 2.2. *In the scattering case* $\mathrm{K_{sc}}(\phi)$ *is generated by the equivalence classes of elements of the subset*

$$\{(\sigma, b) \in A^0_{\mathrm{Id}}(\phi; \mathbb{E});\ \textit{near } \partial M,\ E_+ = E_- = \mathbb{C}^N,\ \sigma = b = \mathrm{Id}\} \subset A^0_{\mathrm{Id}}(\phi; \mathbb{E}). \tag{2.3}$$

Proof. First we show that any invertible joint symbol is homotopic to an element in which, near the boundary, both σ and b are the lifts of some bundle isomorphism from E_+ to E_-. Given an element $(s, b) \in A^0_{\mathrm{Id}}(\phi; \mathbb{E})$ the bundle isomorphism can be taken to be any extension off ∂M of the section b restricted to the zero section (identified with ∂M) of $^{\mathrm{sc}}T^*_{\partial M}(M/B)$. First perturb b to be constant on the (linear) fibres of $^{\mathrm{sc}}T^*_{\partial M}(M/B)$ near the zero section. An extension of the radial expansion of the vector bundle allows it to be deformed, with σ to keep the consistency condition, to be fibre constant in this sense in a neighbourhood of the boundary.

Now, given that σ and b are identified with a bundle isomorphism near the boundary, this isomorphism can be used to modify E_- to be equal to E_+ close to the boundary so that both symbols become the identity on E_+ very close to the boundary without changing the equivalence class. Then E_+ can be complemented to a trivial bundle. □

Once the K-group is identified with the set of equivalence classes as in (2.3), the quantization map can also be arranged to yield pseudodifferential operators, in the ordinary sense, which are equal to the identity in a neighborhood of the boundary. Such operators can be extended to the double of M, across the boundary, to be the identity on the additional copy of M and the Atiyah-Singer index theorem then applies.

Theorem 2.3. *For the scattering structure on a fibration the analytic index for fully elliptic scattering operators on the fibres factors through the Atiyah-Singer index map for the double* $2M = M \cup M^-$

$$\begin{aligned} \mathrm{ind}_a : \mathrm{K_{sc}}(\phi) &\xrightarrow{\sim} \mathrm{K}_c(T^*(M/B); T^*_{\partial M}(M/B)) = \\ &\mathrm{K}_c(T^*(2M/B); T^*(M^-/B)) \longrightarrow \mathrm{K}_c(T^*(2M/B)) \xrightarrow{\mathrm{ind_{AS}}} \mathrm{K}(B). \end{aligned} \tag{2.4}$$

Proposition 2.4. *Suppose that E is a complex vector bundle over the total space of a fibration with scattering structure and* $b \in \mathcal{S}(^{\mathrm{sc}}T^*_{\partial M}(M/B); \hom(E))$ *is such that* $\mathrm{Id} + b$ *is invertible then any family* $\mathrm{Id} + B$, $B \in \Psi^{-\infty}_{\mathrm{sc}}(M/B; E)$ *with* $N(\mathrm{Id} + B) = \mathrm{Id} + b$ *has analytic index equal to the image*

$$[\mathrm{Id} + b] \in \mathrm{K}^1_c(\mathbb{R} \times T^*(\partial M/B)) = \mathrm{K}_c(T^*(\partial M/B)) \xrightarrow{\mathrm{ind_{AS}}} \mathrm{K}(B) \tag{2.5}$$

under the Atiyah-Singer index map for the boundary.

Proof. First we can complement E to be trivial, stabilizing the symbol by the identity. By a small perturbation we can also arrange that the boundary symbol b is compactly supported on ${}^{\mathrm{sc}}T^*_{\partial M}(M/B) = \mathbb{R}\times T^*(\partial M/B)$ and is equal to a bundle map near the zero section. Thus, the varying part of b can be confined to a compact subset of $(0,\infty)\times W$, where W is the boundary of the radial compactification of the vector bundle $\mathbb{R}_s \times T^*(\partial M/B)$ and the first variable is the radial variable. In the deformation in the proof of Lemma 2.2 above, across the corner at infinity and into the interior, the radial variable becomes the normal variable to the boundary, x, in a product decomposition near the boundary with the variation now in $(0,1)$ and W is identified with the boundary of the radial compactification of $T^*(M/B)\big|_{x=\frac12}$. Thus, $\mathrm{Id}+b$ has been identified with a symbol in the conventional sense on the sphere bundle of $T^*((0,1)_x \times \partial M/B)$ reducing to the identity near $x=1$ and some bundle isomorphism near $x=0$. By Bott periodicity of the Atiyah-Singer index map this reduces to the identifications in (2.5). □

Remark 2.1. From this result a families index theorem in K-theory in the setting of Callias' index theorem follows. See also the discussion by Anghel [1] and Kucerovsky [18] for single operators.

3. Perturbations of the identity

The discussion in the previous section of the index in the scattering case allows us to compute the index of Fredholm perturbations of the identity by fibred cusp operators of order $-\infty$, i.e. to give analogues of Proposition 2.4 for all fibred cusp structures. This was done in [32] for the numerical index.

Proposition 3.1. *Suppose that E is a complex vector bundle over the total space of a fibration with fibred cusp structure and $b \in \Psi^{-\infty}_{\Phi-\mathrm{sus}}(\partial M/D;E)$ is such that $\mathrm{Id}+b$ is invertible, then any family $\mathrm{Id}+B$, $B\in\Psi^{-\infty}_{\Phi\text{-}\mathrm{cu}}(M/B;E)$ with $N(\mathrm{Id}+B)=\mathrm{Id}+b$ has analytic index equal to the image*

$$[\mathrm{Id}+b] \in \mathrm{K}^1_c(\mathbb{R}\times T^*(D/B)) = \mathrm{K}_c(T^*(D/B)) \overset{\mathrm{ind_{AS}}}{\longrightarrow} \mathrm{K}(B) \tag{3.1}$$

under the Atiyah-Singer index map for the fibration of D over B.

Proof. First we may use an 'excision' construction to replace M by the simpler manifold $\partial M \times [0,1]_x$. Indeed, taking a product decomposition of M near the boundary and identifying it with a neighbourhood of $x=0$ in the product, the quantization of b may be localized to vanish outside this neighbourhood, i.e. to have kernel vanishing outside the product of this neighbourhood with itself, and so can be identified with an operator on the model product with the same index. Thus it suffices to consider the product case $M=\partial M\times[0,1]$, with a bundle lifted from the boundary and with b trivial at the $x=1$ boundary.

Thus the boundary fibration extends to the whole space as a fibration over $[0,1]\times D$ and the strategy is to reduce the problem to the scattering calculus on this space. Consider a family of smoothing projections Π_N as in Section 1 for the bundle E over the fibration of the boundary given by Φ extended to act trivially in the variable $x\in[0,1]$. Then $\Pi_N b \Pi_N \longrightarrow b$ as $N\to\infty$ uniformly on $\mathbb{R}\times T^*(D/B)$ in view of the rapid decay. Thus we may replace b by $\Pi_N b\Pi_N$ for sufficiently large N and hence assume that it acts on some finite rank subbundle of the smooth sections of $\mathcal{C}^\infty(\partial M/D)$ (namely the range of Π_N) pulled back to $\mathbb{R}\times T^*(D/B)$. Quantizing b to an operator $B \in \Psi^{-\infty}_{\Phi\text{-}\mathrm{cu}}(M;E)$ we may arrange that $\Pi_N B = B = B\Pi_N$ by replacing B by $\Pi_N B \Pi_N$. Note that Π_N is not a fibred cusp pseudodifferential operator, because it kernel is singular on the fibre diagonal, however its composite with an element of $\Psi^{-\infty}_{\Phi\text{-}\mathrm{cu}}(M;E)$ is in the same space; this follows from an examination of the kernels, see Appendix A. Now in fact the same local analysis shows that $\Pi_N B \Pi_N \in \Psi^{-\infty}_{\mathrm{sc}}([0,1]\times D/B; W_N)$ has boundary symbol b, where W_N is the range of Π_N. Thus the result follows from Proposition 2.4. □

4. Analytic classes in KK theory

Baum, Douglas and Taylor, in [30], associate a KK-class with a Dirac operator on a manifold with boundary with Atiyah-Patodi-Singer boundary condition. In this section, we extend, and refine, their construction to the situation of families of fully elliptic fibred cusp operators. This can also be thought as an adaptation to the context of fibred cusp operators of a similar discussion for b-pseudodifferential operators in [26], except that even in that special case, here we associate to a Fredholm b-pseudodifferential operator on X a class in the K-homology of $X/\partial X$, the manifold with the boundary smashed to a point, rather than the absolute (or relative) K-homology (see the final remark of [30]). Although small, this is an important difference in that it is at the heart of our formulation of a families index theorem in K-theory, including the Atiyah-Patodi-Singer case.

For a quick review of KK-theory and the relation with elliptic operators see [30] and [25] and the books [9] and [13] as well as the papers [15], [16], and [17] where KK-theory was initially developed. To describe families of elliptic operators via KK-theory we recall the extra feature introduced in [17].

Definition 4.1. If X is a compact manifold and $\mathcal{A}$ a $\mathbb{Z}_2$-graded C^* algebra then a $\mathcal{C}(X)$-algebra structure on $\mathcal{A}$ is a graded unital homomorphism

$$r : \mathcal{C}(X) \longrightarrow Z(\mathcal{M}(\mathcal{A}))$$

where $Z(\mathcal{M}(\mathcal{A}))$ is the center of $\mathcal{M}(\mathcal{A})$, the multiplier algebra of $\mathcal{A}$, and where $\mathcal{C}(X)$ is trivially graded; in particular this gives $\mathcal{A}$ a $\mathcal{C}(X)$-module structure.

Definition 4.2. Let $(\mathcal{A}, r_{\mathcal{A}})$ and $(\mathcal{A}', r_{\mathcal{A}'})$ be graded $\mathcal{C}(X)$-algebras where X is a compact manifold. Then $\mathbb{E}_X(\mathcal{A},\mathcal{A}')$ is the set of all triples (E,ϕ,F) where E is countably generated graded Hilbert module over $\mathcal{A}'$, ϕ is a graded $*$-homomorphism from $\mathcal{A}$ to $\mathcal{B}(E)$ and F is an operator in $\mathcal{B}(E)$ of degree 1, such that for all $a\in\mathcal{A}$, $b \in \mathcal{A}'$, $e\in E$ and $f\in\mathcal{C}(X)$,

$$\begin{aligned} &\text{(i) } [\phi(a),F] \in \mathcal{K}(E),\\ &\text{(ii) } \phi(a)(F^2-\mathrm{Id}) \in \mathcal{K}(E),\\ &\text{(iii) } \phi(a)(F-F^*) \in \mathcal{K}(E),\\ &\text{(iv) } \phi(a\cdot r_{\mathcal{A}}(f))(e\cdot b) = \phi(a)(e\cdot(r_{\mathcal{A}'}(f)\cdot b)). \end{aligned} \tag{4.1}$$

Here $\mathcal{K}(E)$, defined for instance in [9], is the analog of compact operators for Hilbert modules. The elements of $\mathbb{E}_X(\mathcal{A},\mathcal{A}')$ are called Kasparov $\mathcal{C}(X)$-modules for $(\mathcal{A},\mathcal{A}')$. We denote by $\mathbb{D}_X(\mathcal{A},\mathcal{A}')$ the set of triples in $\mathbb{E}_X(\mathcal{A},\mathcal{A}')$ for which $[F,\phi(a)]$, $(F-F^*)\phi(a)$ and $(F^2-1)\phi(a)$ vanish for all $a\in\mathcal{A}$. The elements of $\mathbb{D}_X(\mathcal{A},\mathcal{A}')$ are called degenerate Kasparov modules.

Condition (iv) is the extra feature needed to deal with families of elliptic operators. It requires equivariance for the $\mathcal{C}(X)$-module structure of $\mathcal{A}$ and $\mathcal{A}'$. One recovers the definition of standard Kasparov modules by dropping (iv).

An element $(E,\phi,F)\in\mathbb{E}_X(\mathcal{A},\mathcal{C}([0,1];\mathcal{A}'))$ generates a family

$$\{(E_t,\phi_t,F_t)\in\mathbb{E}_X(\mathcal{A},\mathcal{A}');t\in[0,1]\}$$

obtained by evaluation at each $t\in[0,1]$. This family and the triple itself will be called a homotopy between (E_0,ϕ_0,F_0) and (E_1,ϕ_1,F_1) and these modules are then said to be homotopic with the relation written

$$(E_0,\phi_0,F_0)\sim_h(E_1,\phi_1,F_1).$$

See for example [9] for a proof that all degenerate Kasparov modules are homotopic to the trivial Kasparov $\mathcal{C}(X)$-module.

Definition 4.3. We denote by $\mathrm{KK}^0_X(\mathcal{A},\mathcal{A}')$ the set of equivalence classes of $\mathbb{E}_X(\mathcal{A},\mathcal{A}')$ under the equivalence relation $\sim_h$ and similarly define $\mathrm{KK}^1_X(\mathcal{A},\mathcal{A}')$ by

$$\mathrm{KK}^1_X(\mathcal{A},\mathcal{A}')=\mathrm{KK}^0_X(\mathcal{A},\mathcal{S}\mathcal{A}')$$

where

$$\mathcal{S}\mathcal{A}'=\{f\in\mathcal{C}([0,1];\mathcal{A}');\ f(0)=f(1)=0\}$$

is the the suspension of the C^* algebra $\mathcal{A}'$.

In the paper of Kasparov [17], the notation $\mathcal{R}\,\mathrm{KK}(X;\mathcal{A},\mathcal{A}')$ is used, we prefer a more compact notation.

As discussed in [9], there are various other useful equivalence relations which give $\mathrm{KK}^0_X(\mathcal{A},\mathcal{A}')$. The set of equivalent classes $\mathrm{KK}^0_X(\mathcal{A},\mathcal{A}')$ is an abelian group with addition given by direct sum

$$[(E_0,\phi_0,F_0)]+[(E_1,\phi_1,F_1)]=[(E_0\oplus E_1,\phi_0\oplus\phi_1,F_0\oplus F_1)].$$

We are now in a position to define the KK-classes associated to fully elliptic fibred cusp operators. In our situation, $X=B$ is the base of the fibration, while $\mathcal{A}$ will be variously $\mathcal{C}_0(M)$, $\mathcal{C}_\Phi(M)$, $\mathcal{C}(D)$, etc., and $\mathcal{A}'=\mathcal{C}(B)$. In particular, we will only consider commutative C^* algebras which are trivially graded and the $\mathcal{C}(B)$-algebra structure will always be the obvious one.

Let $P\in\Psi^m_{\Phi\text{-}\mathrm{cu}}(M/B;\mathbb{E})$ be a family of fully elliptic fibred cusp operators of order m, where $\mathbb{E}$ is a $\mathbb{Z}_2$-graded complex vector bundle on M. Introducing a graded inner product on $\mathbb{E}$ and family of metrics on M/B it follows that P^*P is also a family of fully elliptic operators of order $2m$ with strictly positive symbol and indicial family. By standard pseudodifferential constructions there is an approximate inverse square-root $Q\in\Psi^{-m}_{\Phi\text{-}\mathrm{cu}}(M/B;E_+)$ which is invertible and positive definite such that

$$Q^2\circ P^*P-\mathrm{Id}\in x^\infty\Psi^{-\infty}_{\Phi\text{-}\mathrm{cu}}(M/B;E_+), \tag{4.2}$$

where x is a boundary defining function for ∂M. Consider the family of operators $A=PQ\in\Psi^0_{\Phi\text{-}\mathrm{cu}}(M/B;\mathbb{E})$. It is fully elliptic and by construction it is almost unitary in the sense that

$$A^*A-\mathrm{Id}\in x^\infty\Psi^{-\infty}_{\Phi\text{-}\mathrm{cu}}(M/B;E_+)\text{ and }AA^*-\mathrm{Id}\in x^\infty\Psi^{-\infty}_{\Phi\text{-}\mathrm{cu}}(M/B;E_-) \tag{4.3}$$

are families of compact operators. Let $\mathcal{C}_\Phi(M)$ denote the space of continuous functions on M which are constant along the fibres of Φ, that is

$$\mathcal{C}_\Phi(M)=\{f\in\mathcal{C}(M);f\big|_{\partial M}=g\circ\Phi\text{ for some }g\in\mathcal{C}(D)\}. \tag{4.4}$$

The injection $\mathcal{C}(B) \hookrightarrow \mathcal{C}_\Phi(M)$ gives $\mathcal{C}_\Phi(M)$ a $\mathcal{C}(B)$-algebra structure and $\mathcal{C}(B)$ itself has the $\mathcal{C}(B)$-algebra structure given by the identity map $\mathcal{C}(B) \longrightarrow \mathcal{C}(B)$. Also let $\mathcal{C}^\infty_\Phi(M) \subset \mathcal{C}_\Phi(M)$ denote the subspace of those functions which are smooth, thus

$$\mathcal{C}^\infty_\Phi(M) = \{f \in \mathcal{C}^\infty(M); f\big|_{\partial M} = g \circ \Phi \text{ for some } g \in \mathcal{C}^\infty(D)\}.$$

Let us denote by μ the action of the C^* algebra $\mathcal{C}_\Phi(M)$ through multiplication

$$\mu(f) \in \mathcal{B}(\mathcal{H}),\ \mathcal{H} = \mathrm{L}^2(M/B; \mathbb{E}) = \mathrm{L}^2(M/B; E_+) \oplus \mathrm{L}^2(M/B; E_-),\ f \in \mathcal{C}_\Phi(M).$$

Lemma 4.1. *The triple* $(\mathcal{H}, \mu, \mathcal{F})$ *where*

$$\mathcal{F} = \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix} \in \mathcal{B}(\mathcal{H})$$

gives rise to a well-defined Kasparov module in $\mathbb{E}_B(\mathcal{C}_\Phi(M), \mathcal{C}(B))$ *and hence a class*

$$[P] = [(\mathcal{H}, \mu, \mathcal{F})] \in \mathrm{KK}^0_B(\mathcal{C}_\Phi(M), \mathcal{C}(B)). \tag{4.5}$$

Proof. By Kuiper's theorem, $\mathcal{H}$ is a countably generated $\mathbb{Z}_2$-graded Hilbert module over $\mathcal{C}(B)$. To show that $(\mathcal{H}, \mu, \mathcal{F})$ is a module in the sense of Kasparov we need to check the following properties

(4.6)

(i) $[\mu(f), \mathcal{F}] \in \mathcal{K}(\mathcal{H})$,

(ii) $\mu(f)(\mathcal{F}^2 - \mathrm{Id}) \in \mathcal{K}(\mathcal{H})$,

(iii) $\mu(f)(\mathcal{F} - \mathcal{F}^*) \in \mathcal{K}(\mathcal{H})$,

(iv) $\mu(b_1 f)(h \cdot b_2) = \mu(f)(h \cdot (b_2 b_1))$,

$\forall\ f \in \mathcal{C}_\Phi(M),\ b_1, b_2 \in \mathcal{C}(B)$ and $h \in \mathcal{H}$.

Property (iv) is immediate. Property (iii) follows directly from the fact that $\mathcal{F}^* = \mathcal{F}$. Property (ii) is a consequence of (4.3). To check property (i), we may restrict to $f \in \mathcal{C}^\infty_\Phi(M)$ since these smooth functions are dense in $\mathcal{C}_\Phi(M)$ and the map

$$[\mu(\cdot), \mathcal{F}] : \mathcal{C}_\Phi(M) \longrightarrow \mathcal{B}(\mathcal{H})$$

is continuous.

For any $f \in \mathcal{C}^\infty_\Phi(M)$, $\mu(f) \in \Psi^0_{\Phi\text{-cu}}(M/B; \mathbb{E})$ has symbol f and indicial family which can be identified with $f\big|_{\partial M}$ which is to say a constant multiple of the identity on each fibre, so commuting with the normal operator of any other element. Thus

$$[\mu(f), \mathcal{F}] \in x\Psi^{-1}_{\Phi\text{-cu}}(M/B; \mathbb{E})$$

is a family of compact operators. □

A fully elliptic operator also defines a class in the group $\mathrm{KK}(\mathcal{C}_\Phi(M), \mathcal{C}(B))$; to get Poincaré duality, we need to take into account the $\mathcal{C}(B)$-algebra structure.

Lemma 4.2. *The class* $[P] \in \mathrm{KK}^0_B(\mathcal{C}_\Phi(M), \mathcal{C}(B))$ *associated to a fully elliptic family of fibred cusp pseudodifferential operators by Lemma 4.1 does not depend on the choice of* Q *in* (4.2) *and in fact only depends on the homotopy class of* P *in the space of fully elliptic operators.*

Proof. Any two choices of a family of positive definite approximate square-roots differ by a family in $x^\infty\Psi^{-\infty}_{\Phi\text{-cu}}(M/B; E_+)$ so the resulting families $\mathcal{F}$ differ by compact families and hence define the same element in $\mathrm{KK}^0_B(\mathcal{C}_\Phi(M), \mathcal{C}(B))$.

To prove the second part of the lemma, let $p_t \in \Psi^m_{\Phi\text{-cu}}([0,1] \times M/B; \mathbb{E})$ be a smooth curve of families of fully elliptic operators, where $t \in [0,1]$. Then there exists a smooth curve $Q_t \in \Psi^{-m}_{\Phi\text{-}\mathrm{cu}}([0,1] \times M/B; E_+)$ of invertible approximate inverse square-roots such that

$$Q_t^2 \circ P_t^* P_t - \mathrm{Id} \in x^\infty \Psi^{-\infty}_{\Phi\text{-}\mathrm{cu}}([0,1] \times M/B; \mathbb{E}).$$

Hence, $(\mathcal{H}, \mu, \mathcal{F}_t) \in \mathbb{E}(\mathcal{C}_\Phi(M), \mathcal{C}(B))$ and if $A_t = P_t Q_t$ then

$$\mathcal{F}_t = \begin{pmatrix} 0 & A_t^* \\ A_t & 0 \end{pmatrix} \in \mathcal{B}(\mathcal{H}),$$

defines an operator homotopy between the modules $(\mathcal{H}, \mu, \mathcal{F}_0)$ and $(\mathcal{H}, \mu, \mathcal{F}_1)$. This implies that $[P_0] = [P_1]$ in $\mathrm{KK}^0_B(\mathcal{C}_\Phi(M), \mathcal{C}(B))$. □

This KK-class also behaves in the expected manner under direct sums, so if $P \in \Psi^m_{\Phi\text{-}\mathrm{cu}}(M/B; \mathbb{E})$ and $R \in \Psi^m_{\Phi\text{-}\mathrm{cu}}(M/B; \mathbb{F})$ are families of fully elliptic operators, then

$$[P \oplus R] = [P] + [R] \text{ in } \mathrm{KK}^0_B(\mathcal{C}_\Phi(M), \mathcal{C}(B)).$$

It follows that this construction defines a 'quantization' homomorphism of abelian groups

$$\mathrm{quan} : \mathrm{K}_{\Phi\text{-}\mathrm{cu}}(\phi) \longrightarrow \mathrm{KK}^0_B(\mathcal{C}_\Phi(M), \mathcal{C}(B)). \tag{4.7}$$

The analytical index of the family P factors through this map. Let $c_\Phi : \mathcal{C}(B) \longrightarrow \mathcal{C}_\Phi(M)$ be the inclusion of constant functions along the fibres of $\phi : M \longrightarrow B$. Then, at the level of KK-theory, c_Φ defines a contravariant functor

$$c_\Phi^* : \mathrm{KK}^0_B(\mathcal{C}_\Phi(M), \mathcal{C}(B)) \longrightarrow \mathrm{KK}^0_B(\mathcal{C}(B), \mathcal{C}(B)).$$

Lemma 4.3. *Under the standard identification*

$$\mathrm{KK}^0_B(\mathcal{C}(B), \mathcal{C}(B)) \cong \mathrm{KK}(\mathbb{C}, \mathcal{C}(B)) \cong \mathrm{K}^0(B),$$

there is a commutative diagram

$$\begin{array}{ccc} \mathrm{K}_{\Phi\text{-}\mathrm{cu}}(\phi) & \xrightarrow{\mathrm{quan}} & \mathrm{KK}^0_B(\mathcal{C}_\Phi(M), \mathcal{C}(B)) \\ & \searrow^{\mathrm{ind}_a} & \downarrow c_\Phi^* \\ & & \mathrm{K}(B) \end{array}$$

Proof. This follows from the discussion in [9], more precisely proposition 17.5.5, corollary 12.2.3 and paragraph 8.3.2. It is also a simple consequence of the stabilization of the null space of P, by perturbation, of Lemma 1.1. □

It is also possible to define a quantization map for $\mathrm{K}^1_{\Phi\text{-}\mathrm{cu}}(\phi)$. Given a joint symbol $(\sigma, N) \in A^0_{s\Phi}(s\phi; E_+)$ representing a class in $\mathrm{K}^1_{\Phi\text{-}\mathrm{cu}}(\phi)$, let $P_s \in \Psi^0_{\mathrm{s}\Phi\text{-}\mathrm{cu}}(M \times I/B \times I; E)$ be a family of fully elliptic fibred cusp operators with joint symbol (σ, N) such that $P_s\big|_{B \times \{0,1\}} \equiv \mathrm{Id}$. Here, recall that $s\Phi : \partial M \times I \to D \times I$ is the boundary fibration of (1.19). Let $Q_s \in \Psi^0_{\mathrm{s}\Phi\text{-}\mathrm{cu}}(M \times I/B \times I; E_+)$ be an approximate positive definite inverse square root

$$Q_s^2 \circ P_s^* P_s - \mathrm{Id} \in x^\infty \Psi^{-\infty}_{\mathrm{s}\Phi\text{-}\mathrm{cu}}(M \times I/B \times I, E_+)$$

such that $Q\big|_{B \times \{0,1\}} \equiv \mathrm{Id}$, where x is the boundary defining function for M. Consider the family of operators $A = PQ \in \Psi^0_{\mathrm{s}\Phi\text{-}\mathrm{cu}}(M \times I/B \times I; E_+)$. It is fully elliptic, $A\big|_{B \times \{0,1\}} \equiv \mathrm{Id}$, and by construction it almost unitary in the sense that

$$A^*A - \mathrm{Id} \in x^\infty \Psi^{-\infty}_{\mathrm{s}\Phi\text{-}\mathrm{cu}}(M \times I/B \times I, E_+), \;\; AA^* - \mathrm{Id} \in x^\infty \Psi^{-\infty}_{\mathrm{s}\Phi\text{-}\mathrm{cu}}(M \times I/B \times I, E_-)$$

are families of compact operators which vanish on $B \times \{0,1\}$. Notice that there is a natural inclusion $\mathcal{C}_\Phi(M) \subset \mathcal{C}_{s\Phi}(M \times I)$. Let us denote by μ_s the action of the C^* algebra $\mathcal{C}_\Phi(M)$ through multiplication

$$\begin{aligned}(4.8)\quad \mu_s(f) \in \mathcal{B}(\mathcal{H}_s),\ \mathcal{H}_s &= \mathrm{L}^2(M \times I/B \times I; \mathbb{E}) \\ &= \mathrm{L}^2(M \times I/B \times I; E_+) \oplus \mathrm{L}^2(M \times I/B \times I; E_-)\end{aligned}$$

for $f \in \mathcal{C}_\Phi(M)$, where in this odd context $E_+ = E_-$. Let $\mathcal{SC}(B)$ be the the C^* algebra of continuous functions

$$\mathcal{SC}(B) = \{f \in \mathcal{C}(B \times I); f\big|_{B\times\{0,1\}} \equiv 0\}. \tag{4.9}$$

Lemma 4.4. *The triple $(\mathcal{H}_s, \mu_s, \mathcal{F}_s)$ where*

$$\mathcal{F}_s = \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix} \in \mathcal{B}(\mathcal{H}_s),$$

gives rise to a well-defined Kasparov module $\mathbb{E}_B(\mathcal{C}_\Phi(M), \mathcal{SC}(B))$ and a class

$$[(\sigma, N)] = [P_s] \in \mathrm{KK}^0_B(\mathcal{C}_\Phi(M), \mathcal{SC}(B))$$

which only depends on the class $[(\sigma, N)] \in \mathrm{K}^1_{\Phi\text{-}\mathrm{cu}}(\phi)$

Corollary 4.5. *Under the standard identification*

$$\mathrm{KK}^0_B(\mathcal{C}_\Phi(M), \mathcal{SC}(B)) \cong \mathrm{KK}^1_B(\mathcal{C}_\Phi(M), \mathcal{C}(B)),$$

Lemma 4.4 gives us a well defined quantization map

$$\mathrm{quan} : \mathrm{K}^1_{\Phi\text{-}\mathrm{cu}}(\phi) \longrightarrow \mathrm{KK}^1_B(\mathcal{C}_\Phi(M), \mathcal{C}(B)). \tag{4.10}$$

5. Poincaré duality for cusp operators

We shall prove Theorem 2 in the case of the cusp structure. To do so, we first discuss the short exact sequence (10).

