ETA FORMS AND THE ODD PSEUDODIFFERENTIAL FAMILIES INDEX

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Abstract. Let $A(t)$ be an elliptic, product-type suspended (which is to say parameter-dependant in a symbolic way) family of pseudodifferential operators on the fibres of a fibration ϕ with base Y. The standard example is $A + it$ where A is a family, in the usual sense, of first order, self-adjoint and elliptic pseudodifferential operators and $t \in \mathbb{R}$ is the 'suspending' parameter. Let $\pi_{\mathcal{A}} : \mathcal{A}(\phi) \longrightarrow Y$ be the infinite-dimensional bundle with fibre at $y \in Y$ consisting of the Schwartz-smoothing perturbations, q, making $A_y(t) + q(t)$ invertible for all $t \in \mathbb{R}$. The total eta form, $\eta_{\mathcal{A}}$, as described here, is an even form on $\mathcal{A}(\phi)$ which has basic differential which is an explicit representative of the odd Chern character of the index of the family:

(*) $$d\eta_{\mathcal{A}} = \pi_{\mathcal{A}}^* \gamma_A, \ \mathrm{Ch}(\mathrm{ind}(A)) = [\gamma_A] \in H^{\mathrm{odd}}(Y).$$

The 1-form part of this identity may be interpreted in terms of the τ invariant (exponentiated eta invariant) as the determinant of the family. The 2-form part of the eta form may be interpreted as a B-field on the K-theory gerbe for the family A with (*) giving the 'curving' as the 3-form part of the Chern character of the index. We also give 'universal' versions of these constructions over a classifying space for odd K-theory.

Introduction

Eta forms, starting with the eta invariant itself, appear as the boundary terms in the index formula for Dirac operators [2], [5], [4], [10], [11]. One aim of the present paper is to show that, with the freedom gained by working in the more general context of families of pseudodifferential operators, these forms appear as universal transgression, or connection, forms for the cohomology class of the index. That these forms arise in the treatment of boundary problems corresponds to the fact that boundary conditions amount to the explicit inversion of a suspended (or model) problem on the boundary. The odd index of the boundary family is trivial and the eta form is an explicit trivialization of it in cohomology. To keep the discussion within bounds we work here primarily in the 'odd' setting of a family of self-adjoint elliptic pseudodifferential operators, taken to be of order 1, on the fibres of a fibration of compact manifolds

(1) $$\begin{array}{ccc} Z & \text{---} & M \\ & & \downarrow \phi \\ & & Y, \end{array} \qquad A \in \Psi^1(M/Y; E), \ A^* = A, \ \text{elliptic}$$

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where a smooth, positive, fibre density on M and Hermitian inner product on the bundle have been chosen to define the adjoint. A similar discussion is possible in the more usual 'even' case.

From the fibration and pseudodifferential family an infinite-dimensional bundle of principal spaces, $\mathcal{A}(\phi)$, of invertible perturbations on each fibre, can be constructed:

$$
\begin{array}{ccccc}
G^{-\infty}_{\mathrm{sus}}(\phi;E) & \text{———} & \mathcal{A}(\phi) & \longrightarrow & \Psi^{1,1}_{\mathrm{ps}}(\phi;E) \\
& \searrow^{q_{\mathcal{A}}} & \downarrow^{p_{\mathcal{A}}} & \nearrow_{[A+it]} & \\
& & Y. & &
\end{array}
\tag{2}
$$

The vertical map here does not correspond to a principal bundle in the conventional sense since it has a non-constant bundle of structure groups, $G^{-\infty}_{\mathrm{sus}}(\phi;E)$, with fibre consisting of the invertible suspended smoothing perturbations of the identity on the corresponding fibre of ϕ. The individual groups in this bundle are flat pointed loop groups and hence are classifying for even K-theory. Despite the twisting by fibre diffeomorphisms, the homotopy group $\Pi_0(G^{-\infty}_{\mathrm{sus}}(M/Y;E))$, where $G^{-\infty}_{\mathrm{sus}}(M/Y;E)$ is the space of global smooth sections of $G^{-\infty}_{\mathrm{sus}}(\phi;E)$, is canonically identified with $\mathrm{K}^0(Y)$, see for instance [15]. In this sense $G^{-\infty}_{\mathrm{sus}}(\phi;E)$ is a 'classifying bundle' for the K-theory of Y.

Throughout this paper we use notation such as $\mathcal{B}(\phi)$ for the total space of a bundle over Y associated with a given fibration (1) and $\mathcal{B}(M/Y)$ for the corresponding space of global sections of the bundle. Thus, on the right in (2), $\Psi^{1,1}_{\mathrm{ps}}(\phi;E)$ is the space of product-type suspended pseudodifferential operators on the fibres of ϕ (and acting on sections of the bundle E) – an element of $\Psi^{1,1}_{\mathrm{ps}}(\phi,E)$ is thus a family of pseudodifferential operators acting on smooth sections of E on the fibre above a point $y\in Y$ where the parameter in the family is in $\mathbb{R}$ (the suspension variable t) with 'product symbolic' dependence on this parameter (as indicated by the suffix ps). In the diagram above, $A+it$ is such a family, although we consider a more general situation in the body of the paper.