Consider the collection of invertible joint symbols of cusp pseudodifferential operators,

$$\begin{aligned}(5.1)\quad J^0_{\mathrm{cu}}(\phi, \mathbb{E}) = \big\{&(\sigma, N) \in S^0_{\mathrm{cu}}(M/B); \mathbb{E}) \oplus \Psi^0_{\Phi-\mathrm{sus}}(\partial M/B; \mathbb{E}); \\ &\sigma\big|_{\partial M} = \sigma_0(N),\ (\sigma^{-1}, N^{-1}) \in S^0_{\mathrm{cu}}(M/B); \mathbb{E}^-) \oplus \Psi^0_{\Phi-\mathrm{sus}}(\partial M/B; \mathbb{E}^-)\big\}\end{aligned}$$

on $\mathbb{Z}_2$-graded bundles where $S^0_{\mathrm{cu}}(M/B; \mathbb{E}) = \mathcal{C}^\infty({}^{\mathrm{cu}}S^*(M/B); \hom(\mathbb{E}))$. This naturally maps by restriction to the collection of invertible symbols

$$\begin{aligned}(5.2)\quad G^0_{\mathrm{cu}}(\phi, \mathbb{E}) = \big\{\sigma \in S^0_{\mathrm{cu}}(M/B; \mathbb{E}); \sigma^{-1} \in S^0_{\mathrm{cu}}(M/B; \mathbb{E}^-)\big\}, \\ \sigma : J^0_{\mathrm{cu}}(\phi, \mathbb{E}) \longrightarrow G^0_{\mathrm{cu}}(\phi, \mathbb{E}).\end{aligned}$$

In fact this map is surjective. That is, for every family of elliptic symbols there does exist a family of invertible normal operators which is compatible with it. This is an aspect of the cobordism invariance of the index and is shown in this form in [28].

The clutching construction associates to each invertible symbol an element of the compactly supported K theory of ${}^{\mathrm{cu}}T^*(M/B)$ which is 'absolute' with respect to the boundary of M. As in the boundaryless case, the resulting element is stable under homotopy, bundle isomorphisms of $E_\pm$ and stabilization, so (5.2) descends to a surjective map

$$\mathrm{K}_{\mathrm{cu}}(\phi) \longrightarrow \mathrm{K}_{\mathrm{c}}(T^*(M/B)) \tag{5.3}$$

where we use the fact that the cusp and standard cotangent bundles are isomorphic, with the isomorphism natural up to homotopy. Thus (5.3) is a surjective group homomorphism.

If $E_+ = E_-$ we can consider those invertible suspended families of pseudodifferential operators which are smoothing perturbations of the identity on a fixed bundle over ∂M

$$G^{-\infty}_{\mathrm{sus}}(\phi; E) = \left\{ N \in \mathrm{Id}_E + \Psi^{-\infty}_{\mathrm{sus}}(\partial M/B; E); N^{-1} \in \mathrm{Id}_E + \Psi^{-\infty}_{\mathrm{sus}}(\partial M/B; E) \right\}. \tag{5.4}$$

Let $G^{-\infty}_{\mathrm{sus}}(\phi; *)$ be the union over E and let $G^{-\infty}_{\text{e-sus}}(\phi; *)$ be the subset corresponding to bundles E which bound a bundle over M. Since we may complement a bundle to be trivial, and hence extendible, the stable homotopy classes of elements of $G^{-\infty}_{\text{e-sus}}(\phi; *)$ and $G^{-\infty}_{\mathrm{sus}}(\phi; *)$ are the same.

Lemma 5.1. *Passing to the set of stable homotopy classes, with equivalence also under bundle isomorphisms, the inclusion and restriction maps give the split short exact sequence of Abelian groups* (11)*:*

$$\begin{array}{ccccc} G^{-\infty}_{\text{e-sus}}(\phi; *) & \xrightarrow{i} & J^0_{\mathrm{cu}}(\phi, *) & \xrightarrow{\sigma} & G^0_{\mathrm{cu}}(\phi, *) \\ \downarrow \sim & \swarrow \mathrm{ind}_a & \downarrow \sim & & \downarrow \sim \\ \mathrm{K}(B) & \underset{\mathrm{ind}_a}{\overset{i}{\rightleftarrows}} & \mathrm{K}_{\mathrm{cu}}(\phi) & \underset{\mathrm{inv}}{\overset{\sigma}{\rightleftarrows}} & \mathrm{K}_c(T^*(M/B)). \end{array} \tag{5.5}$$

Proof. That the set of stable homotopy classes, also allowing smooth identification of bundles, of $G^{-\infty}_{\text{e-sus}}(\phi, *)$, and hence also $G^{-\infty}_{\mathrm{sus}}(\phi, *)$, is canonically identified with $\mathrm{K}(B)$ is a standard result (see for instance [12]) when the fibration is trivial, $M = Z \times B$. It remains true in the general case, this can be shown using the families of projections Π_N of Section 1, see also [32].

Since direct sums behave consistently, the resulting maps are group homomorphisms and form a complex, since $G^{-\infty}_{\text{e-sus}}(\phi, *)$ clearly maps to the identity in $G^0_{\mathrm{cu}}(\phi; *)$.

We have already noted the surjectivity of the second map. To see exactness in the middle, suppose that a compatible pair (σ, N) induces a trivial class in $\mathrm{K}_{\mathrm{c}}(T^*(M/B))$. Since stabilization and the action of bundle isomorphisms is the same on the full and symbolic data, we may suppose that σ is homotopic to the identity through elliptic symbols. In particular, $E_+ = E_-$. Adding the homotopy variable as an additional base variable, the surjectivity of the symbol map allows the homotopy of symbols to be lifted to an homotopy of joint symbols, see Remark 1.2. Thus (σ, N) may be deformed by homotopy to $(\mathrm{Id}, \mathrm{Id} + A)$ where necessarily $A \in \Psi^{-1}_{\mathrm{sus}}(\partial M/B; E)$. By a further small perturbation this is homotopic to $\mathrm{Id} + A \in G^{-\infty}_{\text{e-sus}}(\phi, *)$, $A \in \Psi^{-\infty}_{\mathrm{sus}}(\partial M/B; E)$, showing exactness at $\mathrm{K}_{\mathrm{cu}}(\phi)$. Injectivity of the first map follows from the fact that the index map provides a right inverse for it, which is a consequence of the families index of Proposition 3.1. Note that the existence of an invertible family with a given elliptic symbol defines the map inv which shows that the sequence splits. □

There is also a corresponding exact sequence at the level of KK-theory.

Lemma 5.2. *The short exact sequence of C^*-algebras* (12) *leads to a split short exact sequence*

$$\mathrm{KK}^0_B(\mathcal{C}(B),\mathcal{C}(B)) \xrightarrow{\iota^*} \mathrm{KK}^0_B(\mathcal{C}_{\mathrm{cu}}(M),\mathcal{C}(B)) \xrightarrow{s^*} \mathrm{KK}^0_B(\mathcal{C}_0(M),\mathcal{C}(B)). \tag{5.6}$$

Proof. From standard results in KK-theory (cf. theorem 19.5.7 in [9]), the short exact sequence (12) leads to a six-term exact sequence

$$\begin{array}{ccccc} \mathrm{KK}^0_B(\mathcal{C}(B),\mathcal{C}(B)) & \xrightarrow{\iota^*} & \mathrm{KK}^0_B(\mathcal{C}_{\mathrm{cu}}(M),\mathcal{C}(B)) & \xrightarrow{s^*} & \mathrm{KK}^0_B(\mathcal{C}_0(M),\mathcal{C}(B)) \\ \delta\uparrow & & & & \downarrow\delta \\ \mathrm{KK}^1_B(\mathcal{C}_0(M),\mathcal{C}(B)) & \xleftarrow{s^*} & \mathrm{KK}^1_B(\mathcal{C}_{\mathrm{cu}}(M),\mathcal{C}(B)) & \xleftarrow{\iota^*} & \mathrm{KK}^1_B(\mathcal{C}(B),\mathcal{C}(B)) \end{array} \tag{5.7}$$

where both boundary homomorphisms, δ, are obtained by multiplying by a specific element $\delta_\iota \in \mathrm{KK}^1_B(\mathcal{C}(B),\mathcal{C}_0(M))$. More precisely, under the identification of $\mathrm{KK}^1_B(\mathcal{C}(B),\mathcal{C}_0(M))$ with $\mathrm{KK}^0_B(S\mathcal{C}(B),\mathcal{C}_0(M))$, $\delta_\iota = i^*u$ where $u \in \mathrm{KK}^0_B(C_\iota,\mathcal{C}_0(M))$ and $i : S\mathcal{C}(B) \longrightarrow C_\iota$ is the natural inclusion. Here, C_ι is the mapping cone

$$C_\iota = \{(x,f) \in \mathcal{C}_{\mathrm{cu}}(M) \oplus \mathcal{C}_0([0,1)\times B); \iota(x) = f(0)\},\ \ S\mathcal{C}(B) = \mathcal{C}_0((0,1)\times B)$$

and $i(f) = (0,f) \in C_\iota$ for $f \in S\mathcal{C}(B)$. In this situation there is an injective map

$$j : \mathcal{C}_0([0,1)\times B) \ni f \longmapsto (\phi^*(f(0)),f) \in C_\iota$$

so we may interpret i as a map from $S\mathcal{C}(B)$ to $\mathcal{C}_0([0,1)\times B)$. Since $\mathcal{C}_0([0,1)\times B)$ is a contractible C^*-algebra, the class $[i]$ of i in $\mathrm{KK}^0_B(S\mathcal{C}(B),C_\iota)$ is zero. In particular, this means that $i^*u = 0$ since i^*u can be interpreted as the Kasparov product of $[i]$ with u (see for example 18.4.2a in [9]). Thus $\delta = 0$ and we get the short exact sequence (5.6).

To see that this short exact sequence splits, consider the natural injective C^*-homomorphism $i_\phi : \mathcal{C}(B) \longrightarrow \mathcal{C}_{\mathrm{cu}}(M)$. This satisfies $\iota i_\phi = \mathrm{Id}$, so i^*_ϕ is a left inverse for ι^*. □

There is a correspondence between the short exact sequences of Lemmas 5.1 and 5.2. Consider the diagram

$$\begin{array}{ccccc} \mathrm{K}(B) & \xrightarrow{i_*} & \mathrm{K}_{\mathrm{cu}}(\phi) & \xrightarrow{\sigma_*} & \mathrm{K}_{\mathrm{c}}(T^*(M/B)) \\ \mathrm{quan}_{-\infty}\downarrow & & \mathrm{quan}\downarrow & & \downarrow\mathrm{quan}_a \\ \mathrm{KK}^0_B(\mathcal{C}(B),\mathcal{C}(B)) & \xrightarrow{\iota^*} & \mathrm{KK}^0_B(\mathcal{C}_{\mathrm{cu}}(M),\mathcal{C}(B)) & \xrightarrow{s^*} & \mathrm{KK}^0_B(\mathcal{C}_0(M),\mathcal{C}(B)) \end{array} \tag{5.8}$$

where the top row is the short exact sequence of Lemma 5.1 and the bottom row is the short exact sequence of Lemma 5.2. The central map is the quantization map of Lemma 4.3. Similarly, the map quan_a is the quantization map as discussed in [25].

The map $\mathrm{quan}_{-\infty}$ on the left is essentially the same quantization map as in Lemma 4.3. Thus if $N \in G^{-\infty}_{\text{e-sus}}(\phi,E_+)$ there exists $P = \mathrm{Id}+L$, $L \in \Psi^{-\infty}_{\mathrm{cu}}(M/B;E_+)$ with $N(P) = N$. Choose an invertible, positive, approximate inverse square-root, $Q \in \mathrm{Id}+\Psi^{-\infty}_{\mathrm{cu}}(M/B;E_+)$,

$$Q^2 \circ P^*P - \mathrm{Id} \in x^\infty \Psi^{-\infty}_{\Phi\text{-}\mathrm{cu}}(M/B;E_+)$$

and consider $A = PQ$. Let $\mathcal{H}_B$ be the $C(B)$-Hilbert module

$$\mathcal{H}_B = \mathcal{H}_B^+ \oplus \mathcal{H}_B^- = \mathrm{L}^2(M;E) \oplus \mathrm{L}^2(M;E). \tag{5.9}$$

Then $\mathrm{quan}_{-\infty}(N)$ is the KK-class associated to the Kasparov module $(\mathcal{H}_B, \mu_B, \mathcal{F}_B)$ in $\mathbb{E}_B(\mathcal{C}(B), \mathcal{C}(B))$ where

$$\mathcal{F}_B = \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix} \tag{5.10}$$

and $\mu_B : \mathcal{C}(B) \longrightarrow \mathcal{B}(\mathcal{H}_B)$ is just given by multiplication using the $\mathcal{C}(B)$-module structure.

Proposition 5.3. *The diagram* (5.8) *is commutative and* $\mathrm{quan}_{-\infty}$, quan *and* quan_a *are isomorphisms.*

Proof. The fact that quan_a is an isomorphism is established in [25], where the Poincaré duality of [17] is generalized. From the definition of quan and quan_a, it is straightforward to check that the right square of the diagram is commutative.

That $\mathrm{quan}_{-\infty}$ is an isomorphism follows from the identification

$$\mathrm{KK}^0_B(\mathcal{C}(B), \mathcal{C}(B)) \cong \mathrm{KK}^0(\mathbb{C}, \mathcal{C}(B)) \cong \mathrm{K}(B)$$

and that under this $\mathrm{quan}_{-\infty}$ gives the index map.

The commutativity of the left square is not quite obvious since it does not commute in terms of Kasparov modules. Indeed, if $P \in \mathrm{Id} + K$ is as discussed above then $\iota^* \mathrm{quan}_{-\infty}(P)$ is represented by the Kasparov module $(\mathcal{H}_B, \iota^*\mu_B, \mathcal{F}_B)$ with $\mathcal{H}_B$ and $\mathcal{F}_B$ are as in (5.9) and (5.10) whereas $\mathrm{quan}\, i_*(P)$ is represented by the Kasparov module $(\mathcal{H}_B, \mu, \mathcal{F}_B)$ with $\mu : \mathcal{C}_{\mathrm{cu}}(M) \longrightarrow \mathcal{B}(\mathcal{H}_B)$ given by multiplication by $\mathcal{C}_{\mathrm{cu}}(M)$. Since

$$\iota^* \mathrm{quan}_{-\infty}(P) = \iota^*(i_\phi)^* \mathrm{quan}\, i_*(P)$$

as Kasparov modules, the commutativity of the second square of the diagram, $s^* \mathrm{quan}\, i_*(P) = 0$ shows that

$$\iota^* \mathrm{quan}_{-\infty}(P) = \iota^*(i_\phi)^* \mathrm{quan}\, i_*(P) = \mathrm{quan}\, i_*(P) \text{ in } \mathrm{KK}^0_B(\mathcal{C}_{\mathrm{cu}}(M), \mathcal{C}(B))$$

and so these two modules give the same element in the K-group.

This implies that quan is also an isomorphism. □

6. The 6-term exact sequence

As noted above, it is always possible to perturb an elliptic family of cusp operators by a cusp operator of order $-\infty$ so that it becomes invertible and this leads to the short exact sequence (11). In the general fibred cusp case there is an obstruction in K-theory to the existence of such a perturbation and this results in the 6-term exact sequence (22).

Consider a fibration with fibred cusp structure, as in (21), so that the algebra $\Psi^*_{\Phi\text{-}\mathrm{cu}}(M/B)$ of families of fibred cusp operators is well-defined. Let $r_{\partial M}$ denote the inclusion

$$r_{\partial M} : T^*_{\partial M}(M/B) \hookrightarrow T^*(M/B). \tag{6.1}$$

If

$$d\Phi : T_{\partial M}(M/B) \longrightarrow T(D/B) \times \mathbb{R} \tag{6.2}$$

is the (extended) differential of Φ, then identifying the tangent bundles with the cotangent bundles via some choice of metrics, there is a well-defined families index

$$\mathrm{ind}_{\mathrm{AS}} : K^0_c(T^*_{\partial M}(M/B)) \longrightarrow \mathrm{K_c}(T^*(D/B) \times \mathbb{R}) \cong \mathrm{K}^1_{\mathrm{c}}(T^*(D/B)).$$

Proposition 6.1. *An elliptic family, $P \in \Psi^m_{\Phi\text{-}\mathrm{cu}}(M/B; \mathbb{E})$, can be perturbed to be fully elliptic by the addition of some $Q \in \Psi^{-\infty}_{\Phi\text{-}\mathrm{cu}}(M/B; \mathbb{E})$ if and only if the index of the class of its symbol at the boundary*

$$\mathrm{ind}_{\mathrm{AS}}[r^*_{\partial M}\sigma(P)] = 0 \in \mathrm{K}^1_c(T^*(D/B)) \tag{6.3}$$

and then Q can be chosen so that $P + Q$ is invertible.

Proof. By the families index theorem of Atiyah and Singer, there is a suspended perturbation $I(Q)$ of order $-\infty$ such that $I(P) + I(Q)$ is invertible if and only if (6.3) holds. With such a choice $P+Q$ is a fully elliptic family. Since it follows from Proposition 3.1 that the index map on perturbations of the identity is surjective, we may compose on the left with a Fredholm family of the form $\mathrm{Id} + \Psi^{-\infty}_{\Phi\text{-}\mathrm{cu}}(M/B; E_-)$ to get an operator of index zero and then perturb, as in Section 1 to make it invertible. □

In the particular case of a single elliptic cusp operator, the obstruction is in $\mathrm{K}^1(\mathrm{pt}) \cong \{0\}$, so there is no obstruction to such a perturbation. In the case of a family of elliptic cusp operators, the obstruction is in $\mathrm{K}^1(D)$, which is not the trivial group in general. However $\mathrm{ind}_{\mathrm{AS}} \circ r^*_{\partial M}\kappa(P)$ is always zero in $\mathrm{K}^1(D)$ by the cobordism invariance of the index.

There is a parallel discussion in the odd case. The fibration (6.2) also induces an odd index

$$\mathrm{ind}_{\mathrm{AS}} : \mathrm{K}^1_{\mathrm{c}}(T^*_{\partial M}(M/B)) \longrightarrow \mathrm{K}^1_{\mathrm{c}}(T^*(D/B) \times \mathbb{R}) \cong \mathrm{K_c}(T^*(D/B)). \tag{6.4}$$

Proposition 6.2. *Suppose $P \in \Psi^0_{\Phi\text{-}\mathrm{cu}}(M \times [0,1]/D \times [0,1]; E)$ is a family of elliptic operators which is the identity at $t = 0$ and $t = 1$ and let $[\sigma(P)] \in \mathrm{K}^1_{\mathrm{c}}(T^*(M/B))$ be the class of its symbol, then P can be perturbed by $Q \in \Psi^{-\infty}_{\Phi\text{-}\mathrm{cu}}(M \times [0,1]/D \times [0,1]; E)$ with $Q\big|_{D\times\{0,1\}} = 0$ to be invertible if and only if*

$$\mathrm{ind}_{\mathrm{AS}}\, r^*_{\partial M}[\sigma(P)] = 0 \in \mathrm{K}_c(T^*(D/B)).$$

Proof. One could proceed as in the proof of Proposition 6.1. However, there is an alternative proof which is more suggestive in this case. As discussed in [12], one can define the odd index (6.4) in the following way. Notice that $r^*_{\partial M}\sigma(P)$, which is the symbol of the indicial family, gives a class in

$$\mathrm{K}^1_{\mathrm{c}}(T^*_{\partial M}(M/B)) \cong \mathrm{K}^1_{\mathrm{c}}(T^*(\partial M/B) \times \mathbb{R}).$$

If $t \in [0,1]$ denotes the suspension parameter, then by assumption $r^*_{\partial M}\sigma(P)$ is the identity at $t = 0$ and $t = 1$. So there is no obstruction to perturb P by smoothing operators so that one gets a 1-parameter family

$$t \mapsto N(P_t) \in G^0_{\Phi s(1)}(\partial M/B; E) \tag{6.5}$$

such that $N(P_0) = \mathrm{Id}$ and $N(P_1) \in G^{-\infty}_{\Phi s}(\partial M/B; E)$, where $G^0_{\Phi s(1)}(\partial M/B; E)$ is the group of invertible elliptic fibred suspended operators of order 0. Via the identification (if $\dim D = \dim \partial M$, this is only true after stabilization)

$$\pi_0(G^{-\infty}_{\Phi s}(\partial M/B; E)) \cong \mathrm{K}^1_{\mathrm{c}}(T^*(D/B)), \tag{6.6}$$

the connected component in which $N(P_1)$ lies gives a class in $\mathrm{K}^1_{\mathrm{c}}(T^*(D/B))$ which is precisely $\mathrm{ind}_{\mathrm{AS}}[r^*_{\partial M}\sigma(P)]$. In particular, it does not depend on the choice of the 1-parameter family (6.5).

Now, if the index is zero, this means $N(P_1)$ is in the connected component of the identity in $G^{-\infty}_{\Phi s}(\partial M/B;E)$, so via some smooth deformation, it can be arranged that $N(P_1) = \mathrm{Id}$ as well, so P can be perturbed by $Q \in \Psi^{-\infty}_{\Phi\text{-}\mathrm{cu}}(M/B;E)$ with $Q\big|_{D\times\{0,1\}} = 0$, to become fully elliptic. Conversely, if such a perturbation Q exists, then the index is given by the K-class corresponding to $N(P_1+Q_1) = \mathrm{Id}$, which is necessarily zero. □

The obstruction result of Proposition 6.1 and Proposition 6.2 indicates that the short exact sequence of Lemma 5.1 fails to be exact in the more general setting of fibred cusp operators. However, as in the case of the K-theory of a pair of spaces, there is a 6-term exact sequence given by

$$\begin{array}{ccccc} \mathrm{K_c}(T^*(D/B)) & \xrightarrow{i_0} & \mathrm{K}_{\Phi\text{-}\mathrm{cu}}(\phi) & \xrightarrow{\sigma_0} & \mathrm{K_c}(T^*(M/B)) \\ \uparrow{\scriptstyle I_1} & & & & \downarrow{\scriptstyle I_0} \\ \mathrm{K}^1_{\mathrm{c}}(T^*(M/B)) & \xleftarrow{\sigma_1} & \mathrm{K}^1_{\Phi\text{-}\mathrm{cu}}(\phi) & \xleftarrow{i_1} & \mathrm{K}^1_{\mathrm{c}}(T^*(D/B)). \end{array} \tag{6.7}$$

To define i_k for $k \in \mathbb{Z}_2$, consider the group

$$G^{-\infty}_{\Phi s(1+k)}(\partial M/B) \cong \{\mathrm{Id}+Q; Q \in \Psi^{-\infty}_{\Phi s(1+k)}(\partial M/B), \mathrm{Id}+Q \text{ is invertible}\}.$$

Then using spectral sections techniques as in [32] or the projections in Section 1, one has an identification

$$\mathrm{K}^{\text{-k}}_{\mathrm{c}}(T^*(D/B)) \cong \pi_0(G^{-\infty}_{\Phi s(1+k)}(\partial M/B)). \tag{6.8}$$

Strictly speaking, the result is only true provided $\dim X > 0$, but in the case $\dim X = 0$, it is only necessary to allow some stabilization. Any element of $G^{-\infty}_{\Phi s(1+k)}(\partial M/B)$ can be seen as the indicial family of a family of fully elliptic operators with symbol given by the identity. This gives a map

$$\pi_0(G^{-\infty}_{\Phi s(1+k)}(\partial M/B)) \longrightarrow \mathrm{K}^{\text{-k}}_{\Phi\text{-}\mathrm{cu}}(\phi) \tag{6.9}$$

and we define i_k by composing (6.8) with (6.9). The symbol maps have already been defined and the boundary map I_0 is given by

$$I_0 = \mathrm{ind}_{\mathrm{AS}} \circ r^*_{\partial M} : \mathrm{K_c}(T^*(M/B)) \longrightarrow \mathrm{K}^1_{\mathrm{c}}(T^*(D/B)),$$

while I_1 has a similar definition

$$I_1 = \mathrm{ind}_{\mathrm{AS}} \circ r^*_{\partial M} : \mathrm{K}^1_{\mathrm{c}}(T^*(M/B)) \longrightarrow \mathrm{K}^{\text{-}1}_{\mathrm{c}}(T^*(D/B)\times\mathbb{R}) \cong \mathrm{K_c}(T^*(D/B)),$$

where the last identification is by Bott periodicity.

Theorem 6.3. *The diagram* (6.7) *is exact.*

Proof. The exactness at $\mathrm{K}_{\Phi\text{-}\mathrm{cu}}(\phi)$ and $\mathrm{K}^1_{\Phi\text{-}\mathrm{cu}}(\phi)$ follows rather directly from the definition since homotopies of the symbol can be lifted to homotopies of the joint symbol, see the discussion above in the cusp case. Exactness at $\mathrm{K_c}(T^*(M/B))$ and $\mathrm{K}^1_{\mathrm{c}}(T^*(M/B))$ follows from Proposition 6.1 and its suspended version.

The exactness at $\mathrm{K_c}(T^*(D/B))$ can be seen from the alternative definition of the odd index used in the proof of Proposition 6.2. Indeed, suppose that $\alpha \in \mathrm{K_c}(T^*(D/B))$ is in the image of I_1. According to the proof of Proposition 6.2, this means that there is a family of fully elliptic operators $t \mapsto P_t \in \Psi^0_{\Phi\text{-cu}}(M/B;E)$ such that $P_0 = \mathrm{Id}$, $\sigma(P_1) = \mathrm{Id}$ and

$$[N(P_1)] \in \pi_0(G^{-\infty}_{\Phi s}(\partial M/B;E) \cong \mathrm{K_c}^0(T^*(D/B))$$

corresponds to α. The homotopy $t \mapsto (\sigma(P_t), N(P_t))$ between the identity and $(\mathrm{Id}, N(P_1))$ then shows that $i(\alpha) = 0$ in $\mathrm{K}_{\Phi\text{-cu}}(\phi)$.

Conversely, suppose that $i(\alpha) = 0$. If $N(P_1) \in G^{-\infty}_{\Phi s}(\partial M/B;E)$ represents α, then after some stabilization, one can assume that there is a homotopy of invertible joint symbols $t \mapsto (\sigma(P_t), N(P_t))$ between $(\mathrm{Id}, \mathrm{Id})$ and $(\mathrm{Id}, N(P_1))$. The family symbol $t \mapsto \sigma(P_t)$ then defines a class $\beta \in \mathrm{K}^1_{\mathrm{c}}(T^*(M/B))$ such that $I_1(\beta) = \alpha$, which establishes the exactness at $\mathrm{K_c}(T^*(D/B))$.

To prove the exactness at $\mathrm{K}^1_{\mathrm{c}}(T^*(D/B))$, one can proceed in a similar way. Indeed, the diagram

$$\begin{array}{ccccc} \mathrm{K}^0_{\mathrm{c}}(T^*(M/B)) & \xrightarrow{I_0} & \mathrm{K}^{-1}_{\mathrm{c}}(T^*(D/B)) & & \\ \downarrow b & & \downarrow b & \searrow^{i_1} & \\ \mathrm{K}^{-2}_{\mathrm{c}}(T^*(M/B)) & \xrightarrow{I_{-2}} & \mathrm{K}^{-3}_{\mathrm{c}}(T^*(D/B)) & \xrightarrow{j} & \mathrm{K}^1_{\Phi\text{-cu}}(\phi) \end{array} \tag{6.10}$$

commutes, where the vertical arrows are given by Bott periodicity and $I_{-2} = \mathrm{ind_{AS}}\, r^*_{\partial M}$ in terms of the families index map

$$\mathrm{ind_{AS}} : \mathrm{K}^{-2}_{\mathrm{c}}(T^*_{\partial M}(M/B)) \to \mathrm{K}^{-2}_{\mathrm{c}}(T^*(D/B) \times \mathbb{R}) \cong \mathrm{K}^{-3}_{\mathrm{c}}(T^*(D/B))$$

and j is the map (6.9) obtained using the identification (when $\dim D = \dim \partial M$, this is only true after stabilization)

$$\mathrm{K}^{-3}_{\mathrm{c}}(T^*(D/B)) \cong \pi_0(G^{-\infty}_{\Phi s(2)}(\partial M/B)). \tag{6.11}$$

Notice that as opposed to (6.8), no Bott periodicity is involved in (6.11), hence the right triangle in (6.10) is commutative. The fact that the left square is commutative follows from the commutativity of the families index with Bott periodicity (and more generally with the Thom isomorphism).

The exactness of the bottom row of (6.10) can be proved by applying the proof of the exactness at $\mathrm{K_c}(T^*(D/B))$ (really $\mathrm{K}^{-2}_{\mathrm{c}}(T^*(D/B))$) with one extra suspension parameter. Since the Bott periodicity maps in (6.10) are isomorphisms, this implies that

$$\mathrm{K_c}(T^*(M/B)) \xrightarrow{I_0} \mathrm{K}^{-1}_{\mathrm{c}}(T^*(D/B)) \xrightarrow{i_1} \mathrm{K}^1_{\Phi\text{-cu}}(\phi)$$

is also exact in the middle. □

In the scattering case (when stabilization is necessary in the arguments above), using Lemma 2.1, the 6-term exact sequence (6.7) reduces to the exact sequence in K-theory associated to the inclusion of the boundary of $T^*(M/B)$

(6.12)

$$\begin{array}{ccccc} \mathrm{K}^1_{\mathrm{c}}(T^*_{\partial M}(M/B)) & \longrightarrow & \mathrm{K_c}(T^*(M/B), T^*_{\partial M}(M/B)) & \longrightarrow & \mathrm{K_c}(T^*(M/B)) \\ \uparrow & & & & \downarrow \\ \mathrm{K}^1_{\mathrm{c}}(T^*(M/B)) & \longleftarrow & \mathrm{K}^1_{\mathrm{c}}(T^*(M/B), T^*_{\partial M}(M/B)) & \longleftarrow & \mathrm{K_c}(T^*_{\partial M}(M/B)). \end{array}$$

7. Poincaré duality, the general case

Consider the short exact sequence of C^* algebras (23) and the associated 6-term exact sequence

(7.1)
$$\begin{array}{ccccc} \mathrm{KK}^0_B(\mathcal{C}(D),\mathcal{C}(B)) & \xrightarrow{\iota^*} & \mathrm{KK}^0_B(\mathcal{C}_\Phi(M),\mathcal{C}(B)) & \xrightarrow{s^*} & \mathrm{KK}^0_B(\mathcal{C}_0(M),\mathcal{C}(B)) \\ \delta\uparrow & & & & \downarrow\delta \\ \mathrm{KK}^1_B(\mathcal{C}_0(M),\mathcal{C}(B)) & \xleftarrow[s^*]{} & \mathrm{KK}^1_B(\mathcal{C}_\Phi(M,\mathcal{C}(B)) & \xleftarrow[\iota^*]{} & \mathrm{KK}^1_B(\mathcal{C}(D),\mathcal{C}(B)). \end{array}$$

Each term of this sequence can be related to the corresponding term in (6.7) via a quantization map, namely the quantization maps of Section 4

$$\text{quan} : \mathrm{K}^{\mathrm{k}}_{\Phi\text{-}\mathrm{cu}}(\phi) \longrightarrow \mathrm{KK}^k_B(\mathcal{C}_\Phi(M),\mathcal{C}(B)),\ k \in \mathbb{Z}_2, \tag{7.2}$$

and the quantization maps of [25] and of Atiyah and Singer,

$$\text{quan}_r : \mathrm{K}^{\mathrm{k}}_{\mathrm{c}}(T^*(M/B)) \longrightarrow \mathrm{KK}^k_B(\mathcal{C}_0(M),\mathcal{C}(B)), \tag{7.3}$$

$$\text{quan}_{-\infty} : \mathrm{K}^{\mathrm{k}}_{\mathrm{c}}(T^*(D/B)) \longrightarrow \mathrm{KK}^k_B(\mathcal{C}(D),\mathcal{C}(B)). \tag{7.4}$$

Theorem 7.1. *The quantization maps* quan, quan_r, *and* $\text{quan}_{-\infty}$ *are isomorphisms giving a commutative diagram*

$$\begin{array}{ccccc} \cdots\mathrm{K}_c(T^*(D/B)) & \xrightarrow{i} & \mathrm{K}_{\Phi\text{-}\mathrm{cu}}(\phi) & \xrightarrow{\sigma} & \mathrm{K}_c(T^*(M/B))\cdots \\ \downarrow \text{quan}_{-\infty} & & \downarrow \text{quan} & & \downarrow \text{quan}_r \\ \cdots\mathrm{KK}^0_B(\mathcal{C}(D),\mathcal{C}(B)) & \xrightarrow{\iota^*} & \mathrm{KK}^0_B(\mathcal{C}_\Phi(M),\mathcal{C}(B)) & \xrightarrow{s^*} & \mathrm{KK}^0_B(\mathcal{C}_0(M),\mathcal{C}(B))\cdots \end{array}$$

between the 6-term exact sequences (6.7) *and* (7.1). *In particular, this implies the Poincaré duality of Theorem 2.*

The fact that quan_r and $\text{quan}_{-\infty}$ are isomorphisms follows from the Poincaré duality result of Kasparov [17] and its extensions by the first author and Piazza in [25]. Then, given the commutativity of the diagram between the two 6-term exact sequence, the fact that quan is an isomorphism follows from the fives lemma. So it remains to check that the diagram commutes.

For $k \in \mathbb{Z}_2$, the commutativity of

(7.5)
$$\begin{array}{ccc} \mathrm{K}^{\mathrm{k}}_{\Phi\text{-}\mathrm{cu}}(\phi) & \xrightarrow{\sigma} & \mathrm{K}^{\mathrm{k}}_{\mathrm{c}}(T^*(M/B)) \\ \downarrow \text{quan} & & \downarrow \text{quan}_r \\ \mathrm{KK}^k_B(\mathcal{C}_\Phi(M),\mathcal{C}(B)) & \xrightarrow{s^*} & \mathrm{KK}^k_B(\mathcal{C}_0(M),\mathcal{C}(B)) \end{array}$$

is clear from the definition of quan and quan_r. The proof of the commutativity of the four other squares of the diagram is more involved and will occupy the remainder of this section.

Let us first consider the commutativity of

(7.6)
$$\begin{array}{ccc} \mathrm{K}^{\mathrm{k}}_{\mathrm{c}}(T^*(D/B)) & \xrightarrow{i} & \mathrm{K}^{\mathrm{k}}_{\Phi\text{-}\mathrm{cu}}(\phi) \\ \downarrow \text{quan}_{-\infty} & & \downarrow \text{quan} \\ \mathrm{KK}^k_B(\mathcal{C}(D),\mathcal{C}(B)) & \xrightarrow{\iota^*} & \mathrm{KK}^k_B(\mathcal{C}_\Phi(M),\mathcal{C}(B)). \end{array}$$

We will only provide a proof in the even case since the odd case is similar. The first step is to describe the quantization map $\mathrm{quan}_{-\infty}$ in terms of indicial families instead of symbols. Given a class $\alpha \in \mathrm{K_c}(T^*(D/B))$, let $p \in G^{-\infty}_{\Phi s}(\partial M/B; E_+)$ be a representative of this class. Consider the manifold

$$M_c = \partial M \times [0,1]$$

which can be seen as a collar neighborhood ∂M in M. Consider the associated fibration

$$\phi_c = \partial\phi \circ \pi_1 : M_c \longrightarrow B, \tag{7.7}$$

where $\pi_1 : \partial M \times [0,1] \longrightarrow \partial M$ is the projection on the first factor. Since the boundary of M_c has two parts $\partial M_0 = \partial M \times \{0\}$ and $\partial M_1 = \partial M \times \{1\}$, let

$$\Phi_i : \partial M_i \longrightarrow D_i, \ i \in \{0,1\},$$

denote the two boundary fibration maps and let

$$\Phi_c : \partial M_c \longrightarrow D_0 \cup D_1$$

be the total boundary map. Let $P \in (\mathrm{Id} + \Psi^{-\infty}_{\Phi_c\text{-}\mathrm{cu}}(M_c; E_+))$ be a Fredholm operator with indicial family the identity at ∂M_0 and by p at ∂M_1. Then Lemma 4.1 gives an associated KK-class

$$[P] \in \mathrm{KK}^0_B(\mathcal{C}_{\Phi_c}(M_c), \mathcal{C}(B))$$

and the pull-back map

$$d^* : \mathrm{KK}^0_B(\mathcal{C}_{\Phi_c}(M_c), \mathcal{C}(B)) \longrightarrow \mathrm{KK}^0_B(\mathcal{C}(D), \mathcal{C}(B))$$

where $d : \mathcal{C}(D) \hookrightarrow \mathcal{C}_{\Phi_c}(M_c)$ is the obvious inclusion, gives a KK-class

$$d^*[P] \in \mathrm{KK}^0_B(\mathcal{C}(D), \mathcal{C}(B)).$$

Thus, this procedure gives a well-defined quantization map

$$\mathrm{quan}'_{-\infty} : \mathrm{K_c}(T^*(\partial M/D)) \longrightarrow \mathrm{KK}^0_B(\mathcal{C}(D), \mathcal{C}(B)).$$

Lemma 7.2. $\mathrm{quan}'_{-\infty} = \mathrm{quan}_{-\infty}$.

Proof. Let $\alpha \in \mathrm{K_c}(T^*(\partial M/D))$ be given. Following the discussion in §3 we are reduced to the scattering case, so

$$M_c = D \times [0,1]$$

in our construction. Let p be an indicial family representing the K-class α and let P be as above. Let $P_t \in \Psi^0_{\Phi_c\text{-}\mathrm{cu}}(M_c/B; E_+)$, $t \in [0,1]$ be a homotopy through families of fully elliptic operators such that $P_0 = P$ and P_1 has trivial indicial families both at ∂M_0 and ∂M_1 and so with symbol σ having K-class

$$[\sigma] \in \mathrm{K_c}(T^*(M_c/B), T^*_{\partial M_c}(M_c/B)) \cong \mathrm{K_c}(T^*(D \times (0,1)/B))$$

identifying via Bott periodicity with α (see [32] for details). That such a homotopy exists is discussed in [23] and §2. By considering a family of positive definite approximate inverse square roots for P_t, we construct an operator homotopy $(\mathcal{H}, \mu, \mathcal{F}_t)$ of Kasparov modules, which means that P_0 and P_1 define the same element in $\mathrm{KK}^0_B(\mathcal{C}(D), \mathcal{C}(B))$.

On the other hand, let us quantize partially the symbol σ of P_1 in the $T^*[0,1]$ direction to get a family of elliptic operators $\widehat{p}_1$ parameterized by $T^*(D/B)$ and acting on the Hilbert bundle

$$\mathrm{L}^2(M_c/D; E_+) \longrightarrow T^*(D/B)$$

with typical fibre $\mathrm{L}^2([0,1];E_+)$. As discussed in [19], we can interpret

$$\widehat{p}_1 : \mathrm{L}^2(M_c/D;E_+) \longrightarrow \mathrm{L}^2(M_c/D;E_+)$$

as a symbol on a Hilbert bundle. The notion of ellipticity leads to very restrictive conditions in this context. But according to example 1.7 in [19], $\widehat{p}_1$ is elliptic and once it is quantized we get back P_1 modulo compact operators. Assume without loss of generality that σ (and $\widehat{p}_1$) is homogeneous of degree zero in the fibres of $T^*(D/B) \longrightarrow D$ outside a compact neighborhood of the zero section. Assume also without loss of generality that E_+ is a trivial bundle on M_c. Then as discussed in [19], it is possible to deform $\widehat{p}_1$ through a homotopy $\widehat{p}_t$, $t \in [1,2]$ of elliptic symbols so that $\widehat{p}_2$ takes the form

$$\widehat{p}_2 = \widehat{p}_2' \oplus \widehat{p}_2'' : V \oplus V^\perp \longrightarrow \widehat{p}_1 V \oplus (\widehat{p}_1 V)^\perp$$

where V is a sub-bundle of $\mathrm{L}^2(M_c/D;E)$ of finite corank on which $\widehat{p}_1$ is injective and where $\widehat{p}_2'$ is invertible and constant along the fibres of $T^*(D/B) \longrightarrow D$. Thus, when we quantize $\widehat{p}_2$ we get an operator of the form

$$P_2 = P_2' \oplus P_2'' : \mathrm{L}^2(D;V) \oplus \mathrm{L}^2(D;V^\perp) \longrightarrow \mathrm{L}^2(D;\widehat{p}_1 V) \oplus \mathrm{L}^2(D;(\widehat{p}_1 V)^\perp)$$

with P_2' invertible. Using positive definite approximate inverse square roots of $(P_2')^* P_2'$ and $(P_2'')^* P_2''$, we can associate a KK-class $\beta \in \mathrm{KK}^0_B(\mathcal{C}(D),\mathcal{C}(B))$. By homotopy invariance, P_1 gives the same K-class so $\beta = \mathrm{quan}'_{-\infty}(\alpha)$. On the other hand, the Kasparov module coming from P_2' is degenerate, so β can be defined using P_2''. The K-class associated to the symbol of P_2'' is the families index of $\widehat{p}_1$ (cf. [19]) which by the families index of Atiyah-Singer should be precisely $\alpha \in \mathrm{K_c}(T^*(\partial M/D))$. This means that

$$\mathrm{quan}'_{-\infty}(\alpha) = \beta = \mathrm{quan}_{-\infty}(\alpha).$$

□

Thus we can use indicial families instead of symbols to define the quantization map $\mathrm{quan}_{-\infty}$. Consider again the manifold M_c. Since the boundary of M_c has two disconnected parts (which can themselves be disconnected), we can consider, instead of $\mathrm{K}_{\Phi_c\text{-}\mathrm{cu}}(\phi_c)$, the group $\mathrm{K}_{\Phi_1\text{-}\mathrm{cu}}(\phi_c)$ of stabilized homotopy classes of invertible joint symbols with indicial family given by the identity at the boundary face ∂M_0. If we define $\mathcal{C}_{\Phi_1}(M_c)$ to be the C^* algebra of continuous functions

$$\mathcal{C}_{\Phi_1}(M_c) = \left\{ f \in \mathcal{C}(M_c); f\big|_{\partial M_1} = \Phi_1^* g \text{ for some } g \in \mathcal{C}(D_1) \right\}, \tag{7.8}$$

then a quantization map

$$\mathrm{quan}_c : \mathrm{K}_{\Phi_1\text{-}\mathrm{cu}}(\phi_c) \longrightarrow \mathrm{KK}^0_B(\mathcal{C}_{\Phi_1}(M_c),\mathcal{C}(B))$$

can be defined as follows. Given a joint symbol (σ, N) representing a class in $\alpha \in \mathrm{K}_{\Phi_1\text{-}\mathrm{cu}}(\phi_c)$, one considers a family of fibred cusp operators P of order zero with joint symbol given by (σ, N). Deforming σ if needed, we can assume P acts as the identity in a small collar neighborhood of ∂M_0. Then in the usual fashion, one can construct a Kasparov module in $\mathrm{KK}^0_B(\mathcal{C}_{\Phi_1}(M_c),\mathcal{C}(B))$ by considering a positive definite approximate inverse square root to P which acts as the identity in a collar neighborhood of ∂M_0. As in the definition of quan, one can check that the associate KK-class only depends on $\alpha \in \mathrm{K}_{\Phi_1\text{-}\mathrm{cu}}(\phi_c)$.

There is also a natural map

$$i_c : \mathrm{K_c}(T^*(\partial M/D)) \longrightarrow \mathrm{K}_{\Phi_1\text{-}\mathrm{cu}}(\phi_c) \tag{7.9}$$

which to $\alpha \in \mathrm{K_c}(T^*(\partial M/D))$ associates the class $i_c(\alpha)$ represented by any joint symbol with trivial indicial family on ∂M_0, indicial family with K-class given by α on ∂M_1 and with trivial symbol.

Lemma 7.3. *The diagram*

$$\begin{array}{ccc} \mathrm{K}_c(T^*(D/B)) & \xrightarrow{i_c} & \mathrm{K}_{\Phi_1\text{-}\mathrm{cu}}(\phi_c) \\ \big\downarrow{\scriptstyle \mathrm{quan}_{-\infty}} & & \big\downarrow{\scriptstyle \mathrm{quan}_c} \\ \mathrm{KK}^0_B(\mathcal{C}(D),\mathcal{C}(B)) & \xrightarrow{\iota_c^*} & \mathrm{KK}^0_B(\mathcal{C}_{\Phi_1}(M_c),\mathcal{C}(B)) \end{array}$$

is commutative, where $\iota_c : \mathcal{C}_{\Phi_1}(M_c) \longrightarrow \mathcal{C}(D)$ *is the restriction to* ∂M_1.