On the total space of the structure bundle in (2) there is a deRham form, $\mathrm{Ch}_{\mathrm{even}}$, representing the even Chern character, i.e. which pulls back under any section to a representative of the Chern character of the K-class defined by that section. The eta form, $\eta_{\mathcal{A}}$, in this setting is a form on $\mathcal{A}(\phi)$, defined by regularization of the formula for the Chern character on the structure bundle (see (6.4) and (6.6)). Under the action of a section of the structure bundle, this eta form shifts by the pull back of the Chern character up to an exact term

$$
\begin{gathered}
\alpha : G^{-\infty}_{\mathrm{sus}}(M/Y;E)\times_Y \mathcal{A}(\phi)\longrightarrow \mathcal{A}(\phi),\ \alpha^*\eta_{\mathcal{A}} = \mathrm{pr}_2^*\,\eta_{\mathcal{A}} + \mathrm{pr}_1^*\,\mathrm{Ch}_{\mathrm{even}} + d\gamma, \\
\gamma \text{ a smooth form on } G^{-\infty}_{\mathrm{sus}}(M/Y;E)\times_Y \mathcal{A}(\phi), \\
\text{where } \mathrm{pr}_1 : G^{-\infty}_{\mathrm{sus}}(M/Y;E)\times_Y \mathcal{A}(\phi)\longrightarrow G^{-\infty}_{\mathrm{sus}}(M/Y;E) \text{ and} \\
\mathrm{pr}_2 : G^{-\infty}_{\mathrm{sus}}(M/Y;E)\times_Y \mathcal{A}(\phi)\longrightarrow \mathcal{A}(\phi)
\end{gathered}
\tag{3}
$$

are the natural projections. The central result below is:

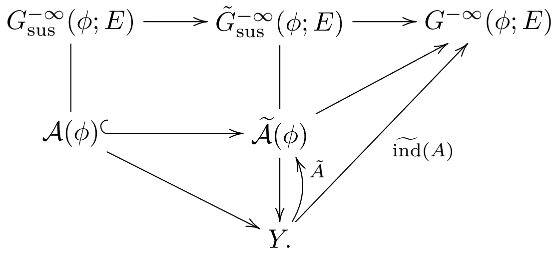
Theorem 1. *The eta form, $\eta_{\mathcal{A}}$, on $\mathcal{A}(\phi)$ has basic differential representing the odd Chern character of the index bundle of the given family A in* (1)

$$
d\eta_{\mathcal{A}} = p^*_{\mathcal{A}}\gamma_A,\ \gamma_A\in\mathcal{C}^\infty(Y;\Lambda^{odd}),\ d\gamma_A = 0,\ \mathrm{Ch}_{odd}(\mathrm{ind}(A)) = [\gamma_A]\in H^{odd}(Y). \tag{4}
$$

Once a choice of connection is made, the form $\gamma_{\mathcal{A}}$, which can be written explicitly in terms of the formal trace of [9] (see (6.13) below), gives a representative of the Chern character of the index class. For the particular case of families of Dirac operators associated to a pseudodifferential bundle, Paycha and Mickelsson, in [13], obtained a related representative of the Chern class using the Wodzicki residue instead of the formal trace.

To prove Theorem 1 we use the smooth delooping sequence for the fibration, which is the top row in the diagram

(5)



Here $\widetilde{\mathcal{A}}(\phi)$ is an extension of $\mathcal{A}(\phi)$ to a bundle of principal spaces (in the same sense as for $\mathcal{A}$) with bundle of structure groups, $\tilde{G}^{-\infty}_{\text{sus}}(\phi; E)$, the half-open (smooth-flat) loop group bundle. This has contractible fibres and hence $\widetilde{\mathcal{A}}(\phi)$ has a section $\widetilde{A}$ as indicated in (5). Taking the quotient by the original structure group, this projects to a section, $\widetilde{\text{ind}}(A)$, of $G^{-\infty}(\phi; E)$ with homotopy class giving the index in $\mathrm{K}^1(Y)$ of the family. Ultimately, (4) follows from the fact that there is a corresponding multiplicativity formula linking $\eta_{\mathcal{A}}$ to an analogous form, $\widetilde{\eta}$. Thus $\widetilde{\eta}$ is a universal transgression form for the delooping sequence, in that it restricts to the Chern character on $G^{-\infty}_{\text{sus}}(\phi; E)$ and $d\widetilde{\eta}$ is basic; it is the pull-back of the odd Chern character Ch_{odd} on $G^{-\infty}(\phi; E)$.