Proof. Let $\alpha \in \mathrm{K_c}(T^*D/B)$ be given. Let $e : \mathcal{C}(D) \hookrightarrow \mathcal{C}_{\Phi_1}(M_c)$ be the obvious injective map of C^* algebras so that $\iota_c \circ e = \mathrm{Id}$. Then

$$\iota_c^* \,\mathrm{quan}_{-\infty}(\alpha) = \iota_c^* e^* \,\mathrm{quan}_c \circ i_c(\alpha). \tag{7.10}$$

Consider the C^* algebra

$$\mathcal{C}_1(M_c) = \left\{ f \in \mathcal{C}(M_c); f\big|_{\partial M_1} = 0 \right\}.$$

From the commutativity of the diagram

$$\begin{array}{ccc} \mathrm{K}_{\Phi_1\text{-}\mathrm{cu}}(\phi_c) & \xrightarrow{\sigma} & \mathrm{K_c}(T^*(M_c/B)) \\ \big\downarrow{\scriptstyle \mathrm{quan}_c} & & \big\downarrow{\scriptstyle \mathrm{quan}_r} \\ \mathrm{KK}^0_B(\mathcal{C}_{\Phi_1}(M_c),\mathcal{C}(B)) & \longrightarrow & \mathrm{KK}^0_B(\mathcal{C}_1(M_c),\mathcal{C}(B)) \end{array} \tag{7.11}$$

and the exactness in the middle of

$$\mathrm{K_c}(T^*(D/B)) \xrightarrow{i_c} \mathrm{K}_{\Phi_1\text{-}\mathrm{cu}}(\phi_c) \xrightarrow{\sigma} \mathrm{K_c}(T^*(M_c/B)), \tag{7.12}$$

$$\mathrm{KK}^0_B(\mathcal{C}(D),\mathcal{C}(B)) \xrightarrow{\iota_c} \mathrm{KK}^0_B(\mathcal{C}_{\Phi_1}(M_c),\mathcal{C}(B)) \longrightarrow \mathrm{KK}^0_B(\mathcal{C}_1(M_c),\mathcal{C}(B)), \tag{7.13}$$

we deduce that there exists $\beta \in \mathrm{KK}^0_B(\mathcal{C}(D),\mathcal{C}(B))$ such that $\iota_c^*\beta = \mathrm{quan}_c \circ i_c(\alpha)$. Since $\iota_c \circ e = \mathrm{Id}$,

$$\beta = e^*\iota_c^*\beta = e^* \,\mathrm{quan}_c \circ i_c(\alpha),$$

which implies by (7.10) that

$$\mathrm{quan}_c \circ i_c(\alpha) = \iota_c^*\beta = \iota_c^*(e^* \,\mathrm{quan}_c \circ i_c(\alpha)) = \iota_c^* \,\mathrm{quan}_{-\infty}(\alpha).$$

□

Lemma 7.4. *For* $k \in \mathbb{Z}_2$*, the diagram* (7.6) *is commutative.*

Proof. As noted above, we will only provide a proof for the case $k = 0$, the case $k = 1$ being similar. Think of M_c as a collar neighborhood of M where ∂M_1 is identified with ∂M. Let

$$j : \mathcal{C}_\Phi M) \longrightarrow \mathcal{C}_{\Phi_1}(M_c)$$

be the restriction map on $M_c \subset M$. Then consider the diagram

$$\begin{array}{ccccc} \mathrm{K_c}(T^*(D/B)) & \xrightarrow{i_c} & \mathrm{K}_{\Phi_1\text{-}\mathrm{cu}}(\phi_c) & \xrightarrow{e_c} & \mathrm{K}_{\Phi\text{-}\mathrm{cu}}(\phi) \\ \big\downarrow{\scriptstyle \mathrm{quan}_{-\infty}} & & \big\downarrow{\scriptstyle \mathrm{quan}_c} & & \big\downarrow{\scriptstyle \mathrm{quan}} \\ \mathrm{KK}^0_B(\mathcal{C}(D),\mathcal{C}(B)) & \xrightarrow{\iota_c^*} & \mathrm{KK}^0_B(\mathcal{C}_{\Phi_1}(M_c),\mathcal{C}(B)) & \xrightarrow{j^*} & \mathrm{KK}^0_B(\mathcal{C}_\Phi M),\mathcal{C}(B)). \end{array} \tag{7.14}$$

where e_c is the extension of a representative in $\mathrm{K}_{\Phi_1\text{-}\mathrm{cu}}(\phi_c)$ by the identity inside M. Clearly, $e_c i_c = i$ and $j^*\iota_c^* = \iota^*$, so the lemma will be proven provided we show the diagram (7.14) is commutative. By the previous lemma, we only need to show that the right square is commutative. Consider the two graded Hilbert modules over $\mathcal{C}(B)$

$$\mathcal{H}_1 = \mathrm{L}^2(M_c;\mathbb{E}) = \mathrm{L}^2(M_c;E_+)\oplus \mathrm{L}^2(M_c;E_-) \tag{7.15}$$

$$\mathcal{H}_2 = \mathrm{L}^2(\overline{M\setminus M_c};\mathbb{E}) = \mathrm{L}^2(\overline{M\setminus M_c};E_+)\oplus \mathrm{L}^2(\overline{M\setminus M_c};E_-). \tag{7.16}$$

Then given $\alpha\in \mathrm{K}_{\Phi_1\text{-}\mathrm{cu}}(\phi_c)$, we can represent $j^*\operatorname{quan}_c(\alpha)$ by a Kasparov module of the form $(\mathcal{H}_1, j^*\mu,\mathcal{F}_1)$ and at the same time $\operatorname{quan} e_c(\alpha)$ can be represented by a Kasparov module of the form $(\mathcal{H}_1\oplus\mathcal{H}_2, j^*\mu\oplus l^*\nu, \mathcal{F}_1\oplus\mathcal{F}_2)$ where

$$l:\mathcal{C}_\Phi M)\longrightarrow \mathcal{C}(\overline{M\setminus M_c})$$

is the restriction map,

$$\mu:\mathcal{C}_{\Phi_1}(M_c)\longrightarrow \mathcal{B}(\mathcal{H}_1) \tag{7.17}$$

$$\nu:\mathcal{C}(\overline{M\setminus M_c})\longleftrightarrow \mathcal{B}(\mathcal{H}_2) \tag{7.18}$$

are the obvious actions given by multiplication and

$$\mathcal{F}_2 = \begin{pmatrix} 0 & \mathrm{Id}\\ \mathrm{Id} & 0\end{pmatrix}.$$

Since $(\mathcal{H}_2,\nu,\mathcal{F}_2)$ is a degenerate Kasparov module,

$$j^*\operatorname{quan}_c(\alpha) = \operatorname{quan} e_c(\alpha).$$

□

Finally we need to show that the diagram

$$\begin{array}{ccc} \mathrm{K}_{\mathrm{c}}^{\mathrm{k}}(T^*(M/B)) & \xrightarrow{\ I\ } & \mathrm{K}_{\mathrm{c}}^{\mathrm{k}\text{-}1}(T^*(D/B)) \\ \Big\downarrow{\scriptstyle \operatorname{quan}_r} & & \Big\downarrow{\scriptstyle \operatorname{quan}_{-\infty}} \\ \mathrm{KK}_B^k(\mathcal{C}_0(M),\mathcal{C}(B)) & \xrightarrow{\ \delta\ } & \mathrm{KK}_B^{1-k}(\mathcal{C}(D),\mathcal{C}(B)) \end{array} \tag{7.19}$$

is commutative for $k\in\mathbb{Z}_2$, which requires an understanding of the boundary homomorphism δ. Let us again limit our attention to the even case $k=0$, the case $k=1$ being similar. Recall that in topological K-theory (see for instance [2]), the boundary homomorphism of the 6-term exact sequence associated to a pair of spaces (X,Y) is defined via the cone space $X\cup CY$ which is obtained from X and

$$CY = Y\times[0,1]/Y\times\{1\}$$

by identifying $Y\subset X$ with $Y\times\{0\}\subset CY$. There is an exact sequence

$$\mathrm{K}(X\cup CY,X)\xrightarrow{m^*}\widetilde{\mathrm{K}}(X\cup CY)\xrightarrow{k^*}\widetilde{\mathrm{K}}(X).$$

Under the identifications

$$\widetilde{\mathrm{K}}^{-1}(Y)\cong \mathrm{K}(X\cup CY,X),\ \mathrm{K}(X,Y)\cong\widetilde{\mathrm{K}}(X\cup CY),$$

this becomes

$$\widetilde{\mathrm{K}}^{-1}(Y)\xrightarrow{\delta}\mathrm{K}(X,Y)\longrightarrow\widetilde{\mathrm{K}}(X)$$

where δ is the boundary homomorphism of the 6-term exact sequence associated to the pair (X,Y).

In KK-theory, one can define the boundary homorphism in a similar way, introducing the mapping cone

$$\mathcal{C}_\iota = \{(x,f) \in \mathcal{C}_\Phi(M) \oplus \mathcal{C}_0([0,1) \times D);\ \iota(x) = f(0)\}\,, \tag{7.20}$$

where

$$\mathcal{C}_0([0,1) \times D) = \left\{ f \in \mathcal{C}([0,1] \times D); f\big|_{\{1\}\times D} = 0 \right\}$$

is the C^* closure of $\mathcal{C}_c([0,1) \times D)$. There is a natural inclusion

$$e : \mathcal{C}_0(M) \ni x \longmapsto (x,0) \in \mathcal{C}_\iota \tag{7.21}$$

which induces a map

$$e^* : \mathrm{KK}^0_B(\mathcal{C}_\iota, \mathcal{C}(B)) \longrightarrow \mathrm{KK}^0_B(\mathcal{C}_0(M), \mathcal{C}(B)). \tag{7.22}$$

This map is an isomorphism as can be seen by interpreting the map e^* as multiplication (on the left) by $[e] \in \mathrm{KK}^0_B(\mathcal{C}_0(M), \mathcal{C}_\iota)$ associated to the C^* homomorphism e (see[9]). Then $[e]$ is a KK-equivalence between $\mathcal{C}_0(M)$ and $\mathcal{C}_\iota$, which is to say that it has an inverse $u \in \mathrm{KK}^0_B(\mathcal{C}_\iota, \mathcal{C}_0(M))$ with respect to the Kasparov product.

Given this one can then define the boundary homomorphism as

$$\delta = j^*(e^*)^{-1} : \mathrm{KK}_B(\mathcal{C}_0(M), \mathcal{C}(B)) \longrightarrow \mathrm{KK}_B(\mathcal{S}\mathcal{C}(D), \mathcal{C}(B)) \cong \mathrm{KK}^1_B(\mathcal{C}(D), \mathcal{C}(B)) \tag{7.23}$$

where $j : \mathcal{S}\mathcal{C}(D) \hookrightarrow \mathcal{C}_\iota$ is the natural inclusion. In terms of Kasparov product, δ is multiplication on the left by $j^*u \in \mathrm{KK}_B(\mathcal{S}\mathcal{C}(D), \mathcal{C}_0(M))$.

In the present context, it is possible to give an alternative definition of the mapping cone which is especially useful. Consider as before a collar neighborhood $M_c = \partial M \times [0,1]$ of ∂M with ∂M identifies with $\partial M_1 = \partial M \times \{1\}$. Then

$$\mathcal{C}_\iota \cong \left\{ f \in \mathcal{C}_0(M); f\big|_{M_c} = (\Phi \times \mathrm{Id})^* g \text{ for some } g \in \mathcal{C}_0(D \times [0,1)) \right\} \tag{7.24}$$

by identifying $M' = \overline{M/M_c}$ with M. The isomorphism (7.24) gives

$$\varepsilon : \mathcal{C}_\iota \hookrightarrow \mathcal{C}_0(M). \tag{7.25}$$

The map e in (7.21) is then described by

$$e : \mathcal{C}_0(M) \xrightarrow{\sim} \mathcal{C}_0(M') \hookrightarrow \mathcal{C}_\iota.$$

Lemma 7.5. *The boundary homomorphism δ of the 6-term exact sequence* (7.1) *is given by the pull-back map*

$$j^*\varepsilon^* : \mathrm{KK}^0_B(\mathcal{C}_0(M), \mathcal{C}(B)) \longrightarrow \mathrm{KK}^0_B(\mathcal{S}\mathcal{C}(D), \mathcal{C}(B)).$$

Proof. If σ is a symbol on $T^*(M/B)$ then $e^*\varepsilon^* \operatorname{quan}_r(\sigma)$ can be obtained by quantizing the symbol

$$\sigma' = \sigma\big|_{T^*(M'/B)}$$

on $T^*(M'/B)$ and making the identification $\mathcal{C}_0(M') \cong \mathcal{C}_0(M)$. Since σ' and σ are homotopic when M' and M are identified, they correspond to the same K-class. Since σ was arbitrary, this shows $\operatorname{quan}_r = e^*\varepsilon^* \operatorname{quan}_r$. Consequently,

$$(e^*)^{-1} \operatorname{quan}_r = (e^{-*})^{-1} e^*\varepsilon^* \operatorname{quan}_r = \varepsilon^* \operatorname{quan}_r.$$

Since quan_r is an isomorphism, $(e^*)^{-1} = \varepsilon^*$ and it follows that $\delta = j^*(e^*)^{-1} = j^*\varepsilon^*$. □

After identifying the tangent bundle with the cotangent bundle via some metric, The boundary fibration induces a fibration

$$\pi_\Phi : T^*(M_c/B) \longrightarrow T^*(D \times [0,1]/B). \tag{7.26}$$

Let

$$\mathrm{ind}_{\pi_\Phi} : \mathrm{K_c}(T^*(M_c/B)) \longrightarrow \mathrm{K_c}(T^*(D \times [0,1]/B)) \tag{7.27}$$

be the topological index family map associated to this fibration.

Lemma 7.6. *The diagram*

$$\begin{array}{ccccc}
\mathrm{K}_c(T^*(M/B)) & \xrightarrow{\ r^*\ } & \mathrm{K}_c(T^*(M_c/B)) & \xrightarrow{\ \mathrm{ind}_{\pi_\Phi}\ } & \mathrm{K}_c(T^*(D \times [0,1]/B)) \\
\big\downarrow{\scriptstyle \mathrm{quan}_r} & & \big\downarrow{\scriptstyle \mathrm{quan}_r} & & \big\downarrow{\scriptstyle \mathrm{quan}_r} \\
\mathrm{KK}_B(\mathcal{C}_0(M), \mathcal{C}(B)) & \xrightarrow{\ \varepsilon_c^*\ } & \mathrm{KK}_B(\mathcal{C}_0(M_c), \mathcal{C}(B)) & \xrightarrow{\ \varepsilon_d^*\ } & \mathrm{KK}_B(\mathcal{SC}(D), \mathcal{C}(B))
\end{array}$$

is commutative, where

$$\varepsilon_c : \mathcal{C}_0(M_c) \hookrightarrow \mathcal{C}_0(M), \quad \varepsilon_d : \mathcal{SC}(D) \hookrightarrow \mathcal{C}_0(M_c), \quad r : T^*(M_c/B)) \hookrightarrow T^*(M/B),$$

are the natural inclusions.

Proof. The commutativity of the first square is clear. The proof of the commutativity of the second square is essentially a consequence of the Atiyah-Singer families index theorem. Indeed, it allows us to define ind_{π_Φ} as an analytical families index. Then, using quantization of symbols over Hilbert bundles as in [19], one can represent (cf. the proof of Lemma 7.2) $\varepsilon_d^* \, \mathrm{quan}_r(\alpha)$ by a Kasparov module of the form

$$(\mathcal{H}_1 \oplus \mathcal{H}_2, \mu, \mathcal{F}_1 \oplus \mathcal{F}_2)$$

where

$$\mathcal{H}_1 = \mathrm{L}^2(D \times [0,1]; V_+) \oplus \mathrm{L}^2(D \times [0,1]; V_-),$$
$$\mathcal{H}_2 = \mathrm{L}^2(D \times [0,1]; V_+^\perp) \oplus \mathrm{L}^2(D \times [0,1]; V_-^\perp),$$

and $V_\pm$ is a sub-bundle of $\mathrm{L}^2(M_c/(D \times [0,1]); E_\pm)$ with V_+ and V_- of same finite corank. As usual,

$$\mu : \mathcal{C}_0(D \times [0,1]) \longrightarrow \mathcal{B}(\mathcal{H}_i)$$

denotes multiplication. One can do this in such a way that $(\mathcal{H}_1, \mu, \mathcal{F}_1)$ is degenerate and $(\mathcal{H}_2, \mu, \mathcal{F}_2)$ represents $\mathrm{quan}_r \circ \mathrm{ind}_{\pi_\Phi}(\alpha)$, which establishes the commutativity of the left square. □

Lemma 7.7. *We have the identity* $\delta \, \mathrm{quan}_r = \mathrm{quan}_r \, \kappa I_0$ *where*

$$\kappa : \mathrm{K}_c^{-1}(T^*(D/B)) \longrightarrow \mathrm{K}_c(T^*(D \times [0,1]/B))$$

is the canonical isomorphism induced from the homotopy equivalence

$$(T^*([0,1]), \infty) \sim (\mathbb{R}, \infty).$$

Proof. Since clearly $\varepsilon_d^* \varepsilon_c^* = j^* \varepsilon^* = \delta$, we deduce from Lemma 7.6 that

$$\delta \, \mathrm{quan}_r = \varepsilon_d^* \varepsilon_c^* \, \mathrm{quan}_r = \mathrm{quan}_r \, \mathrm{ind}_{\pi_\Phi} \, r^*,$$

so we can conclude from the fact that $\mathrm{ind}_{\pi_\Phi} \circ r^* = \kappa I_0$, which is a consequence of the commutativity of the topological index map with the Thom isomorphism. □

Thus, Lemma 7.7 reduces the proof of the commutativity of (7.19) in the even case to the following statement.

Lemma 7.8. *The diagram*

$$\begin{array}{ccc} \mathrm{K}_c^{-1}(T^*(D/B)) & \xrightarrow{\ \kappa\ } & \mathrm{K}_c(T^*(D\times[0,1]/B)) \\ \Big\downarrow{\scriptstyle \mathrm{quan}_{-\infty}} & & \Big\downarrow{\scriptstyle \mathrm{quan}_r} \\ \mathrm{KK}_B(\mathcal{C}(D),\mathcal{SC}(B)) & \longrightarrow & \mathrm{KK}_B(\mathcal{SC}(D),\mathcal{C}(B)) \end{array}$$

is commutative, where the bottom arrow is Bott periodicity in KK-theory.

Proof. This essentially follows from the family version of proposition 11.2.5 in [13] and the alternative definition of odd quantization using self-adjoint operators. □

In the odd case, a similar argument reduces the proof of the commutativity of (7.19) to the following lemma.

Lemma 7.9. *The diagram*

$$\begin{array}{ccc} \mathrm{K}_c(T^*(D/B)) & \longrightarrow & \mathrm{K}_c(T^*(D\times(0,1)\times[0,1])/[B\times(0,1)])) \\ \Big\downarrow{\scriptstyle \mathrm{quan}_{-\infty}} & & \Big\downarrow{\scriptstyle \mathrm{quan}_r} \\ \mathrm{KK}^0_B(\mathcal{C}(D),\mathcal{C}(B)) & \longrightarrow & \mathrm{KK}^0_B(\mathcal{SC}(D),\mathcal{SC}(B)) \end{array}$$

is commutative, where the top and bottom arrows are Bott periodicity in K-theory and KK-theory respectively.

Proof. This essentially follows from the family version of proposition 11.2.5 in [13] and the alternative definition of odd quantization using self-adjoint operators. □

8. Adiabatic passage to the cusp case

As noted in the introduction, and confirmed by the properties of the 6-term exact sequence above, it is the 'cusp' case amongst the possible fibred cusp structures on a give fibration which is universal.

Proposition 8.1. *For any fibration with fibred cusp structure there is a natural map, given by an adiabatic limit,* $q_{\mathrm{ad}}:\mathrm{K}_{\Phi\text{-}\mathrm{cu}}(\phi)\longrightarrow\mathrm{K}_{\mathrm{cu}}(\phi)$ *into the cusp K-group which commutes with the analytic index and gives a commutative diagram*

(8.1)
$$\begin{array}{ccccc} & & \mathrm{K}_{\Phi\text{-}\mathrm{cu}}(\phi) & & \\ & \swarrow{\scriptstyle \mathrm{ind}_a} & \Big\downarrow{\scriptstyle q_{\mathrm{ad}}} & \searrow{\scriptstyle \sigma} & \\ \mathrm{K}(B) & \xleftarrow[\mathrm{ind}_a]{} & \mathrm{K}_{\mathrm{cu}}(\phi) & \xrightarrow[\sigma]{} & \mathrm{K}_c(T^*(M/B)). \end{array}$$

Proof. Given a family $P\in\Psi^0_{\Phi\text{-}\mathrm{cu}}(M;\mathbb{E})$ we construct an adiabatic family $P(\epsilon)\in\Psi^0_{\Phi\text{-ad,cu}}(M;\mathbb{E})$ which are cusp pseudodifferential operators for $\epsilon>0$ but degenerate, in the adiabatic limit, to the fibred cusp operator P at $\epsilon=0$. However, it is perhaps better to think of the construction as being in the opposite direction. The definition and properties of such adiabatic fibred cusp operators are discussed in Appendix C where they are defined directly through their Schwartz kernels which are defined on the space given by blow-ups in (C.10), in our case with the finer fibration of the boundary being the given one, Φ, and the coarser fibration being $\partial\phi$ corresponding to the cusp structure.

Thus there are three boundary faces of the double space in (C.10) which meet the diagonal (and the kernels are supposed to vanish rapidly at all other boundary faces). At the 'old' face $\epsilon = 0$ we wish to recover the given operator P by the map (C.15); this fixes the kernel precisely on that face. The adiabatic front face can be constructed, as in (C.10), by the last blow-up and it is fibred over the lifted variable

$$\tau = \frac{x-\epsilon}{x+\epsilon} \in [-1,1] \tag{8.2}$$

representing the adiabatic passage, with fibre which is just the front face for the fibred cusp calculus. On the fibre at $\tau = 1$ the kernel is already fixed to be that of P. Thus, we may simply choose to extend the kernel to the whole adiabatic face to be independent of the 'angle' τ. This fixes the kernel of the adiabatic normal operator and also fixes the boundary value, corresponding to $\tau = -1$, of the kernel on the front face for the cusp blow-up as $\epsilon \downarrow 0$. Simply extend it to that face as a conormal distribution with respect to the diagonal; in fact we also choose an extension which is conormal to the diagonal in the interior and consistent with these boundary values (and also the requirement of the vanishing at the 'non-diagonal' boundaries).

This fixes an element $P(\epsilon) \in \Psi^0_{\Phi\text{-ad,cu}}(M;\mathbb{E})$. The choices ensure that in the sense of (C.15)

$$A(P(\epsilon)) = P \text{ and } \operatorname{ad}(P)^{-1} \text{ exists,} \tag{8.3}$$

since the invertibility here comes from the invertibility of the normal operator of P. Now, for $\epsilon \le \delta$, for some $\delta > 0$, the hypotheses of Proposition C.1 apply and show that $P(\delta) \in \Psi^0_{\mathrm{cu}}(M;\mathbb{E})$ is fully elliptic. Thus we can define

$$q_{\mathrm{ad}} : \mathrm{K}_{\Phi\text{-}\mathrm{cu}}(\phi) \ni [P] \longmapsto [P(\delta)] \in \mathrm{K}_{\mathrm{cu}}(\phi) \tag{8.4}$$

provided the independence of choices is shown.

Clearly the image in (8.4) is independent of the choice of δ small and the choices in the construction can all be related by homotopies. Similarly an homotopy of P through fully elliptic fibred cusp operators lifts to a homotopy of $P(\delta)$ in fully elliptic cusp operators and the behaviour under stabilization and composition with bundle isomorphism is appropriate to guarantee that (8.4) is well defined. The construction also ensures that this adiabatic limit commutes with the passage to the symbol, giving commutativity in (8.1). □

Under the Poincaré duality of Section 7, the corresponding adiabatic map in KK-theory is given by pull-back. Let $\iota_{\mathrm{ad}} : \mathcal{C}_{\mathrm{cu}}(M) \hookrightarrow \mathcal{C}_{\Phi}(M)$ be the natural inclusion.

Proposition 8.2. *The diagram*

$$\begin{array}{ccc} \mathrm{K}_{\Phi\text{-}\mathrm{cu}}(\phi) & \xrightarrow{q_{\mathrm{ad}}} & \mathrm{K}_{\mathrm{cu}}(\phi) \\ \downarrow{\scriptstyle \mathrm{quan}} & & \downarrow{\scriptstyle \mathrm{quan}} \\ \mathrm{KK}^0_B(\mathcal{C}_{\Phi}(M), \mathcal{C}(B)) & \xrightarrow{\iota^*_{\mathrm{ad}}} & \mathrm{KK}^0_B(\mathcal{C}_{\mathrm{cu}}(M), \mathcal{C}(B)) \end{array}$$

is commutative.

Proof. If $s : \mathcal{C}_0(M) \hookrightarrow \mathcal{C}_{\mathrm{cu}}(M)$ and $i : \mathcal{C}(B) \hookrightarrow \mathcal{C}_{\mathrm{cu}}(M)$ are the natural inclusions, then from the fact that q_{ad} preserves the index and the homotopy class of the symbol, we see that

$$i^*\iota_{\mathrm{ad}}^* \operatorname{quan} = i^* \operatorname{quan} q_{\mathrm{ad}} \quad \text{and} \quad s^*\iota_{\mathrm{ad}}^* \operatorname{quan} = s^* \operatorname{quan} q_{\mathrm{ad}}.$$

Thus, the result follows from the split short exact sequence of Lemma 5.2. □

9. The extension of fibred cusp structures

The definition of the topological index uses an embedding of a given family of cusp structures into a product setting. In fact we show how to embed a general fibred cusp structure (since this can also be used to define the index without using Proposition 8.1). We shall use as 'model' structure (for a given base manifold B) the products

$$\begin{aligned} \widetilde{M} &= B \times \mathbb{B}^{p+1},\ p > 0,\ \text{for cusp structures and} \\ \widetilde{M} &= B \times \mathbb{B}^{p+1} \times \mathbb{S}^q,\ p, q > 0 \text{ for fibred cusp structures.} \end{aligned} \tag{9.1}$$

The model fibration, being just projection onto the first factor will be written

$$\pi : \widetilde{M} \longrightarrow B \tag{9.2}$$

and in the case of fibred cusp structures the model boundary fibration is the projection onto the first two factors (restricted to the boundary)

$$\pi' : \partial\widetilde{M} \longrightarrow \widetilde{D} = B \times \mathbb{S}^p. \tag{9.3}$$

Definition 9.1. A fibration with fibred cusp structure $\tilde{\phi} : \widetilde{M} \longrightarrow B$ is an *extension* of a given fibration with fibred cusp structure $\phi : M \longrightarrow B$ if $i : M \hookrightarrow \widetilde{M}$ and $j : D \hookrightarrow \widetilde{D}$ have the following properties

(1) i embeds M as a 'product type' submanifold in the sense that it is an embedding and

$$i(\partial M) = \partial\widetilde{M} \cap i(M) \tag{9.4}$$

and the pull-back under i of a defining function for $\partial\widetilde{M}$ is a defining function for ∂M.

(2) $\phi = \pi \circ i$.

(3) $j \circ \Phi = \pi' \circ i$.

Notice that the inclusion j is completly specified by the inclusion i.

Proposition 9.1. *For any family of fibred cusp structures as in* (21) *there is an extension*

$$i : M \longrightarrow \widetilde{M} = B \times \mathbb{B}^{p+1} \times \mathbb{S}^q \tag{9.5}$$

provided that p and q are large enough. In the case of cusp structures there is an extension $i : M \longrightarrow B \times \mathbb{B}^{p+1}$, for p sufficiently large.

Proof. First consider the cusp case. By Whitney's Embedding Theorem, we may embed M in Euclidean space of sufficiently large dimension

$$i' : M \longrightarrow \mathbb{R}^k. \tag{9.6}$$

To replace this by an embedding of the desired product type into the interior of a closed ball, consider the embedding

$$(x, i') : M \longrightarrow [0,1) \times \mathbb{R}^k, \tag{9.7}$$

where $x \in \mathcal{C}^\infty M)$ is the boundary defining function (it may be assumed without loss of generality that $0 \le x \le 1$.) But $[0,1) \times \mathbb{R}^k$ can be identified, using a smooth map e, with an open neighbourhood of a piece of the boundary of $\mathbb{B}^{p+1}$, $p = k$. It follows directly that this is an embedding of product type in the sense of (1). Then let

$$i = (\phi, e(x, i')) : M \longrightarrow B \times \mathbb{B}^{p+1} \tag{9.8}$$

be the product of this map with M as a map into $B \times \mathbb{B}^{p+1}$. This is still an embedding with property (1) and has the property (2) (and so (3) as well).

In the case of a fibred cusp structure, we can start with the constructions above for the embedding of the underlying cusp structure. Now, take an additional embedding of D into a Euclidean space

$$d : D \hookrightarrow \mathbb{R}^k$$

for convenience again $\mathbb{R}^k$ by increasing k if necessary. Using a product structure near the boundary this fibration can be extended inwards near the boundary to a fibration over $[0, \epsilon)_x \times D$ and we can consider the smooth map

$$(\mathrm{Id}, d \circ \Phi) : [0, \epsilon)_x \times \partial M \longrightarrow [0, \epsilon) \times \mathbb{R}^k. \tag{9.9}$$

Taking $0 < \epsilon < 1$ small enough and using radial retraction on $\mathbb{R}^k$, this can be extended to a smooth map

$$(x, d') : M \longrightarrow [0,1) \times \mathbb{R}^k.$$

For the overall embedding we can then take

$$i = (\phi, e'(x, d'), e'' \circ i') : M \longrightarrow B \times \mathbb{B}^{p+1} \times \mathbb{S}^q, \quad p = q = k, \tag{9.10}$$

with

$$e' : [0,1) \times \mathbb{R}^k \times \mathbb{R}^k \hookrightarrow \mathbb{B}^{p+1},\ p = k,\ q = k$$
$$e" : \mathbb{R}^k \hookrightarrow \mathbb{S}^q,$$

being identifications of open sets. We can also take $j : D \hookrightarrow B \times \mathbb{S}^p$ to be

$$j = (\phi, e''(0, d')).$$

Since i' is an embedding, this is also and satisfies (1) – only the boundary is mapped into the boundary, since x is a boundary defining function of M. Now, (2) holds for the same reason as before and (3) follows from the definition of j. □

10. Multiplicativity

Recall one form of the 'multiplicativity property' for the index of pseudodifferential operators as introduced by Atiyah and Singer. Consider an iterated fibration, for the moment of compact manifolds without boundary but later allowing Z to have a boundary

$$
\begin{array}{ccc}
F & \longrightarrow & M' \\
 & & \big\downarrow \phi' \\
Z & \longrightarrow & M \\
 & & \big\downarrow \phi \\
 & & B.
\end{array}
\qquad \partial F = \emptyset \tag{10.1}
$$

Taking a connection on ϕ' allows the fibre cotangent bundle $T^*(M'/M)$ to be identified (naturally up to homotopy) as a subbundle of $T^*(M'/B)$ which then splits

$$
T^*(M'/B) = T^*(M'/M) \oplus (\phi')^* T^*(M/B). \tag{10.2}
$$

Choose cutoff functions χ_1 and χ_2 which are homogeneous of degree 0, smooth outside the zero section and are respectively supported outside the two summands and equal to one in a conic neighbourhood of the other with $\chi_1 + \chi_2 = 1$.

Let $A \in \Psi^0(M/B;\mathbb{E})$ and $B \in \Psi^0(M'/M;\mathbb{G})$ be families of elliptic pseudodifferential, then the matrix

$$
\begin{pmatrix} \chi_1\sigma(B) & -\chi_2\sigma(A)^* \\ \chi_2\sigma(A) & \chi_1\sigma(B) \end{pmatrix} \tag{10.3}
$$

acting from sections of $(\mathbb{E}\otimes\mathbb{G})_+ = E_+\otimes G_+ \oplus E_-\otimes G_-$ to $(\mathbb{E}\otimes\mathbb{G})_- = E_-\otimes G_+ \oplus E_+\otimes G_-$ is a family of elliptic symbols for the overall fibration $M' \longrightarrow B$.

Proposition 10.1. *(Atiyah and Singer) If the family B has trivial one-dimensional index in* $\mathrm{K}(M)$ *then any operator in* $\Psi^0(M'/B;\mathbb{E}\otimes\mathbb{G})$ *with symbol* (10.3) *has index in* $\mathrm{K}(B)$ *equal to that of A.*

Proof. To prove this we construct a natural 'product type' calculus for the top fibration which includes the fibrewise pseudodifferential operators and an appropriate class of lifts of operators on the fibres of the lower fibration and then do a deformation to the 'true' pseudodifferential calculus.

To ease the notational burden, at least initially, we consider the case of the numerical index and suppose that the second fibration, ϕ has just one fibre. The general case will be proved as part of the extension below to the cusp calculus. The product-type algebra is discussed in Appendix A. It contains the algebra of pseudodifferential operators on the total space, M' but has a double order filtration and we denote the filtered spaces $\Psi^{m,m'}_{\phi'-\mathrm{p}}(M';\mathbb{H})$ for a $\mathbb{Z}_2$-graded vector bundle $\mathbb{H} = (H_+, H_-)$ over M';

$$
\Psi^m(M';\mathbb{H}) \subset \Psi^{m,m}_{\phi'-\mathrm{p}}(M',\mathbb{H}) \ \forall\ m \in \mathbb{Z} \tag{10.4}
$$

(there is no particular problem with real order, but since we do not need them, we shall not bother with them here.)

The basic properties of these operator are similar to those of regular pseudodifferential operators

$$
\begin{gathered}
\Psi^{m_2,m_2'}_{\phi'-\mathrm{p}}(M',\mathbb{H}_1) \circ \Psi^{m_1,m_1'}_{\phi'-\mathrm{p}}(M',\mathbb{H}_2) \subset \Psi^{m_1+m_2,m_1'+m_2'}_{\phi'-\mathrm{p}}(M',\mathbb{H}), \\
\text{provided } (\mathbb{H}_1)_+ = (\mathbb{H}_2)_-,\ \mathbb{H}_+ = (\mathbb{H}_2)_+,\ \mathbb{H}_- = (\mathbb{H}_1)_-.
\end{gathered} \tag{10.5}
$$

The main difference between this product-type algebra and the usual one is that there are two related symbol maps. The 'usual' principal symbol map is modified to a short exact sequence

$$
\begin{gathered}
\Psi^{m-1,m'}_{\phi'-\mathrm{p}}(M',\mathbb{H}) \longrightarrow \Psi^{m,m'}_{\phi'-\mathrm{p}}(M',\mathbb{H}) \xrightarrow{\sigma_{m,m'}} \mathcal{S}^{m,m'}_{\phi'-\mathrm{p}}(M',\mathbb{H}), \\
\mathcal{S}^{m,m'}_{\phi'-\mathrm{p}}(M',\mathbb{H}) = \mathcal{C}^\infty([S^*M',\phi'^*S^*(M)];\hom(\mathbb{H})\otimes_+ N^{-m}\otimes_+ N_{\mathrm{ff}}^{-m'}).
\end{gathered} \tag{10.6}
$$

Here, S^*M' is the sphere bundle of T^*M' (best thought of as the boundary of the radial compactification of this bundle) which is then blown up at the submanifold given by the corresponding sphere bundle at infinity of the cotangent bundle of the base. Thus $[S^*M',\phi'^*S^*(M)]$ is the 'old' boundary hypersurface of the manifold with corners $[\overline{T^*M'},\phi'^*S^*(M)]$. The bundle N is the normal bundle in this sense and N_{ff} is the normal bundle to the front face, thought of as a trivial bundle over $[S^*M',\phi'^*S^*M]$. Thus these factors just represent the growth order of symbols at the boundaries. This 'standard' symbol is multiplicative in the obvious sense that

$$
\sigma_{m_1+m_2,m_1'+m_2'}(BA) = \sigma_{m_2,m_2'}(B)\sigma_{m_1,m_1'}(A) \tag{10.7}
$$

More interestingly, there is a non-commutative symbol, which has much in common with the indicial family for the fibred cusp calculus; it is called here the *base family.* Since $S^*M \longrightarrow M$ is a fibration we can lift the fibres of $\phi' : M' \longrightarrow M$ to give a fibration $S^*_{\phi'}M \longrightarrow S^*M$ which has the same typical fibre as ϕ'. This second symbol gives a short exact sequence

$$
\Psi^{m,m'-1}_{\phi'-\mathrm{p}}(M',\mathbb{H}) \hookrightarrow \Psi^{m,m'}_{\phi'-\mathrm{p}}(M',\mathbb{H}) \overset{L_{m,m'}}{\longrightarrow} \Psi^{m'}(S^*_{\phi'}M/S^*M;\mathbb{H}\otimes_+ N^{-m}). \tag{10.8}
$$

Here N is the inverse homogeneity bundle on T^*M, which is to say the normal bundle to the boundary of the radial compactification $\overline{T^*M}$. Again this sequence is multiplicative in the sense that

$$
L_{m_1+m_2,m_1'+m_2'}(BA) = L_{m_2,m_2'}(B)L_{m_1,m_1'}(A) \tag{10.9}
$$

under (10.5).

Although these two maps are separately surjective they are related through the symbol map on the image of (10.8). Namely the combination of the two maps gives a short exact sequence

$$
\begin{gathered}
\Psi^{m-1,m'-1}_{\phi'-\mathrm{p}}(M',\mathbb{H}) \hookrightarrow \Psi^{m,m'}_{\phi'-\mathrm{p}}(M',\mathbb{H}) \xrightarrow{(\sigma,L)} A^{m,m'}_{\phi'-\mathrm{p}}(M',\mathbb{H}) \\
A^{m,m'}_{\phi'-\mathrm{p}}(M',\mathbb{H}) = \big\{(a,A)\in \Psi^{m'}(S^*_\psi X/S^*X;\mathbb{H}\otimes_+ N^{-m})\oplus \mathcal{S}^{m,m'}_{\phi'-\mathrm{p}}(M',\mathbb{H}); \\
\sigma_{m'}(A) = a\big|_{\partial[S^*M',\psi^*S^*(M)]}\big\}.
\end{gathered} \tag{10.10}
$$

There are appropriate Sobolev spaces on which these operators are bounded. Since we are interested principally in the case of operators of order 0 it suffices to note that

$$
\begin{gathered}
\Psi^{0,0}_{\phi'-\mathrm{p}}(M';\mathbb{H}) \ni A : L^2(M';H_+) \longrightarrow L^2(M';H_-) \\
\text{is bounded and is compact iff} \\
\sigma_{0,0}(A) = 0,\ N_{0,0}(A) = 0.
\end{gathered} \tag{10.11}
$$

From this and standard constructions it follows that A in (10.11) is Fredholm iff

$$
\exists\ \sigma_{0,0}(A)^{-1} \in \mathcal{S}^{0,0}_{\phi'-\mathrm{p}}(M',\mathbb{H}^-) \text{ and } N_{0,0}(A)^{-1} \in \Psi^{m'}(S^*_{\phi'}M'/S^*M;\mathbb{H}^-). \tag{10.12}
$$

As well as (10.4) the fibrewise pseudodifferential operators may also be considered as product-type operators

$$\begin{gathered}\Psi^m(M'/M;\mathbb{H}) \subset \Psi^{0,m}_{\phi'-\mathrm{p}}(M';\mathbb{H}),\\ \sigma_{0,m}(A)=\sigma_m(A),\ N_0(A)=A\ \forall\ A\in\Psi^m(M'/B;\mathbb{H}).\end{gathered} \tag{10.13}$$

In particular such an operator of order 0 is Fredholm on L^2 if and only it is invertible, at least if the fibration has non-trivial base.

It is also important to see that pseudodifferential operators on the base are in a sense included in the product type pseudodifferential operators on the total space. Suppose that E is a $\mathbb{Z}_2$-graded vector bundle over M which is embedded in $\mathcal{C}^\infty(M';\mathbb{F})$ for some $\mathbb{Z}_2$-bundle $\mathbb{F}$. Thus there is a pair of families of finite rank, smoothing, projections $\pi_\pm \in \Psi^{-\infty}(M'/M;F_\pm)$ projecting onto the fibre of $E_\pm$ over each point of M. Then suppose that $A\in\Psi^m(M;\mathbb{E})$ is some pseudodifferential operator. With the identification of bundles, we may identify A as an operator from $\mathcal{C}^\infty(M';F_+)$ to $\mathcal{C}^\infty(M';F_-)$ and then

$$A=\pi_-A\pi_+\in\Psi^{m,-\infty}_{\phi'-\mathrm{p}}(M';\mathbb{F}). \tag{10.14}$$

In this case

$$N_m(A)=\pi_-\sigma_m(A)\pi_+\in\Psi^{-\infty}(S^*_{\phi'}M'/S^*M;\mathbb{F}\otimes_+ N_{-m}). \tag{10.15}$$

Under the hypothesis of the Proposition that we are trying to prove – for the moment only in the case of a single operator – we have a fibre family of trivial index of rank one. So, by smoothing perturbation we may assume that this $B\in\Psi^0(M'/M;\mathbb{G})$ is surjective and has a null bundle which is a trivial line bundle. Taking the $\mathbb{Z}_2$-bundle from the base we may form the extended operator which we denote

$$\begin{gathered}\begin{pmatrix}B & 0\\ 0 & B^*\end{pmatrix}\in\Psi^0(M'/M;\mathbb{E}\otimes\mathbb{G})\subset\Psi^{0,0}_{\phi'-\mathrm{p}}(M';\mathbb{E}\otimes\mathbb{G}),\\ \mathbb{E}\otimes\mathbb{H}=(E_+\otimes G_+\oplus E_-\otimes G_-, E_+\otimes G_-\oplus E_-\otimes G_+).\end{gathered} \tag{10.16}$$

Since the lifted bundles $E_\pm$ are trivial on each fibre, this has null bundle precisely E_+ as a subbundle of $\mathcal{C}^\infty M;E_+\otimes H_+)$ and cokernel bundle given by E_- as a subbundle of $\mathcal{C}^\infty(E_+\otimes H_-)$. Thus we can interpret A as mapping this null bundle into $E_-\otimes G_+$ and so form the operator

$$\begin{pmatrix}B & 0\\ A & B^*\end{pmatrix}\in\Psi^{0,0}_{\phi'-\mathrm{p}}(M';\mathbb{E}\otimes\mathbb{G}). \tag{10.17}$$

Directly from the definition, this is a Fredholm operator on L^2. Since it has null space just the null space of A as a subspace of the null space of B and cokernel the complement of the range of A as a subspace of the complement of the range of B^* we conclude directly that

$$\operatorname{ind}_{\mathrm{a}}\begin{pmatrix}B & 0\\ A & B^*\end{pmatrix}=\operatorname{ind}_{\mathrm{a}}(A). \tag{10.18}$$

Now we proceed to deform this operator as an elliptic family in $\Psi^{0,0}_{\phi'-\mathrm{p}}(M';\mathbb{E}\otimes\mathbb{G})$. First choose an element $\tilde{A}\in\Psi^0(M';\mathbb{E}\otimes G_+)$ with symbol $\chi_2\sigma(A)\otimes\mathrm{Id}_{G_+}$. As an element of $\Psi^{0,0}_{\phi'-\mathrm{p}}(M';\mathbb{E}\otimes G_+)$ its symbol is the lift of $\chi_2\sigma(A)$ and its base family is the lift of this symbol to the front face, interpreted as a bundle map. Consider the 1-parameter family

$$\begin{pmatrix} B & -s\tilde{A}^* \\ s\tilde{A}+(1-s)A & B^* \end{pmatrix} \in \Psi^{0,0}_{\phi'-\mathrm{p}}(M';\mathbb{E}\otimes\mathbb{G}), s\in[0,1]. \tag{10.19}$$

This is fully elliptic, i.e. both symbol and base family are invertible for all $s\in[0,1]$. For the symbol itself this is rather clear since it is the family

$$\begin{pmatrix} \sigma(b) & -s\chi_2\sigma(a)^* \\ s\chi_2\sigma(a) & \sigma(b)^* \end{pmatrix}. \tag{10.20}$$

On the other hand the base family is

$$\begin{pmatrix} B & -s\sigma(A)^* \\ s\sigma(A)+(1-s)\pi_-\sigma(A)\pi_+ & B^* \end{pmatrix} \tag{10.21}$$

as an operator on each fibre of ϕ', lifted to S^*M. Thus, $\sigma(A)$ is simply being extended from the null space of B to the whole bundle and this family is invertible for all $s\in[0,1]$.

Then we may choose $\tilde{B}\in\Psi^0(M';E_+\otimes\mathbb{G})$ with symbol $\chi_1\sigma(B)\otimes\mathrm{Id}_{E_+}$ and similarly extend the operator(10.19) at $s=1$ to a 1-parameter elliptic family

$$\begin{pmatrix} (1-r)B+r\tilde{B} & -\tilde{A}^* \\ \tilde{A} & (1-r)B^*+r\tilde{B}^* \end{pmatrix} \in \Psi^{0,0}_{\phi'-\mathrm{p}}(M';\mathbb{E}\otimes\mathbb{G}), r\in[0,1]. \tag{10.22}$$

Again this is a fully elliptic family and at $r=1$ reduces to an element of $\Psi^0(M';\mathbb{E}\otimes\mathbb{G})$ which has index equal to that of A and symbol given by (10.3). □

Remark 10.1. In this construction it is already clear that the K-class of the symbol of the resulting operator only depends on the K-classes of the symbols of A and B and so this defines a map

$$M_b:\mathrm{K_c}(T^*(M/B))\longrightarrow\mathrm{K_c}(T^*(M'/B)). \tag{10.23}$$

It should also be noted that if the operator (or family of operators) A is actually invertible then the lifted family is invertible and this invertibility can be maintained through the homotopy back to the standard algebra. This is used below for full ellipticity in the cusp setting.

Despite this, in general a fully elliptic family of product-type cannot be deformed through elliptics into the 'standard' subspace since this involves deforming the base family, through invertibles, to bundle maps and there may be an obstruction to this in K-theory.

Next we consider the corresponding construction in the cusp case; the main obstacle to this extension is notational! Thus we have an iterated fibration. Here $\phi:M\longrightarrow B$ is our usual fibration of compact manifolds with fibre a compact manifold with boundary. The 'top' fibration $\phi':M'\longrightarrow M$ is assumed to have compact fibre a manifold without boundary (in the application to lifting it is a sphere.) Again we are given an elliptic family $B\in\Psi^0(M'/M;\mathbb{G})$ which is surjective and has a trivial 1-dimensional null bundle where $\mathbb{G}$ is some $\mathbb{Z}_2$-graded bundle over M'.

We refer to Appendix B for a discussion of product-type cusp operators in this setting. These are quite analogous to the product-type pseudodifferential operators discussed above. Such an operator $P\in\Psi^{m,m'}_{\phi'-\mathrm{p,cu}}(M'/B;\mathbb{F})$ for any $\mathbb{Z}_2$-graded bundle $\mathbb{F}$ over M' acts on (weighted) smooth sections as in (1.3) but there are now *three* 'symbol' maps. The 'commutative' symbol corresponds to the usual cusp symbol modified in a way analogous to that of the symbol in the product-type pseudodifferential calculus in (10.6)

$$\begin{gathered}\Psi^{m-1,m'}_{\phi'-\mathrm{p,cu}}(M'/B;\mathbb{F}) \hookrightarrow \Psi^{m,m'}_{\phi'-\mathrm{p,cu}}(M'/B;\mathbb{F}) \overset{\sigma_{m,m'}}{\longrightarrow} \mathcal{S}^{m,m'}_{\phi'-\mathrm{p,cu}}(M'/B;\mathbb{F}),\\ \mathcal{S}^{m,m'}_{\phi'-\mathrm{p,cu}}(M'/B;\mathbb{F}) = \mathcal{C}^\infty([{}^{\mathrm{cu}}S^*M', (\phi')^{*\mathrm{cu}}S^*(M/B)]; N^{-m}\otimes N_{\mathrm{ff}}^{-m'}\otimes \hom(\mathbb{F})).\end{gathered} \tag{10.24}$$

Secondly is the indicial family, corresponding to the map (1.6) for the usual cusp calculus, but now taking values in the suspended version of the product-type calculus for the restriction of the fibration to the preimage of the boundary of M

$$x\Psi^{m,m'}_{\phi'-\mathrm{p,cu}}(M'/B;\mathbb{F}) \longrightarrow \Psi^{m,m'}_{\phi'-\mathrm{p,cu}}(M'/B;\mathbb{F}) \overset{N}{\longrightarrow} \Psi^{m,m'}_{\mathrm{sus},\partial\phi'-\mathrm{p}}(\partial M'/B;\mathbb{F}). \tag{10.25}$$

Finally there is an analogue of the non-commutative symbol in (10.8), namely a short exact sequence

$$\Psi^{m,m'-1}_{\phi'-\mathrm{p,cu}}(M'/B;\mathbb{F}) \longrightarrow \Psi^{m,m'}_{\phi'-\mathrm{p,cu}}(M'/B;\mathbb{F}) \overset{L}{\longrightarrow} \Psi^{m'}({}^{\mathrm{cu}}S^*_{\phi'}M/{}^{\mathrm{cu}}S^*M;\mathbb{F}\otimes_+ N^{-m}) \tag{10.26}$$

taking values in the pseudodifferential operators on the fibres of ϕ' but lifted to the fibrewise cusp cosphere bundle ${}^{\mathrm{cu}}S^*(M/B)$. Whilst separately surjective these three maps are related and combine to give a 'full symbol algebra' consisting of the elements with the compatibility conditions

$$\begin{aligned}A^{m,m'}_{\phi'-\mathrm{p,cu}}(M'/B) = \big\{(a,I,\beta) \in \\ \mathcal{S}^{m,m'}_{\phi'-\mathrm{p,cu}}(M'/B;\mathbb{F}) \times \Psi^{m,m'}_{\mathrm{sus},\partial\phi'-\mathrm{p}}(\partial M'/B;\mathbb{F}) \times \Psi^{m'}({}^{\mathrm{cu}}S^*_{\phi'}M/{}^{\mathrm{cu}}S^*M;\mathbb{F}\otimes_+ N^{-m});\\ a\big|_{\partial} = \sigma(I),\ a\big|_{\mathrm{ff}} = \sigma(\beta),\ N(\beta) = L(I)\big\}.\end{aligned} \tag{10.27}$$

This space itself corresponds to the short exact sequence

$$x\Psi^{m-1,m'-1}_{\phi'-\mathrm{p,cu}}(M'/B;\mathbb{F}) \longrightarrow \Psi^{m,m'}_{\phi'-\mathrm{p,cu}}(M'/B;\mathbb{F}) \overset{(\sigma,N,L)}{\longrightarrow} A^{m,m'}_{\phi'-\mathrm{p,cu}}(M'/B,\mathbb{F}) \tag{10.28}$$

which captures compactness and Fredholm properties on the appropriate Sobolev spaces. For us it suffices to note that

$$\begin{gathered}A \in \Psi^{0,0}_{\phi'-\mathrm{p,cu}}(M'/B;\mathbb{F}) \Longrightarrow A : L^2(M'/B;F_+) \longrightarrow L^2(M'/B;F_-) \text{ is bounded and}\\ A \text{ is compact} \Longleftrightarrow \sigma(A)=0,\ N(A)=0,\ L(A)=0\\ A \text{ is Fredholm} \Longleftrightarrow\ \exists\ (\sigma(A),\ N(A),\ L(A))^{-1} \in A^{-m,-m'}_{\phi'-\mathrm{p,cu}}(M'/B,\mathbb{F}^-).\end{gathered} \tag{10.29}$$

In the latter case, we say A is *fully elliptic*. As with the product-type pseudodifferential operators in the boundaryless case, it is important that the three different spaces of pseudodifferential operators, namely the ordinary pseudodifferential operators on the fibres of the smaller fibration, the ordinary cusp pseudodifferential operators on the fibres of the larger fibration and the cusp pseudodifferential operators on the base fibration all lift into this larger space:

$$\begin{aligned}\Psi^m(M'/M;\mathbb{F}) &\subset \Psi^{0,m}_{\phi'-\mathrm{p},\mathrm{cu}}(M'/B;\mathbb{F})\\ \Psi^m_{\mathrm{cu}}(M'/B;\mathbb{F}) &\subset \Psi^{m,m}_{\phi'-\mathrm{p},\mathrm{cu}}(M'/B;\mathbb{F})\\ \Psi^m_{\mathrm{cu}}(M/B;\mathbb{E}) &\subset \Psi^{m,-\infty}_{\phi'-\mathrm{p},\mathrm{cu}}(M'/B;\mathbb{F})\end{aligned} \tag{10.30}$$

where $\mathbb{E}$ is a $\mathbb{Z}_2$ graded vector bundle embedded as a subbundle of $\mathcal{C}^\infty(M'/M;\mathbb{F})$ as a bundle over M.

Proposition 10.2. *For an iterated fibration with cusp structure as in* (10.1)*, if an elliptic family $B \in \Psi^0(M'/M;\mathbb{F})$ is surjective and has trivial one-dimensional null bundle H then for any fully elliptic element $A \in \Psi^0_{\mathrm{cu}}(M/B;\mathbb{E})$ the operator*

$$\begin{pmatrix} B & 0\\ A & B^* \end{pmatrix} \in \Psi^{0,0}_{\phi'-\mathrm{p},\mathrm{cu}}(M'/B;\mathbb{F}\otimes\mathbb{E}), \tag{10.31}$$

where the inclusions are through (10.30)*, is a fully elliptic element with the same index as A in* $\mathrm{K}(B)$ *and furthermore this element is deformable, through fully elliptic elements, into $\Psi^0_{\mathrm{cu}}(M'/B;\mathbb{F}\otimes\mathbb{E})$. This results again in a map*

$$M_b : \mathrm{K}_{\mathrm{cu}}(\phi) \longrightarrow \mathrm{K}_{\mathrm{cu}}(\phi\circ\phi') \tag{10.32}$$

which only depends on $b = [\sigma(B)] \in \mathrm{K}_c(T^*(M'/M))$.

Proof. As noted above, the first part of the result, that (10.31) gives a fully elliptic element of the product-type calculus on the total fibration, and that (10.18) again holds, follows as in the proof of Proposition 10.1 with only minor notational changes.

For the second, deformation, part of the statement we need to proceed with more care. First, the arguments of Proposition 10.1 apply unchanged in the sense that (10.19) and (10.22) together give a symbolically elliptic deformation, indeed the computation of the symbol (although now for a family in the base) is the same as there, as is the computation of the base family – the latter is still given by (10.21) provided the symbol of A is interpreted as the cusp symbol. Thus we have constructed a family P_t, in the product-type cusp calculus, where we can relabel and change parameterization to make the family smooth in $t \in [0,1]$. This family is elliptic for all t, has invertible base family for all t, is Fredholm at $t = 0$ and is in the ordinary cusp calculus at $t = 1$. So, consider the indicial family of P_t. This is a suspended family of product-type pseudodifferential operators (acting on the fibres of a fibration) which is fully elliptic, i.e. is elliptic and has invertible base family, and which is invertible at $t = 0$. Since it is also invertible for large values of the suspending variable, it follow by standard arguments that it can be perturbed by a family of smoothing operators (see Remark 10.1) which vanishes near infinity in the suspension variable and vanishes near $t = 0$ to be invertible for all values of t. Modifying the family in this way results in a deformation as desired.

That this construction is symbolic as far as B is concerned and 'fully symbolic' as far as A is concerned, in the sense of (10.32), follows readily from the construction. Namely it certainly behaves well under stabilization and homotopy, with the homotopy parameter simply being added to the base variables. So it remains to show that it is stable under different regularizations of the null bundle of B; in fact any two such stabilizations are homotopic. □

Finally we comment on the generalization of this result to the fibred cusp case although this is not used below. Thus we again consider an iterated fibration (10.1) where now the second fibration has a fibred cusp structure, $\Phi : \partial M \longrightarrow D$. There are two extreme possibilities for a fibred cusp structure for the overall fibration. First, it is always possible to 'add' the fibres of $\partial M' \longrightarrow \partial M$ to the fibres of the boundary, i.e. to take the boundary fibration for the overall fibration to be

$$\Phi' = \Phi \circ \phi'. \tag{10.33}$$

Proposition 10.3. *For an iterated fibration as in* (10.1) *where the second fibration has a fibred cusp structure and the top fibration has the fibred cusp structure* (10.33)*, if an elliptic family* $B \in \Psi^0(M'/M; \mathbb{F})$ *is surjective and has trivial one-dimensional null bundle* H *then for any fully elliptic element* $A \in \Psi^0_{\Phi\text{-}\mathrm{cu}}(M/B; \mathbb{E})$ *the operator*

$$\begin{pmatrix} B & 0 \\ A & B^* \end{pmatrix} \in \Psi^{0,0}_{\phi'-\mathrm{p},\Phi'\text{-}\mathrm{cu}}(M'/B; \mathbb{F} \otimes \mathbb{E}), \tag{10.34}$$

where the inclusions are through (10.30)*, is a fully elliptic element with the same index as* A *in* $\mathrm{K}(B)$ *and furthermore this element is deformable, through fully elliptic elements, into* $A_B \in \Psi^0_{\Phi'\text{-}\mathrm{cu}}(M'/B; \mathbb{F} \otimes \mathbb{E})$*. This results again in a map*

$$M_b : \mathrm{K}_{\Phi\text{-}\mathrm{cu}}(\phi) \longrightarrow \mathrm{K}_{\Phi\circ\phi'\text{-}\mathrm{cu}}(\phi \circ \phi') \tag{10.35}$$

which only depends on $b = [\sigma(B)] \in \mathrm{K}_c(T^*(M'/M))$*.*

Proof. The arguments in the proof of Proposition 10.2 go through essentially without change, the only difference being the appearance of more suspension parameters in the normal operator. ☐

There is a second extreme possibility, other than (10.33), for the fibred cusp structure on the top fibration in (10.1), corresponding to the fibres for $\Phi' : M' \longrightarrow D'$ being of the same dimension as for Φ in which case all the 'new' boundary variables are in the base of the fibration. Then there is a commutative diagram of fibrations

$$\begin{array}{ccccccc} F & \text{———} & \partial M' & \xrightarrow{\Phi'} & D' & \text{———} & F \\ & & \downarrow{\scriptstyle \phi'} & & \downarrow & & \\ & & \partial M & \xrightarrow{\Phi} & D & & \end{array} \tag{10.36}$$

In this case there is a difficulty in extending the construction above, in that it is not quite clear what the family B should be. Approached directly B would need to be a family which is adiabatic in the boundary variable. Since the index theory for such families has not been properly developed we have chosen to proceed more indirectly by using Proposition 8.1. However, it is possible instead to use Proposition 10.3 above and then make an adiabatic limit back to the case where the extra variables are in the base of the fibration. Since this involves a rather delicate homotopy, and is not used below, the details are omitted.

11. Lifting and excision

Next we consider the lifting construction for cusp K-theory with respect to an extension of a given fibration, see Definition 9.1.

Proposition 11.1. *If $\tilde{\phi} : \widetilde{M} \longrightarrow B$ is an extension of a fibration $\phi : M \longrightarrow B$ with cusp structure in the sense of Definition 9.1 then there is a well-defined lifting homomorphism*

$$(\tilde{\phi}/\phi)^! : \mathrm{K}_{\mathrm{cu}}(\phi) \longrightarrow \mathrm{K}_{\mathrm{cu}}(\tilde{\phi}) \tag{11.1}$$

which induces a commutative diagram (17) *where the map on the right is an absolute form of the lifting map of Atiyah and Singer. When $\widetilde{M} = \mathbb{B}^{2N+1} \times B$, this reduces to pull-back under Poincaré duality*

$$\begin{array}{ccc} \mathrm{K}_{\mathrm{cu}}(\tilde{\phi}) & \xrightarrow{\text{quan}} & \mathrm{KK}^0_D(\mathcal{C}_{\mathrm{cu}}(\widetilde{M}), \mathcal{C}(B)) \\ {\scriptstyle (\tilde{\phi}/\phi)^!}\uparrow & & \uparrow{\scriptstyle R^*} \\ \mathrm{K}_{\mathrm{cu}}(\phi) & \xrightarrow{\text{quan}} & \mathrm{KK}^0_D(\mathcal{C}_{\mathrm{cu}}(M), \mathcal{C}(B)) \end{array} \tag{11.2}$$

where $R : \mathcal{C}_{\mathrm{cu}}(\widetilde{M}) \to \mathcal{C}_{\mathrm{cu}}(M)$ is the restriction map.

Proof. The normal bundle to M in $\widetilde{M}$ is a real vector bundle which models the extended fibration near M. Thus, if we take the one-point compactification of the fibres we obtain a bundle of spheres $\pi : S_V \longrightarrow M$ which gives an extension of the original fibration but is also an iterated fibration in the sense of Proposition 10.2. Moreover, following the approach of Atiyah and Singer, there is a 'Bott element', which we can realize as a family

$$B \in \Psi^0(S_V/M; \mathbb{L}) \tag{11.3}$$

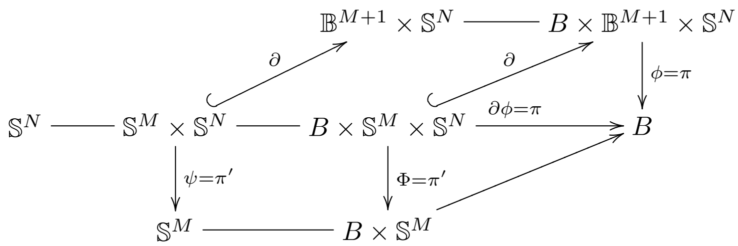
for a $\mathbb{Z}_2$-graded bundle $\mathbb{L}$ over S_V with L_+ and L_- identified near the section at infinity and such that $B - \mathrm{Id}$ has kernel with support disjoint from the section at infinity (in either factor). Now, we can apply Proposition 10.2 to 'lift' each fully elliptic element of $\Psi^0_{\mathrm{cu}}(M/B; \mathbb{E})$ to a fully elliptic element of $\Psi^0_{\mathrm{cu}}(S_V/B; \mathbb{F})$. A brief review of the construction shows that the condition that all operators be equal to the identity near the section at infinity can be maintained throughout. In view of this property of the kernel, including an identification of F_+ and F_- near the section at infinity, these operators may be trivially extended into $\Psi^0_{\mathrm{cu}}(\widetilde{M}/B; \mathbb{F})$ to be fully elliptic. The nature of this construction shows that this does indeed construct a map (11.1) and that it leads to a commutative diagram (17). When $\widetilde{M} = \mathbb{B}^{2N+1} \times B$, the commutativity of diagram (11.2) follows from proposition 12.1 and the fact that $\tilde{\phi} = \phi \circ R$. □

Note that the Poincaré duality isomorphism and (11.2) show that the lifting map is independent of choices and is well-behaved under composition. This is also relatively easy to see directly.

12. Product with a ball and a sphere

We have shown in Proposition 9.1 that a fibred cusp structure over B can always be 'trivialized' by embedding it in one of the product cases with both fibrations being projections, in which case the diagram (21) becomes

(12.1)



with π being projection onto the leftmost factor and π' off the rightmost.

Proposition 12.1. *If $M > 0$ and $N \geq 0$ in the product spaces in* (12.1)*, then*

$$\mathrm{K}_{\pi\text{'-cu}}(\pi) \cong \begin{cases} \mathrm{K}(B) & M+N \text{ even} \\ \mathrm{K}(B) \oplus \mathrm{K}(B) & M+N \text{ odd} \end{cases}. \tag{12.2}$$

In the cusp case with $\pi : B \times \mathbb{B}^{M+1} \to B$, we have

$$\mathrm{K}_{\mathrm{cu}}(\pi) \cong \begin{cases} \mathrm{K}(B) & M \text{ even} \\ \mathrm{K}(B) \oplus \mathrm{K}(B) & M \text{ odd} \end{cases}.$$

Proof. From the Poincaré duality isomorphism, Theorem 2, we can use KK-theory to perform the computation. Now, for any product $M = B \times Z$ in which the overall fibration is the projection onto the left factor and the boundary fibration is the product with some fibration of the boundary of $\psi : \partial Z \longrightarrow D$,

$$\mathcal{C}_{\mathrm{Id}\times\psi}(B \times Z) = \mathcal{C}(B)\hat{\otimes}\mathcal{C}_{\psi}(Z) \tag{12.3}$$

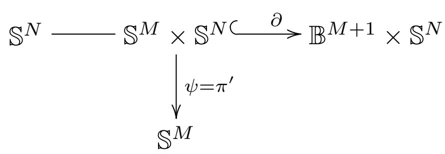
is the completed tensor product. From the general properties of KK-theory it follows that

$$\begin{aligned} \mathrm{KK}_B(\mathcal{C}_{\mathrm{Id}\times\psi}(B \times Z), \mathcal{C}(B)) &= \mathrm{KK}_B(\mathcal{C}(B)\hat{\otimes}\mathcal{C}_{\psi}(Z), \mathcal{C}(B)) \\ &\cong \mathrm{KK}(\mathcal{C}_{\psi}(Z), \mathcal{C}(B)) \\ &\cong \mathrm{K}(B) \otimes \mathrm{KK}(\mathcal{C}_{\psi}(Z), \mathbb{C}), \end{aligned} \tag{12.4}$$

where in the last step we used the Künneth formula for KK-theory (see for instance Theorem 23.1.3 in [9]) and we used the fact (proved below) that $\mathrm{KK}(\mathcal{C}_{\psi}(Z), \mathbb{C})$ is a free $\mathbb{Z}$-module. Thus it suffices to consider the case in which the base in (12.1) is reduced to a point and to show then that $\mathrm{K}_{\pi\text{'-cu}}(\pi)$ reduces to one or two copies of $\mathbb{Z}$ according to parity. In fact the general argument, with the base factor retained, is not much more complicated.

Thus we need only to consider

(12.5)



and compute the fibred-cusp K-theory in this case.

First consider the cusp case, where there is no factor of $\mathbb{S}^N$. The same argument applies to reduce the computation to the case of $\mathbb{B}^{M+1}$. So consider the (split) short exact sequence (10), proved in Section 5. Since $K(\{\mathrm{pt}\}) = \mathbb{Z}$, essentially by definition, it suffices to check that

$$\mathrm{K_c}(\mathbb{B}^{M+1} \times \mathbb{R}^{M+1}) = \begin{cases} \{0\} & M \text{ even} \\ \mathbb{Z} & M \text{ odd.} \end{cases} \tag{12.6}$$

Here, recall that the ball is taken to be closed, so the K-theory is absolute in that factor and compactly supported in the Euclidean factor. Since the ball is contractible, the only source for K-classes is

$$\mathrm{K_c}(\mathbb{R}^{M+1}) \equiv \mathrm{K_c}(\mathbb{B}^{M+1} \times \mathbb{R}^{M+1})$$

so (12.6) follows.

In the fibred cusp case, set $Z_{M,N} = \mathbb{B}^{M+1} \times \mathbb{S}^N$. The obstruction to perturbing an elliptic operator so that it becomes fully elliptic lies in

$$\mathrm{K}(T^*\mathbb{S}^M \times \mathbb{R}) \cong \mathrm{K}(\mathbb{S}^M \times \mathbb{R}^{M+1}).$$

Consider the 6-term exact sequence coming from the inclusion

$$r_{\partial\mathbb{B}^{M+1}} : T^*\mathbb{S}^M \times \mathbb{R} \cong T^*(\mathbb{B}^{M+1})\big|_{\partial\mathbb{B}^{M+1}} \hookrightarrow T^*(\mathbb{B}^{M+1}),$$

namely

$$\begin{array}{ccccc} \mathrm{K_c^0}(\mathbb{R}^{2M+2}) & \longrightarrow & \mathrm{K_c^0}(T^*(\mathbb{B}^{M+1})) & \xrightarrow{r^*_{\partial\mathbb{B}^{M+1}}} & \mathrm{K_c^0}(T^*(\mathbb{S}^M) \times \mathbb{R}) \\ \uparrow & & & & \downarrow \\ \mathrm{K_c^1}(T^*(\mathbb{S}^M) \times \mathbb{R}) & \xleftarrow{r^*_{\partial\mathbb{B}^{M+1}}} & \mathrm{K_c^1}(T^*(\mathbb{B}^{M+1})) & \longleftarrow & \mathrm{K_c^1}(\mathbb{R}^{2M+2}) \end{array} \tag{12.7}$$

using the identification

$$\mathrm{K_c^k}(T^*(\mathbb{B}^{M+1}), T^*(\mathbb{S}^M) \times \mathbb{R}) \cong \mathrm{K_c^k}(\mathbb{R}^{2M+2}) \cong \begin{cases} \mathbb{Z} & k = 0, \\ \{0\} & k = 1. \end{cases} \tag{12.8}$$

From (12.6) and (12.8) and using the fact that

$$\mathrm{K_c^k}(\mathbb{R}^{2M+2}) \longrightarrow \mathrm{K_c^k}(T^*(\mathbb{B}^{M+1})) \tag{12.9}$$

maps to zero, we get that

$$K_c^0(T^*\mathbb{S}^M) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & M \text{ even} \\ \mathbb{Z} & M \text{ odd} \end{cases}, \quad \mathrm{K}_c^1(T^*\mathbb{S}^M) \cong \begin{cases} \{0\} & M \text{ even} \\ \mathbb{Z} & M \text{ odd.} \end{cases} \tag{12.10}$$

Thus, in particular, when M is even, there is no obstruction to the existence of a smoothing perturbation that makes an elliptic fibred cusp operator fully elliptic. So consider the image of

$$I_1 : \mathrm{K_c^1}(T^*(\mathbb{B}^{M+1} \times \mathbb{S}^N)) \longrightarrow \mathrm{K_c^0}(T^*(\mathbb{S}^M)).$$

Since $\pi' : \mathbb{S}^M \times \mathbb{S}^N \longrightarrow \mathbb{S}^M$ extends to

$$P' : \mathbb{B}^{M+1} \times \mathbb{S}^N \longrightarrow \mathbb{B}^{M+1}$$

by projecting on the right factor

$$I_1 = \mathrm{ind}_{\pi'} r^*_{\partial(\mathbb{B}^{M+1} \times \mathbb{S}^N)} = r^*_{\partial(\mathbb{B}^{M+1})} \mathrm{ind}_{P'}.$$

But $\mathrm{ind}_{P'}$ is clearly surjective, so the image of I_1 is the same as the image of $r^*_{\partial(\mathbb{B}^{M+1})}$. Thus, we conclude from the 6-term exact sequence (12.7) that

$$I_1(\mathrm{K}^1_{\mathrm{c}}(T^*(\mathbb{B}^{M+1}\times\mathbb{S}^N)))\cong\begin{cases}\mathbb{Z} & M \text{ even}\\ \{0\} & M \text{ odd.}\end{cases}$$

Thus, when M is even, there is a short exact sequence

$$\mathbb{Z}\longrightarrow \mathrm{K}_{\pi\text{'-}\mathrm{cu}}(\mathbb{B}^{M+1}\times\mathbb{S}^N)\longrightarrow \mathrm{K}_{\mathrm{c}}(T^*(\mathbb{B}^{M+1}\times\mathbb{S}^N)).$$

Since

$$K^0_c(T^*(\mathbb{B}^{M+1}\times\mathbb{S}^N))\cong\begin{cases}\{0\} & N \text{ even}\\ \mathbb{Z} & N \text{ odd}\end{cases}$$

is a free $\mathbb{Z}$-module, this sequence splits and hence

$$\mathrm{K}_{\pi\text{'-}\mathrm{cu}}(\mathbb{B}^{M+1}\times\mathbb{S}^N)\cong\mathbb{Z}\oplus\mathrm{K}(T^*(\mathbb{B}^{M+1}\times\mathbb{S}^N))\cong\begin{cases}\mathbb{Z} & N \text{ even}\\ \mathbb{Z}\oplus\mathbb{Z} & N \text{ odd.}\end{cases}$$

When M is odd, we have instead the short exact sequence

$$K^0_c(T^*\mathbb{S}^M)\longrightarrow \mathrm{K}_{\pi\text{'-}\mathrm{cu}}(\mathbb{B}^{M+1}\times\mathbb{S}^N)\longrightarrow\ker(I_0)$$

which splits by sending an element of $\ker(I_0)$ to a full symbol with null index. Thus we conclude that

$$\mathrm{K}_{\pi\text{'-}\mathrm{cu}}(\mathbb{B}^{M+1}\times\mathbb{S}^N)\cong\ker(I_0)\oplus\mathbb{Z}\cong K^0_c(T^*(\mathbb{B}^{M+1}\times\mathbb{S}^N))$$

since the space of obstruction is $\mathrm{K}^1_{\mathrm{c}}(T^*\mathbb{S}^M)\cong\mathbb{Z}$ and I_0 is surjective in this case, as one can see from the 6-term exact sequence (12.7). Using again (12.10)

$$\begin{aligned}\mathrm{K}_{\pi\text{'-}\mathrm{cu}}(\mathbb{B}^{M+1}\times\mathbb{S}^N)&\cong\mathrm{K}_c(T^*(\mathbb{B}^{M+1}\times\mathbb{S}^N))\\&\cong\mathrm{K}_c(T^*\mathbb{S}^N\times\mathbb{R}^{M+1})\\&\cong\mathrm{K}_c(T^*\mathbb{S}^N)\cong\begin{cases}\mathbb{Z}\oplus\mathbb{Z} & N \text{ even,}\\ \mathbb{Z} & N \text{ odd.}\end{cases}\end{aligned}\tag{12.11}$$

□

13. The topological index

Consider a fibration as in (21). Let

$$i: M\hookrightarrow B\times\mathbb{B}^{p+1},$$

with p even, be an embedding as in Proposition 9.1, where the fibration structure on $B\times\mathbb{B}^{p+1}$ is given by the projection on the right factor

$$\pi: B\times\mathbb{B}^{p+1}\longrightarrow B.$$

Proposition 11.1 gives a well-defined lifting homomorphism

$$(\pi/\phi)^!:\mathrm{K}_{\mathrm{cu}}(\phi)\longrightarrow\mathrm{K}_{\mathrm{cu}}(\pi).\tag{13.1}$$

By the proof of Proposition 12.1,

$$\mathrm{K}_{\mathrm{cu}}(\pi)\overset{\text{quan}}{\longrightarrow}\mathrm{KK}_B(\mathcal{C}_{\partial\pi}(B\times\mathbb{B}^{p+1}),\mathcal{C}(B))\overset{\text{ind}}{\longrightarrow}\mathrm{K}(B)$$

is an isomorphism.

Definition 13.1. the topological index map $\mathrm{ind}_{\mathrm{t}} : \mathrm{K}_{\Phi\text{-}\mathrm{cu}} \longrightarrow \mathrm{K}(B)$ is defined by

$$\mathrm{ind}_{\mathrm{t}} = \mathrm{ind} \circ \mathrm{quan} \circ (\pi/\phi)^{!} \circ q_{\mathrm{ad}} \ .$$

The fact that it does not depend on the choice of the embedding follows from the stability of the lifting map under repeated embedding, but in any case will follow from Theorem 1!

Proof of Theorem 1. The theorem follows from the commutativity of the diagram (17) stated in Proposition 11.1, and the commutativity of diagram (25) stated in Proposition 8.1. □

14. Families of Atiyah-Patodi-Singer type

The Atiyah-Patodi-Singer index theorem of [4] was originally proved with the idea of obtaining a generalization of Hirzebruch's signature theorem for the case of manifolds with boundary. Since then, variants of the proof of the theorem were obtained, for instance in [10] and [22]. An extension to the family case was discussed in [7], [8] and in [26], [27]. In all these proofs, including the original one, heat kernel techniques for Dirac operators play an important rôle. As opposed to the Atiyah-Singer index theorem, the Atiyah-Patodi-Singer index theorem was originally restricted to Dirac operators, but using trace functional techniques, a pseudodifferential generalization was obtained in [24] in the setting of cusp operators. Such a generalization was also discussed by Piazza in [31] for b-pseudodifferential operators.

In [11], Dai and Zhang provided an interesting proof of the Atiyah-Patodi-Singer index theorem by embedding the manifold with boundary into a large ball. Relating the Dirac operator of interest with one defined on the large ball via a careful analysis, they were able to take advantage of the simple topology of the ball to get the Atiyah-Patodi-Singer index theorem for the original operator. This is certainly closely related to our constructions above, however, the methods of [11] are essentially analytical and no K-theory is involved.

We proceed to briefly recall the setting of Atiyah-Patodi-Singer boundary problem and its reformulation in terms of cusp pseudodifferential operators, leading to a K-theory index. Let Z be an even dimensional Riemannian manifold with nonempty boundary $\partial Z = X$ with a Riemannian metric which is of product type near the boundary, so there is a neighborhood $X \times [0, 1) \subset Z$ of the boundary in which the metric takes the form

$$g = du^2 + h_X \tag{14.1}$$

where $u \in [0, 1)$ is the coordinate normal to the boundary and h_X is the pull-back of a metric on X via the projection $X \times [0, 1) \longrightarrow X$. Let $\mathbb{E}$ be a Hermitian vector bundle over Z with a Clifford module structure for the metric (14.1) and with a unitary Clifford connection which is constant in the normal direction near the boundary under a product trivialization. This defines a generalized Dirac operator

$$\eth : \mathcal{C}^{\infty}(Z, \mathbb{E}) \longrightarrow \mathcal{C}^{\infty}(Z, \mathbb{E}).$$

In the neighborhood $X \times [0, 1) \subset Z$ of the boundary described above, it takes the form

$$\eth^{+} = \gamma \left(\frac{\partial}{\partial u} + A \right) \tag{14.2}$$

where $\gamma = cl(u) : \mathbb{E}\big|_X \longrightarrow \mathbb{E}\big|_X$ is given by Clifford multiplication by the normal differential and $A : \mathcal{C}^\infty(X, E_+|_X) \longrightarrow \mathcal{C}^\infty(X, E_+|_X)$ is a Dirac operator on X such that

$$\gamma^2 = -\operatorname{Id},\ \gamma^* = -\gamma,\ A\gamma = -\gamma A,\ A^* = A. \tag{14.3}$$

Consider the spectral boundary condition

$$\varphi \in \mathcal{C}^\infty(Z, \mathbb{E}),\ P(\varphi\big|_X) = 0, \tag{14.4}$$

where P is the projection onto the nonnegative spectrum of A. Then

$$\eth^+ : W^1_P \longrightarrow \mathrm{L}^2(Z; E_-) \tag{14.5}$$

is a Fredholm operator, where

$$W^1_P = \left\{ f \in \mathrm{H}^1(X; E_+); P(f\big|_X) = 0 \right\}$$

is a subspace of $\mathrm{H}^1(X; E_+)$, the Sobolev space of order 1.

Atiyah, Patodi and Singer show in [4] that the index of $\eth^+$ is given by

$$\operatorname{ind}(\eth^+) = \int_Z \widehat{A}(Z) \operatorname{Ch}'(\mathbb{E}) - \frac{h + \eta}{2}$$

where $\operatorname{Ch}'(\mathbb{E})$ is the twisting Chern of $\mathbb{E}$, $\widehat{A}$ is the $\widehat{A}$-genus, $h = \dim\ker A$ and η is the eta invariant of A.

As discussed in [4], one can alternatively describe the index problem by adding a cylindrical end to the manifold with boundary Z. More precisely, Z may be enlarged by attaching the half-cylinder $(-\infty, 0) \times X$ to the boundary X of Z. Call the resulting manifold Z'. The metric, being a product near the boundary, can be naturally extended to this half-cylinder, which makes the resulting manifold a complete Riemannian manifold. The bundle, Clifford structure, connection and hence the Dirac operator also have natural translation-invariant extensions to Z' using the product structure near the boundary. On Z', it is possible to think of $\eth$ as a cusp operator by compactifying to a compact manifold with boundary (diffeomorphic to the original Z) by replacing u by the variable

$$x = -\frac{1}{u} \in [0, 1) \tag{14.6}$$

for $u \in (-\infty, -1)$. The extension down to $x = 0$, gives the manifold with boundary $\overline{Z'}$, and x is a boundary defining function for $\partial\overline{Z'} \cong X$ which defines a cusp structure. Let us denote by $\eth_{\mathrm{cu}}$ the natural extension of $\eth$ to $\overline{Z'}$. Near the boundary of $\overline{Z'}$, $\eth_{\mathrm{cu}}$ takes the form

$$\eth_{\mathrm{cu}} = \gamma \left(x^2 \frac{\partial}{\partial x} + A \right) \tag{14.7}$$

and so is clearly an elliptic cusp differential operator.

Lemma 14.1. *If A is invertible, then*

$$\eth^+_{\mathrm{cu}} : \mathrm{H}^1(\overline{Z'}; E_+) \longrightarrow \mathrm{L}^2(\overline{Z'}; E_-)$$

is Fredholm and has the same index has the operator (14.5).

Proof. Recall ([21]) that a cusp operator is Fredholm if and only if it is elliptic and its indicial family is invertible. The indicial family of $\eth^+_{\mathrm{cu}}$ is given by

$$e^{-i\frac{\tau}{x}} \eth^+_{\mathrm{cu}} e^{i\frac{\tau}{x}}\big|_{x=0} = \gamma(A - i\tau),\ \tau \in \mathbb{R}.$$

Since A is self-adjoint, this is invertible for all τ if and only if A is invertible. Thus, $\eth_{\mathrm{cu}}$ is Fredholm if and only if A is invertible. That $\eth^+_{\mathrm{cu}}$ has the same index as (14.5) then follows from Proposition 3.11 in [4] and Proposition 9 in [21]. □

When A is not invertible, it is still possible to relate (14.5) with a Fredholm cusp operator. In fact, since this is basically the problem we encounter when we consider the family version, let us immediately generalize to this context. For the family case consider a fibration, (3), of a manifold with boundary and we assume now that $\eth$ is a family of Dirac operators parameterized by B which as before are of product near the boundary of ∂M. Thus, there is an associated family A of self-adjoint Dirac operators on the boundary. The main difficulty in the family case is that interpreted directly, the spectral boundary condition need not be smooth. In [26], this difficulty was overcome by introducing the notion of a spectral section.

Definition 14.1. A spectral section for a family of elliptic self-adjoint operators

$$A \in \mathrm{Diff}^1(\partial M/B; \mathbb{E}|_{\partial M})$$

is a family of self-adjoint projections $P \in \Psi^0(\partial M/B; \mathbb{E}|_{\partial M})$ such that for some smooth function $R : B \longrightarrow [0, \infty)$ (depending on P) and every $b \in B$,

$$A_b f = \lambda f \Longrightarrow \begin{cases} P_b f = f & \text{if } \lambda > R(b), \\ P_b f = 0 & \text{if } \lambda < -R(b). \end{cases}$$

Such a spectral section always exists for the boundary family and any such choice gives a smooth family of boundary problems problem

$$\eth^+ : W^1_P \longrightarrow \mathrm{L}^2(M/B; E_-) \tag{14.8}$$

where

$$W^1_P = \left\{ f \in \mathrm{H}^1(M/B; E_+); P(f\big|_{\partial M}) = 0 \right\}.$$

The family $\eth^+$ in (14.8) is Fredholm so has a well defined families index. As before, one can attach a cylindrical end and get a new fibration $\phi : M' \longrightarrow B$ where the family of operators $\eth$ naturally extends. By making the change of variable $x = -\frac{1}{u}$, one get the a family of cusp operators $\eth_{\mathrm{cu}}$ which takes the form (14.7) near the boundary.

Lemma 14.2. *There exists $Q \in \Psi^{-\infty}_{\mathrm{cu}}(M/B; E_+, E_-)$ such that $\eth^+_{\mathrm{cu}} + Q$ is a fully elliptic (hence Fredholm) family with the same family index as* (14.8).

Proof. This is carried out in section 8 of [26] in the context of b-pseudodifferential operators instead of cusp operators. Since the relationship corresponds to the introduction of the transcendental variable $1/u$ (for cusp) instead of e^u (for b-) one can check that the argument continues to hold for cusp operators with only minor modifications. □

Proof of Theorem 3. Let $(\eth, P)$ be as in the proposition. Define $[(\eth, P)] \in \mathrm{K}_{\mathrm{cu}}(\phi)$ to be the K-class associated to the operator $\eth^+_{\mathrm{cu}} + Q$ of Lemma 14.2. Then by the lemma

$$\mathrm{ind}(\eth, P) = \mathrm{ind}_{\mathrm{a}}([(\eth, P)]) = \mathrm{ind}_{\mathrm{t}}([(\eth, P)]).$$

That $[(\eth, P)]$ is canonically defined, that this, does not depend on the choice of Q in Lemma 14.2, is a consequence of the relative families index theorem of [28]. Thus, given the Poincaré duality of Theorem 2 and the commutativity of diagram (7), Theorem 3 follows. □

Appendix A. Product-type pseudodifferential operators

There are many variants of pseudodifferential operators 'of product type' to be found in the literature, see Shubin [34], Rodino [33], Melrose-Uhlmann [29], Hörmander [14]. Here we describe, succinctly, a particularly natural algebra of such operators associated to a fibration (and for families to an iterated fibration). First consider the local setting.

On $\mathbb{R}^{d_1}_y \times \mathbb{R}^{d_2}_z$ we consider operators corresponding to this product thought of as a fibration over the first factor. The class of symbols admitted is determined by the $\mathcal{C}^\infty$ structure on the manifold with corners

$$X_{d_1,d_2} = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \times [\overline{\mathbb{R}^{d_1+d_2}}; \partial\overline{\mathbb{R}^{d_1} \times \{0\}}]. \tag{A.1}$$

That is, take the radial compactification of $\mathbb{R}^{d_1+d_2}$ and blow up the boundary (at infinity) of the radial compactification of the subspace $\mathbb{R}^{d_1} \times \{0\}$. Let ρ, $\rho_{\mathrm{ff}} \in \mathcal{C}^\infty(X_{d_1.d_2})$ be defining functions for the two boundary hypersurfaces, the first being the 'old hypersurface' at infinity and the second that produced by the blow up.

Now, if $a \in \rho^{-m}\rho_{\mathrm{ff}}^{-m'}\mathcal{C}^\infty_c(X_{d_1,d_2})$ then it satisfies the estimates

$$|\partial^\alpha_y\partial^\beta_z\partial^\gamma_\eta\partial^\delta_\zeta a| \le C_{\alpha,\beta,\gamma,\delta}(1+|\zeta|)^{m-m'-|\delta|}(1+|\eta|+|\zeta|)^{m'-|\gamma|}, \tag{A.2}$$

as follows by noting that one can take $\rho = (1+|\zeta|^2)^{-\frac{1}{2}}$ and $\rho_{\mathrm{ff}} = (1+|\zeta|^2)^{\frac{1}{2}}(1+|\eta|^2+|\zeta|^2)^{-\frac{1}{2}}$. This gives the overall weight in (A.2) with no differentiation. The vector fields

$$\partial_{y_j},\ \partial_{z_k},\ \eta_i\partial_{\eta_j},\ \zeta_l\partial_{\zeta_k},\ \zeta_k\partial_{\eta_j}$$

all lift to be smooth on $\mathbb{R}^{d_1+d_2} \times \overline{\mathbb{R}^{d_1+d_2}}$ and tangent to the boundary and within the boundary to the submanifold blown up in (A.1) so

$$\partial^\alpha_y\partial^\beta_z\partial^\gamma_\eta\partial^\delta_\zeta a \in \rho^{-m+\gamma+\delta}\rho_{\mathrm{ff}}^{-m'+\gamma}\mathcal{C}^\infty_c(X_{d_1,d_2}); \tag{A.3}$$

this leads directly to the estimates (A.2). Consider the kernels on $\mathbb{R}^{2d_1+2d_2}$ defined by Weyl quantization of symbols

(A.4)

$$A(y,z,y',z') = (2\pi)^{-d_1-d_2}\int e^{i(y-y')\cdot\eta+(z-z')\cdot\zeta}a(\frac{y+y'}{2},\frac{z+z'}{2},\eta,\zeta)d\eta d\zeta \text{ where}$$
$$a = a_1+a_2+a_3,\ a_1 \in \rho^{-m}\rho_{\mathrm{ff}}^{-m'}\mathcal{C}^\infty_c(X_{d_1,d_2}),$$
$$a_2 \in \rho^\infty\mathcal{S}(\mathbb{R}^{2d_2};\rho_\eta^{-m'}\mathcal{C}^\infty_c(\mathbb{R}^{d_1}\times\overline{\mathbb{R}^{d_1}})),\ a_3 \in \mathcal{S}(\mathbb{R}^{2d_1+2d_2}).$$

Note that the three terms in the amplitude in (A.4) are really of the same type and the second and third can be included in the first, except that the support conditions are relaxed to rapid decay at infinity. Thus, the third class of symbols corresponds to Schwartz kernels. The second class corresponds to Schwartz functions in z, z' with values in the the classical pseudodifferential operators of order m' on $\mathbb{R}^{d_1}$ and with kernels having bounded support in $y+y'$. Since these kernels are actually Schwartz if the singularity at $y=y'$ is cut out, the effect of the second two terms is simply to admit the kernels which are Schwartz functions of z, z' with values in the pseudodifferential kernels of order m' on $\mathbb{R}^{d_1}$ with bounded singular support (in y) and Schwartz tails. Similarly addition of these two terms 'completes' the first term in admitting appropriate tails at infinity to ensure that

Proposition A.1. *The operators with kernels as in* (A.4) *act on* $\mathcal{S}(\mathbb{R}^{d_1+d_2})$ *and form a bifiltered algebra with the orders* $m, m' \in \mathbb{Z}$*; omitting the first term in* (A.4) *gives an ideal, as does omitting the first two terms. The filtration is delineated by two symbol maps*

$$\begin{aligned} \sigma_{m,m'}(A) &= \rho^m \rho_{\mathrm{ff}}^{m'} a_1\big|_{\rho=0} \in \mathcal{C}^\infty_c(\mathbb{R}^{d_1+d_2} \times [\mathbb{S}^{d_1+d_2-1}; \mathbb{S}^{d_1-1}]; N_{m,m'}) \\ L(A) &= (2\pi)^{-d_2} \int e^{i(z-z')\cdot\zeta}(b_1+b_2)(y, \frac{z+z'}{2}, \hat{\eta}, \zeta) d\zeta, \\ & \qquad b_1 = (\rho_{\mathrm{ff}}^m a_1)\big|_{\rho_{\mathrm{ff}}=0},\ b_2 = a_2\big|_{\mathbb{S}^{d_1-1}} \end{aligned} \tag{A.5}$$

which are homomorphisms into the algebras of functions and parameterized pseudodifferential operators on $\mathbb{R}^{d_2}$ *respectively.*

Proof. All these conclusions follow from the standard methods for proving the composition formula for pseudodifferential operators on Euclidean space, i.e. some form of stationary phase. The fact that the two symbol maps are homomorphism follows by oscillatory testing. □

Now, these objects can be transferred to a general fibration of compact manifolds by localization. Thus, the kernels in (A.4) are smooth outside the *two* submanifolds

$$\{y = y', z = z'\} \cup \{y = y'\} \tag{A.6}$$

and the singularity is determined by the Taylor series of a_1 at the boundary of X_{d_1,d_2} and the Taylor series of a_2 at the boundary of the ball. Furthermore these singularities are locally determined in the sense that the singularity on the diagonal near a point $(\bar{y}, \bar{z})$ is determined by a_1 near $(\bar{y}, \bar{z}, \eta, \zeta)$ and by a_2 near $(\bar{z}, \bar{z}, \bar{y}, \eta)$. The singularity near a point on $y = y'$ away from the diagonal is determined by a_1 and a_2 near that point $y = \bar{y}$. So, for a fibration we can associate an algebra of operators using such localizations.

Definition A.1. If $\tilde{\phi} : \widetilde{M} \longleftrightarrow M$ is a fibration of compact manifolds without boundary then $\Psi^{m',m}_{\tilde{\phi}-\mathrm{p}}(\widetilde{M}; \mathbb{E})$ for a $\mathbb{Z}_2$-graded vector bundle over $\widetilde{M}$ consists of those operators $A : \mathcal{C}^\infty(\widetilde{M}; E_+) \longrightarrow \mathcal{C}^\infty(\widetilde{M}; E_-)$ which have Schwartz kernels on $\widetilde{M}^2$ which under local trivializations of the bundles and densities are matrices of distributions which are

1. Smooth away from the fibre diagonal.
2. Near a point of the complement of the diagonal in the fibre diagonal are given by smooth functions of the fibre variables with values in the classical pseudodifferential kernels of order m' on the base.
3. Near a point of the diagonal are of the form (A.4).

As a consequence of Proposition A.1 we then conclude that the scalar operators in $\Psi^{m',m}_{\tilde{\phi}-\mathrm{p}}(\widetilde{M})$ form a bigraded algebra for $m, m' \in \mathbb{Z}$ and that there are corresponding global symbol homomorphisms giving surjective, and multiplicative, maps

$$\begin{aligned} \sigma &: \Psi^{m',m}_{\tilde{\phi}-\mathrm{p}}(\widetilde{M}) \longrightarrow \mathcal{C}^\infty(\overline{[S^*\widetilde{M}; \tilde{\phi}^* S^* M]}; N_{m',m}) \\ L &: \Psi^{m',m}_{\tilde{\phi}-\mathrm{p}}(\widetilde{M}) \longrightarrow \Psi^m(\pi^*(\widetilde{M}/M); \mathbb{E} \otimes_+ N_{m'}). \end{aligned} \tag{A.7}$$

In the first map, $N_{m',m}$ is a trivial real bundle corresponding to the coefficients $\rho^{-m'}$ and ρ^{-m} in the symbols and in the second sequence, the pseudodifferential operators act on the fibres of the lift of the fibration from M to S^*M using the projection $\pi : S^*M \to M$ and there is an additional trivial line bundle corresponding to the factor $\rho_{\mathrm{ff}}^{-m'}$. As noted, both maps are surjective, but together they are constrained precisely by the fact that the symbol of $L(A)$, as a family, is given by $\sigma(A)$ evaluated at the front face of the blow up of $S^*\widetilde{M}$.

Apart from these composition properties, and their natural generalizations to the case of families over a second fibration, it is important to note certain inclusions.

First,

$$\Psi^m(\widetilde{M};\mathbb{E}) \subset \Psi^{m,m}_{\tilde{\phi}-\mathrm{p}}(\widetilde{M};\mathbb{E}). \tag{A.8}$$

Indeed, the definition of $\Psi^{m,m}_{\tilde{\phi}-\mathrm{p}}(\widetilde{M};\mathbb{E})$ is modelled on one of the standard definitions of the usual pseudodifferential operators, so it is enough to refer back to the model case. Directly from the definition of the symbols in (A.4), the classical symbols of order m, which are just elements of $\rho^{-m}\mathcal{C}^\infty_c(\mathbb{R}^{d_1+d_2} \times \overline{\mathbb{R}^{d_1+d_2}})$, lift to product type symbols of order (m,m) under the blow up. This leads immediately to (A.8). Note that in this case, the symbol in the product-type sense is just the lift of the symbol in the usual sense (actually loosing no information by continuity) and the base family is simply again the symbol, evaluated on the lift of the cosphere bundle from the base and interpreted as acting as bundle isomorphisms (so local operators) on the fibres.

The second inclusion is of pseudodifferential operators acting on the fibres of the fibration. By definition the kernels of these operators are smooth families in the base variables, with values in the classical pseudodifferential operators on the fibres; as operators on smooth sections over the total space they are therefore of the form of a product of a classical conormal distribution with a delta section on the fibre diagonal. Working locally this reduces to (A.4) with a_1 or a_2 actually independent of η; it follows directly that

$$\Psi^m(\widetilde{M}/M;\mathbb{E}) \subset \Psi^{0,m}_{\tilde{\phi}-\mathrm{p}}(\widetilde{M};\mathbb{E}). \tag{A.9}$$

The symbol map reduces to the lift of the symbol on the fibres and the base family of such an operator is the operator itself.

The third inclusion is simply of pseudodifferential operators on the base but acting on a bundle $\mathbb{E}$ which is embedded as a subbundle of $\mathcal{C}^\infty(\widetilde{M}/M;\mathbb{F})$ for some bundle $\mathbb{F}$ over $\widetilde{M}$. This embedding corresponds to a family of smoothing projections of finite rank $\pi_\pm \subset \Psi^{-\infty}(\widetilde{M}/M;\mathbb{F})$ and the kernel can then be written, somewhat formally, as $\pi_- \cdot K(A) \cdot \pi_+$. Again this is everywhere locally of the form (A.4), with the fibre part of order $-\infty$ and it follows that

$$\Psi^m(M;\mathbb{E}) \subset \Psi^{m,-\infty}_{\tilde{\phi}-\mathrm{p}}(\widetilde{M};\mathbb{F}). \tag{A.10}$$

In this case, the operator being of principal order $-\infty$, the symbol is zero at any order, but the base family is simply $\pi_-\sigma(A)\pi_+$ acting on sections of $\mathbb{F}$.

We pass over without extensive comment the extension of this construction to define families of operators, with respect to an overall fibration, and more generally suspended families in which the cotangent variables of the base are symbolic variables in the operators. These latter parameters can always be incorporated into the operators as the duals of additional Euclidean (base) variables in which the operators are translation invariant.

Appendix B. Product-type fibred cusp operators

The discussion above of operators of product-type can be extended to fibred cusp pseudodifferential operators. In such an extension, as in the subsequent one to an adiabatic limit, the extension is based on the principle that the product-type operators above are defined through a geometric class of distributions, the product-type conormal distributions for the pair of embedded submanifolds

$$\mathrm{Diag} \subset \mathrm{Diag}_{\tilde{\phi}}\,.$$

So, to generalize these operators to another setting it is only necessary to start with a space of operators associated with the conormal distributions on an embedded submanifold, replacing Diag, and to replace these in turn by an appropriate class of product-type distributions.

This is precisely the case with the fibred cusp operators defined and discussed in [21] for a compact manifold with boundary M with a given fibration of the boundary $\Phi : \partial M \longrightarrow D$ and a choice of normal trivialization of the fibres. The latter choice can be represented by a choice of boundary defining function $x \in \mathcal{C}^\infty(M)$. Then the product M^2 on which kernels are normally defined, is replace by a blown-up version of it. Namely first the corner is blown up

$$M^2_{\mathrm{b}} = [M^2; (\partial M)^2] \tag{B.1}$$

(when the boundary is not connected all products of pairs of boundary components should be blown up.) Within the new, or front, face of this manifold with corners the fibration and choice of normal trivialization combine to define a submanifold

$$\Gamma_{\Phi\text{-}\mathrm{cu}} \subset \mathrm{ff}(M^2_{\mathrm{b}}). \tag{B.2}$$

Namely $\Gamma_{\Phi\text{-}\mathrm{cu}}$ is the fibre diagonal in the boundary variables intersected with the submanifold $\frac{x-x'}{x+x'} = 0$ where it should be observed that this *is* a smooth function on M^2_{b} and that the resulting submanifold does only depend on the data giving the fibred cusp structure. Then the kernels for fibred cusp pseudodifferential operators are simply the standard conormal distributional sections with respect to the lifted diagonal of an appropriate smooth bundle over

$$\mathrm{Diag}_{\Phi\text{-}\mathrm{cu}} \subset M^2_{\Phi\text{-}\mathrm{cu}} = [M^2_{\mathrm{b}}; \Gamma_{\Phi\text{-}\mathrm{cu}}]. \tag{B.3}$$

The lifted diagonal is an embedded p-submanifold, i.e. has a product type decomposition at the boundaries. Thus the definition of the kernels as the conormal distributions (which are also required to vanish to infinite order at boundary faces not meeting the diagonal) is meaningful.

Here we consider the case of a fibration over the manifold with boundary, $\tilde{\phi} : \widetilde{M} \longrightarrow M$ which has compact fibres without boundary. We suppose that M has a fibred cusp structure as above and that the fibres of $\tilde{\phi}$ are to be treated as part of the boundary fibres, i.e. we take the fibred cusp structure on $\widetilde{M}$ given by the fibration

$$\widetilde{\Phi} : \partial M \longrightarrow D,\ \widetilde{\Phi} = (\partial\tilde{\phi}) \circ \Phi \tag{B.4}$$

and with the normal trivialization given by lifting an admissible defining function on M.

In the construction of $M^2_{\Phi\text{-}\mathrm{cu}}$, each blow up is near the corners of M^2 and the procedure is local with respect to the fibration of the boundary. That is, if Φ is locally trivialized to a product $O \times X$, consistent with the fibred cusp structure, where O is an open set in $[0,\infty)\times\mathbb{R}^l$ then over the preimage of this set the stretched product is just

$$M^2_{\Phi\text{-}\mathrm{cu}} \simeq O^2_{\mathrm{sc}} \times X^2. \tag{B.5}$$

Here, $O^2_{\mathrm{sc}} = O^2_{\mathrm{Id}\text{-}\mathrm{cu}}$ is the stretched product in the scattering case, that is, the case where the boundary map is the identity. Thus, when the extra fibration is added it follows that, again locally near boundary points and in small open sets $U, U' \subset X$ (so that $\tilde{\phi}$ is also locally trivialized)

$$\widetilde{M}^2_{\tilde{\Phi}\text{-}\mathrm{cu}} \simeq O^2_{\mathrm{sc}} \times U \times U' \times Z^2. \tag{B.6}$$

In fact $\widetilde{M}^2_{\tilde{\Phi}\text{-}\mathrm{cu}}$ fibres over $M^2_{\Phi\text{-}\mathrm{cu}}$ with fibre which is Z^2. This shows that the geometric situation of the lifted diagonal and the lifted fibre diagonal, which is just the diagonal in $M_{\Phi\text{-}\mathrm{cu}}$, is completely analogous to the product-type setting discussed above.

Definition B.1. If $\tilde{\phi} : \widetilde{M} \longrightarrow M$ is a fibration with fibres compact manifolds without boundary over a compact manifold with fibred cusp structure then the space $\Psi^{m',m}_{\tilde{\phi}-\mathrm{p},\Phi\text{-}\mathrm{cu}}(\widetilde{M};\mathbb{E})$ is given by the space of product-type conormal distributions, as in Definition A.1, with respect to the lifted diagonal and fibred diagonal, and vanishing to infinite order at all boundary faces which do not meet these p-submanifolds.

As shown in [21], the composition properties of fibred cusp operators follow from geometric considerations. Namely if the product is interpreted as a push-forward for an appropriately defined triple space (as in [21]) then locally for the fibred cusp calculus the problem is the same uniformly up to the boundary, and hence follows from the discussion above. Thus we conclude that

$$\Psi^{m',m}_{\tilde{\phi}-\mathrm{p},\Phi\text{-}\mathrm{cu}}(\widetilde{M};\mathbb{E}) \circ \Psi^{p',p}_{\tilde{\phi}-\mathrm{p},\Phi\text{-}\mathrm{cu}}(\widetilde{M};\mathbb{F}) \subset \Psi^{m'+p',m+p}_{\tilde{\phi}-\mathrm{p},\Phi\text{-}\mathrm{cu}}(\widetilde{M};\mathbb{G}) \tag{B.7}$$

provided the product makes sense, i.e. $E_+ = F_-$, $G_+ = E_+$, $G_- = F_-$. Furthermore there are now three 'symbol homomorphisms'. Two of these are modified versions of the corresponding homomorphism for the fibred cusp calculus. Thus, the symbol map now takes values, as in (A.7), in sections of the appropriate bundle over a blown-up version of the fibred-cusp cosphere bundle

$$\sigma : \Psi^{m',m}_{\tilde{\phi}-\mathrm{p},\Phi\text{-}\mathrm{cu}}(\widetilde{M}) \longrightarrow \mathcal{C}^\infty(\overline{[{}^{\mathrm{cu}}S^*\widetilde{M}; \tilde{\phi}^{*\mathrm{cu}}S^*M]}; N_{m',m}). \tag{B.8}$$

Similarly the indicial operator, which corresponds geometrically to the restriction of the kernel to the final front face in the blown-up space, now takes values in product-type and suspended operators on the boundary

$$N : \Psi^{m',m}_{\tilde{\phi}-\mathrm{p},\Phi\text{-}\mathrm{cu}}(\widetilde{M};\mathbb{E}) \longrightarrow \Psi^{m,m'}_{\Phi-\mathrm{sus},\tilde{\phi}-\mathrm{p}}(\partial\widetilde{M};\mathbb{E}). \tag{B.9}$$

The base map can also be defined by oscillatory testing

$$L : \Psi^{m',m}_{\tilde{\phi}-\mathrm{p},\Phi\text{-}\mathrm{cu}}(\widetilde{M};\mathbb{E}) \longrightarrow \Psi^{m}_{\Phi\text{-}\mathrm{cu}}(\pi^*(\widetilde{M}/M); \mathbb{E}\otimes_+ N_{m'}). \tag{B.10}$$

Each of these maps delineates a filtration of the algebra, corresponding to the order m, the degree of boundary vanishing x and the order m'. They are separately surjective and jointly, in pairs or all together, subject only to the natural compatibility conditions

$$\sigma(N) = \sigma_\partial,\ \sigma(L) = \sigma_{\mathrm{ff}},\ N(L) = L(N). \tag{B.11}$$

Appendix C. Adiabatic limit in the fibres

A notion of the adiabatic limit of pseudodifferential operators was introduced in [20]. Namely, for a fibration of compact manifolds, for the moment without boundary, $\tilde{\phi} : \widetilde{M} \longrightarrow M$, one can consider a sense in which pseudodifferential operators on $\widetilde{M}$ degenerate, by localizing in the base variables, to become families of pseudodifferential operators on the fibres of $\tilde{\phi}$.

To do so, let $\epsilon \in [0, 1]$ be the 'adiabatic parameter.' If we consider kernels given by conormal distributions with respect to the submanifold

$$\operatorname{Diag} \times [0, 1] \subset \widetilde{M}^2 \times [0, 1] \tag{C.1}$$

we simply arrive at the ϵ-parameterized pseudodifferential operators on $\widetilde{M}$. On the other hand, if we blow up the fibre diagonal at $\epsilon = 0$, introducing

$$\widetilde{M}^2_{\tilde{\phi}\text{-ad}} = [\widetilde{M}^2 \times [0, 1]; \operatorname{Diag}_{\tilde{\phi}} \times \{0\}] \xrightarrow{\beta} \widetilde{M}^2 \times [0, 1] \tag{C.2}$$

we obtain a manifold with corners with two important boundary faces (we ignore $\epsilon = 1$ as being 'regular'), the 'old boundary' being the proper lift, $\beta^{\#}\{\epsilon = 0\}$ and the new 'front face' produced by the blow up. The diagonal has proper lift to a smooth p-submanifold

$$\beta^{\#}(\operatorname{Diag} \times [0, 1]) \subset \widetilde{M}^2_{\tilde{\phi}\text{-ad}} \tag{C.3}$$

which meets the boundary only in the front face. The space of adiabatic pseudodifferential operators is then

$$\Psi^m_{\tilde{\phi}\text{-ad}}(\widetilde{M}; \mathbb{E}) = \{K \in I^{m-\frac{1}{4}}(\widetilde{M}^2_{\tilde{\phi}\text{-ad}}, \beta^{\#}(\operatorname{Diag} \times [0, 1]); \hom(\mathbb{E}) \otimes \Omega_{\tilde{\phi}\text{-ad}}); \\ K \equiv 0 \text{ at } \beta^{\#}\{\epsilon = 0\}\}. \tag{C.4}$$

In fact a neighbourhood of $\beta^{\#}(\operatorname{Diag} \times [0, 1])$ in $\widetilde{M}^2_{\tilde{\phi}\text{-ad}}$ is diffeomorphic to a neighbourhood of $\operatorname{Diag} \times [0, 1]$ in $\widetilde{M}^2 \times [0, 1]$ so we may legitimately think of these kernels as having the 'same singularities' as the ordinary pseudodifferential families but with a different action. The (trivial) density line bundle in (C.4) takes care of factors that arise even for the identity.

These operators compose in the expected way and have two symbols. The first is the usual symbol, now taking values in sections of the appropriate bundle over the 'adiabatic cosphere bundle' (which is a bundle over $\widetilde{M} \times [0, 1]$) and giving a short exact sequence

$$\Psi^{m-1}_{\tilde{\phi}\text{-ad}}(\widetilde{M}; \mathbb{E}) \longrightarrow \Psi^m_{\tilde{\phi}\text{-ad}}(\widetilde{M}; \mathbb{E}) \xrightarrow{\sigma} S^m_{\tilde{\phi}\text{-ad}}(\widetilde{M} \times [0, 1]; \hom(\mathbb{E})). \tag{C.5}$$

Secondly there is a symbol representing the limit at $\epsilon = 0$. It is a suspended family of pseudodifferential operators on the fibres of the fibration with (symbolic) parameters in the rescaled cotangent bundle of the base

$$\epsilon \Psi^m_{\tilde{\phi}\text{-ad}}(\widetilde{M}; \mathbb{E}) \longrightarrow \Psi^m_{\tilde{\phi}\text{-ad}}(\widetilde{M}; \mathbb{E}) \xrightarrow{\tilde{\phi}\text{-ad}} \Psi^m_{\tilde{\phi}-\text{sus}}(\widetilde{M}/M; \hom(\mathbb{E})). \tag{C.6}$$

Now, in the multiplicativity construction in Section 10 we use a quite analogous construction to pass from fibred cusp operators with respect to a fibration and a boundary fibration to fibred cusp operators for the same fibration and a finer boundary fibration, that is with smaller fibres. Thus we are 'converting' some fibre variables in the boundary to base variables. The main step is to carry this out on one fibre, so we can consider the model case of a compact manifold with boundary Z with an iterated fibration of its boundary giving a commutative diagram

$$\begin{array}{ccc} \partial\widetilde{M} & \xrightarrow{\Phi} & D, \\ & \underset{\widetilde{\Phi}}{\searrow} \quad \nearrow & \\ & \widetilde{D} & \end{array} \tag{C.7}$$

so the map from $\widetilde{D}$ to D is also a fibration.

The construction of the fibred cusp calculus on X is briefly discussed above. Adding an adiabatic parameter $\epsilon \in [0,1]$ we can consider the ϵ-parameterized fibred cusp calculus with respect to $\widetilde{\Phi}$ (and some choice of fibred cusp structure) by replacing (B.3) by

$$Z^2_{\Phi\text{-}\mathrm{cu}} \times [0,1] = [Z^2_{\mathrm{b}} \times [0,1]; \Gamma_{\Phi\text{-}\mathrm{cu}} \times [0,1]]. \tag{C.8}$$

Now, at $\epsilon = 0$ we consider the finer boundary fibration given by $\widetilde{\Phi}$, with a consistent fibred cusp structure (meaning such that there is a global boundary defining function which induces both). Note that the corresponding 'lifted' fibre diagonal in the boundary is then a p-submanifold

$$\Gamma_{\widetilde{\Phi}\text{-}\mathrm{cu}} \subset \Gamma_{\Phi\text{-}\mathrm{cu}} \Longrightarrow \Gamma_{\widetilde{\Phi}\text{-}\mathrm{cu}} \cap \{\epsilon = 0\} \subset \Gamma_{\Phi\text{-}\mathrm{cu}} \times [0,1]. \tag{C.9}$$

In particular this means that the proper lift of this manifold under the blow up in (C.8) is again a p-submanifold which can be blown up, giving the new compact manifold with corners

$$\begin{aligned} Z^2_{\Phi\text{-}\mathrm{cu},\widetilde{\Phi}\text{-}\mathrm{ad}} &= [Z^2_{\Phi\text{-}\mathrm{cu}} \times [0,1]; \Gamma_{\widetilde{\Phi}\text{-}\mathrm{cu}} \cap \{\epsilon = 0\}] \\ &= [Z^2_{\mathrm{b}} \times [0,1]; \Gamma_{\Phi\text{-}\mathrm{cu}} \times [0,1], \Gamma_{\widetilde{\Phi}\text{-}\mathrm{cu}} \cap \{\epsilon = 0\}]. \end{aligned} \tag{C.10}$$

The kernels of the operators we consider will be defined on this manifold. Ignoring $\epsilon = 1$ as we shall, there are three boundary faces which meet the proper lift of the diagonal (with a factor of $[0,1]$), which as usual is a p-submanifold. Two of these are the boundary hypersurfaces produced by the blow-ups in (C.10), the first is essentially the front face of the Φ-fibred cusp calculus with an extra factor of $[0,1]$ but also blown up at $\epsilon = 0$ corresponding to the $\widetilde{\Phi}$-fibred cusp calculus. The second is the front face produced by the last blow-up over $\epsilon = 0$ and the third is the proper lift of $\epsilon = 0$, this is simply the manifold with corners $X^2_{\widetilde{\Phi}\text{-}\mathrm{cu}}$.

Now, the space of $\widetilde{\Phi}$-adiabatic, Φ-fibred cusp pseudodifferential operators on Z, $\Psi^m_{\widetilde{\Phi}\text{-ad},\Phi\text{-}\mathrm{cu}}(Z)$ is identified with the conormal distributions, of an appropriate density bundle, on $Z^2_{\Phi\text{-}\mathrm{cu},\widetilde{\Phi}\text{-ad}}$ with respect to the proper lift of the diagonal and vanishing to infinite order at all boundary faces which do not meet this lift. These operators map $\mathcal{C}^\infty(\widetilde{M}_{\mathrm{ad}}; E_+)$, to $\mathcal{C}^\infty(\widetilde{M}_{\mathrm{ad}}; E_-)$ where

$$\widetilde{M}_{\mathrm{ad}} = [\widetilde{M} \times [0,1]; \partial\widetilde{M} \times \{0\}] \tag{C.11}$$

and compose in the usual way.

Corresponding to this definition and the discussion above of boundary hypersurfaces, there are four 'symbol maps', the symbol, the indicial operator, the adiabatic symbol and the adiabatic operator.

The first two of these correspond to the symbol for the Φ-fibred cusp calculus and its indicial operator, with dependence on the parameter ϵ with a change of uniformity at $\epsilon = 0$. The third symbol is the real transition between the Φ- and $\widetilde{\Phi}$-fibred cusp calculi, and the last is simply the limiting $\widetilde{\Phi}$-fibred cusp operator itself. More precisely, the 'usual' symbol becomes

$$\text{(C.12)} \qquad \Psi^{m-1}_{\widetilde{\Phi}\text{-ad},\Phi\text{-cu}}(\widetilde{M};\mathbb{E}) \longrightarrow \Psi^{m}_{\widetilde{\Phi}\text{-ad},\Phi\text{-cu}}(\widetilde{M};\mathbb{E}) \xrightarrow{\sigma} S^{m}_{\widetilde{\Phi}\text{-ad},\Phi\text{-cu}}(\widetilde{M}_{\text{ad}};\mathbb{E}).$$

Note that the symbols here are section of the tensor product of $\hom(\mathbb{E})$ with a density bundle over the modified cosphere bundle to $\widetilde{M}_{\text{ad}}$. This coshere bundle is associated to the cotangent bundle dual to the tangent bundle of $\widetilde{M}$, rescaled near the boundary, with locally generating sections over $\widetilde{M}\times[0,1]$ near a boundary point of the corner given by the vector fields (which lift to be smooth on $\widetilde{M}_{\text{ad}}$)

$$x^2\partial_x,\ x\partial_y,\ (x^2+\epsilon^2)^{\frac12}\partial_{y'},\ \partial_z$$

where x is the normal variable, the y's are base variables for Φ, the y''s are the extra base variables for $\widetilde{\Phi}$ (so fibre variables for Φ) and the z's are fibre variables for $\widetilde{\Phi}$.

The normal, or indicial operator corresponds to the restriction of the kernel to the front face of $\widetilde{M}^2_{\Phi\text{-cu}} \times [0,1]$ after part of its boundary is blown up in the last step in the construction, (C.10). Since this can be related directly to the adiabatic construction for the fibre calculus of Φ with respect to $\widetilde{\Phi}$ it becomes a map into the corresponding adiabatic (and suspended) calculus

$$\text{(C.13)} \qquad \tilde{x}\Psi^{m}_{\widetilde{\Phi}\text{-ad},\Phi\text{-cu}}(\widetilde{M};\mathbb{E}) \longrightarrow \Psi^{m}_{\widetilde{\Phi}\text{-ad},\Phi\text{-cu}}(\widetilde{M};\mathbb{E}) \xrightarrow{N} \Psi^{m}_{\Phi-\text{sus},\widetilde{\Phi}\text{-ad}}(\partial\widetilde{M};\mathbb{E}).$$

Note that the factor $\tilde{x} \in \mathcal{C}^\infty(\widetilde{M}_{\text{ad}})$ is a defining function for the boundary after the last blow-up, it can be taken to be $x(x^2+\epsilon^2)^{-\frac12}$.

The transitional, adiabatic normal operator corresponds to the restriction of the kernel to the front face produced in the final blow-up in (C.10) and takes values in a suspended space of pseudodifferential operators on the fibres of $\widetilde{\Phi}$ giving a short exact sequence

(C.14)

$$(x^2+\epsilon^2)^{\frac12}\Psi^{m}_{\widetilde{\Phi}\text{-ad},\Phi\text{-cu}}(\widetilde{M};\mathbb{E}) \longrightarrow \Psi^{m}_{\widetilde{\Phi}\text{-ad},\Phi\text{-cu}}(\widetilde{M};\mathbb{E}) \xrightarrow{\text{ad}} \Psi^{m}_{\widetilde{\Phi}-\text{sus}(V)}(\partial\widetilde{M};\mathbb{E}),$$

where $V \to [-1,1]_\tau \times \widetilde{D}$ is is the null bundle of the restriction ${}^{\widetilde{\Phi}}T_{\partial\widetilde{M}}M \to T_{\partial\widetilde{M}}\widetilde{M}$ and

$$\tau = \frac{x-\epsilon}{x+\epsilon} \in [-1,1]$$

is a variable on the adiabatic front face. Finally, the fourth map, into the finer fibred cusp calculus gives a short exact sequence

$$\text{(C.15)} \qquad \tilde{\epsilon}\Psi^{m}_{\widetilde{\Phi}\text{-ad},\Phi\text{-cu}}(\widetilde{M};\mathbb{E}) \longrightarrow \Psi^{m}_{\widetilde{\Phi}\text{-ad},\Phi\text{-cu}}(\widetilde{M}/B;\mathbb{E}) \xrightarrow{A} \Psi^{m}_{\widetilde{\Phi}\text{-cu}}(\widetilde{M};\mathbb{E}).$$

Here $\tilde{\epsilon} = \epsilon(x^2+\epsilon^2)^{-\frac12}$ is a defining function for the limiting boundary in $\widetilde{M}_{\text{ad}}$.

Notice that

(C.16)

$$P \in \Psi^{m}_{\widetilde{\Phi}\text{-ad},\Phi\text{-cu}}(\widetilde{M}/B;\mathbb{E}),\ N(P)=0,\ \text{ad}(P)=0 \Longleftrightarrow P \in x\Psi^{m}_{\widetilde{\Phi}\text{-ad},\Phi\text{-cu}}(\widetilde{M}/B;\mathbb{E}).$$

The various compatibility conditions are then given by

$$\begin{gathered} A\circ\sigma(P)=\sigma\circ A(P),\quad \sigma\circ N(P)=N\circ\sigma(P),\quad \operatorname{ad}\circ\sigma(P)=\sigma\circ\operatorname{ad}(P),\\ \operatorname{ad}\circ N(P)=\operatorname{ad}(P)|_{\tau=-1}\quad\text{and}\quad N\circ A(P)=\operatorname{ad}(P)|_{\tau=1}\,. \end{gathered} \tag{C.17}$$

In particular there is no compatibility condition between $A(P)$ and $N(P)$.

As noted above there is in fact a fifth map corresponding to restriction to $\epsilon=1$ (or really any positive value of ϵ)

$$(1-\epsilon)\Psi^m_{\widetilde{\Phi}\text{-ad},\Phi\text{-cu}}(\widetilde{M};\mathbb{E})\longrightarrow\Psi^m_{\widetilde{\Phi}\text{-ad},\Phi\text{-cu}}(\widetilde{M}/B;\mathbb{E})\xrightarrow{\big|_{\epsilon=1}}\Psi^m_{\Phi\text{-cu}}(\widetilde{M};\mathbb{E}). \tag{C.18}$$

In some sense, $P\in\Psi^m_{\widetilde{\Phi}\text{-ad},\Phi\text{-cu}}(\widetilde{M};\mathbb{E})$ should be interpreted as a homotopy between $A(P)\in\Psi^m_{\widetilde{\Phi}\text{-cu}}(\widetilde{M};\mathbb{E})$ and $P\big|_{\epsilon=1}\in\Psi^m_{\Phi\text{-cu}}(\widetilde{M};\mathbb{E})$. When both $A(P)$ and $P\big|_{\epsilon=1}$ are Fredholm operators, one would expect them to have the same index provided P is a homotopy through Fredholm operators. The next proposition makes this statement more precise.

Proposition C.1. *If $P\in\Psi^m_{\widetilde{\Phi}\text{-ad},\Phi\text{-cu}}(\widetilde{M};\mathbb{E})$ exists with $\sigma(P)$, $N(P)$ and $\operatorname{ad}(P)$ invertible in their pseudodifferential (or symbol) calculi then $P\big|_{\epsilon=1}$ and $A(P)$ are both fully elliptic and have the same (families) index.*

Proof. The compatibility conditions between $\sigma(P)$, $N(P)$ and $\operatorname{ad}(P)$ are such that if each is invertible within the calculus of pseudodifferential operators then the inverses are compatible. Thus, under this condition a parametrix can be constructed for P. As usual, a symbolic parametrix can be improved to a full parametrix in the sense that $Q\in\Psi^{-m}_{\widetilde{\Phi}\text{-ad},\Phi\text{-cu}}(\widetilde{M};\mathbb{E}^-)$ satisfies

$$QP-\operatorname{Id}\in x^\infty\Psi^{-\infty}_{\widetilde{\Phi}\text{-ad},\Phi\text{-cu}}(\widetilde{M};E_+),\ PQ-\operatorname{Id}\in x^\infty\Psi^{-\infty}_{\widetilde{\Phi}\text{-ad},\Phi\text{-cu}}(\widetilde{M};E_-). \tag{C.19}$$

Note that we do *not* achieve vanishing at $\epsilon=0$ since we have not assumed that $A(P)$ is invertible. In fact the errors in (C.19) are just smoothing operators, smooth in ϵ and with kernels vanishing to infinite order at the boundary. Following the discussion of the index in Section 1 the index bundle (for a family of such operators) can be stabilized to a bundle over $[0,1]\times B$. Indeed, this follows from the fact that

$$x^\infty\Psi^{-\infty}_{\widetilde{\Phi}\text{-ad},\Phi\text{-cu}}(\widetilde{M};E_+)=\mathcal{C}^\infty([0,1],\dot{\Psi}^{-\infty}(\widetilde{M};E_+))$$

where $\dot{\Psi}^{-\infty}(\widetilde{M};E_+)=x^\infty\Psi^{-\infty}_{\widetilde{\Phi}\text{-cu}}(\widetilde{M};E_+)$ does not depend on the choice of the boundary fibration Φ. It follows from this that the families index at $\epsilon=0$ and $\epsilon=1$ are the same and the former is the index bundle for $A(P)$, the latter for $P\big|_{\epsilon=1}$. □

References

[1] N. Anghel, *On the index of Callias-type operators*, Geom. Funct. Anal. **3** (1993), no. 5, 431–438. MR MR1233861 (94m:58213)

[2] M.F. Atiyah, *K-theory*, Benjamin, 1967.

[3] M.F. Atiyah and R. Bott, *The index theorem for manifolds with boundary*, Proc. Sympos. Differential Anal. (1964), 175–186.

[4] M.F. Atiyah, V.K. Patodi, and I. M. Singer, *Spectral asymmetry and Riemann geometry, I*, Math. Proc. Cambridge Philos. Soc **77** (1975), 43–69.

[5] M.F. Atiyah and I.M. Singer, *The index of elliptic operators. I*, Ann. of Math. (2) **87** (1968), 484–530. MR 38 #5243

[6] ———, *The index of elliptic operators. IV*, Ann. of Math. (2) **93** (1971), 119–138. MR 43 #5554
[7] J.-M. Bismut and J. Cheeger, *Families index for manifolds with boundary, superconnections and cones*, Invent. Math. **89** (1990), 91–151.
[8] J.-M. Bismut and D. Freed, *Eta-invariants and their adiabatic limits*, J.A.M.S. **2** (1989), 33–70.
[9] B. Blackadar, *K-theory for operator algebras*, Springer, New York, 1986.
[10] J. Cheeger, *Spectral geometry of singular Riemann spaces*, J. Diff. Geom. **18** (1983), 575–651.
[11] Xianzhe Dai and Weiping Zhang, *Real embeddings and the Atiyah-Patodi-Singer index theorem for Dirac operators*, Asian J. Math. **4** (2000), no. 4, 775–794.
[12] C.L. Epstein, R. B. Melrose, and G. Mendoza, *The Heisenberg algebra, index theory and homology*, In preparation.
[13] N. Higson and J. Roe, *Analytic K-homology*, Oxford University Press, Oxford, 2000.
[14] L. Hörmander, *The analysis of linear partial differential operators. vol. 3*, Springer-Verlag, Berlin, 1985.
[15] G.G. Kasparov, *Topological invariants of elliptic operators, I: K-homology*, Math. USSR Izvestija **9** (1975), 751–792.
[16] ———, *The operator K-functor and extensions of C*-algebras*, Math. USSR Izvestija **16** (1981), 513–572.
[17] ———, *Equivariant KK-theory and the Novikov conjecture*, Invent. Math. **91** (1988), 147–201.
[18] Dan Kucerovsky, *A short proof of an index theorem*, Proc. Amer. Math. Soc. **129** (2001), no. 12, 3729–3736 (electronic). MR MR1860509 (2002g:58031)
[19] G. Luke, *Pseudodifferential operators on Hilbert bundles*, Journal of Differential Equations **12** (1972), 566–589.
[20] R. Mazzeo and R.B. Melrose, *The adiabatic limit, Hodge cohomology and Leray's spectral sequence for a fibration*, J. Differential Geom. **31** (1990), no. 1, 185–213. MR 90m:58004
[21] ———, *Pseudodifferential operators on manifolds with fibred boundaries*, Asian J. Math. **2** (1999), no. 4, 833–866.
[22] R.B. Melrose, *The Atiyah-Patodi-Singer index theorem*, A. K. Peters, Wellesley, Massachusetts, 1993.
[23] ———, *Geometric scattering theory*, Cambridge University Press, Cambridge, 1995. MR 96k:35129
[24] R.B. Melrose and V. Nistor, *Homology of pseudodifferential operators I. manifold with boundary.*
[25] R.B. Melrose and P. Piazza, *Analytic K-theory for manifolds with corners*, Adv. in Math **92** (1992), 1–27.
[26] ———, *Families of Dirac operators, boundaries and the b-calculus*, J. Differential Geom. **46** (1997), no. 1, 99–180. MR MR99a:58144
[27] ———, *An index theorem for families of Dirac operators on odd-dimensional manifolds with boundary*, J. Differential Geom. **46** (1997), no. 2, 287–334. MR MR99a:58145
[28] R.B. Melrose and F. Rochon, *Families index for pseudodifferential operators on manifolds with boundary*, IMRN **22** (2004), 1115–1141.
[29] R.B. Melrose and G. A. Uhlmann, *Lagrangian intersection and the Cauchy problem*, Comm. Pure Appl. Math. **32** (1979), no. 4, 483–519. MR 81d:58052
[30] M.E. Taylor P. Baum, R.G. Douglas, *Cycles and relative cycles in analytic K-theory*, J. Diff. Geom. **30** (1989), no. 3, 761–804.
[31] P. Piazza, *On the index of elliptic operators on manifolds with boundary*, J. Funct. Anal. **117** (1993), 308–359.
[32] F. Rochon, *Bott periodicity for fibred cusp operators*, J. Geom. Anal. **15** (2005), no. 4.
[33] Luigi Rodino, *A class of pseudo differential operators on the product of two manifolds and applications*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **2** (1975), no. 2, 287–302. MR MR0397800 (53 #1656)
[34] M.A. Shubin, *Pseudodifferential operators and spectral theory*, Springer-Verlag, Berlin-Heidelberg-New York, 1987, (Nauka, Moscow, 1978).

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