In §1 the smooth delooping sequence for K-theory is described. The universal Chern forms on the odd and even classifying spaces are constructed in §2; the regularization to a universal eta form on the half-open loop group is carried out in §3. The constructions of Chern forms is extended to the classifying bundle given by a fibration in §4. The bundle of invertible perturbations for a self-adjoint elliptic family, or more generally an elliptic family of product-type suspended operators, is introduced in §5 and in §6 the eta forms are generalized to this case and further extended in §7. The index formula, Theorem 1, is proved in §8. The realization of the exponentiated eta invariant, the τ-invariant, as a determinant is discussed in §9 and the adiabatic determinant of a doubly-suspended family is discussed in §10. This is used to construct a smooth and primitive form of the determinant line bundle over the even classifying space in §11. The K-theory gerbe is realized as a bundle gerbe in the sense of Murray [12] in §12 and the geometric version of this gerbe for an elliptic family is described in §13.

The relationship between the eta forms as introduced here and the eta forms of Bismut-Cheeger [3] in the Dirac case will be discussed elsewhere.

1. Delooping sequence

We first consider the 'universal' case with constructions directly over classifying spaces. Despite the infinite-dimensional base, this setting is a little simpler than the geometric case of a fibration since there is no twisting by diffeomorphisms. Let

Z be a compact manifold with $\dim Z > 0$ and let $E \longrightarrow Z$ be a complex vector bundle over it. The algebra of smoothing operators on sections of E is

$$\Psi^{-\infty}(Z;E) = \mathcal{C}^\infty(Z^2; \operatorname{Hom}(E) \otimes \Omega_R)$$

where $\Omega_R = \pi_R^*\Omega$ is the pull-back of the density bundle by the projection $\pi_R : Z \times Z \longrightarrow Z$ onto the right factor and $\operatorname{Hom}(E) = \pi_R^* E' \otimes \pi_L^* E$ with $\pi_L : Z \times Z \longrightarrow Z$ the projection onto the left factor. The product is given by the integral

$$(A \circ B)(z,z') = \int_Z A(z,z'')B(z'',z'). \tag{1.1}$$

The topological group

$$G^{-\infty}(Z;E) = \big\{A \in \Psi^{-\infty}(Z;E);\ \exists\ (\operatorname{Id} + A)^{-1} = \operatorname{Id} + B,\ B \in \Psi^{-\infty}(Z;E)\big\} \tag{1.2}$$

is an open dense subset and is classifying for odd K-theory. The 'suspended' (or flat-pointed loop) group

$$G^{-\infty}_{\mathrm{sus}}(Z;E) = \big\{A \in \mathcal{S}(\mathbb{R}_\tau \times Z^2; \operatorname{Hom}(E) \otimes \Omega_R); A(\tau) \in G^{-\infty}(Z;E)\big\} \tag{1.3}$$

is therefore classifying for even K-theory. It is an open (and dense) subspace of the Schwartz functions on $\mathbb{R}$ with values in $\Psi^{-\infty}(Z;E)$.

Thus, for any other manifold X, the sets of equivalence classes of (smooth) maps reducing to the identity outside a compact set under (smooth) homotopy through such maps are the K-groups:

$$\begin{aligned}
\mathrm{K}^1_{\mathrm{c}}(X) &= \big\{f \in \mathcal{C}^\infty(X; G^{-\infty}(Z;E)); f = \operatorname{Id} \text{ on } X \setminus K,\ K \Subset X\big\}/\sim,\\
\mathrm{K}^0_{\mathrm{c}}(X) &= \big\{f \in \mathcal{C}^\infty(X; G^{-\infty}_{\mathrm{sus}}(Z;E)); f = \operatorname{Id} \text{ on } X \setminus K,\ K \Subset X\big\}/\sim .
\end{aligned}$$

By definition, Schwartz functions are 'flat at infinity' and we introduce a larger space of functions which are Schwartz at $-\infty$ but more generally 'flat to a constant' at $+\infty$ and the corresponding group

$$\begin{aligned}
\tilde{G}^{-\infty}_{\mathrm{sus}}(Z;E) = \{A \in \mathcal{C}^\infty(\mathbb{R}_\tau \times Z^2; \operatorname{Hom}(E) \otimes \Omega_R); \lim_{\tau \to -\infty} A(\tau) = 0,\\
\frac{dA(\tau)}{d\tau} \in \mathcal{S}(\mathbb{R}_\tau \times Z^2; \operatorname{Hom}(E) \otimes \Omega_R),\ A(\tau) \in G^{-\infty}(Z;E)\ \forall\ \tau \in [-\infty,\infty]\}.
\end{aligned} \tag{1.4}$$

Thus A can be recovered from its derivative,

$$A(\tau) = \int_{-\infty}^{\tau} \frac{dA(s)}{ds} ds. \tag{1.5}$$

Moreover, there is a well-defined map 'restriction to $\tau = \infty$',

$$R_\infty : \tilde{G}^{-\infty}_{\mathrm{sus}}(Z;E) \longrightarrow G^{-\infty}(Z;E) \tag{1.6}$$

which is surjective since $G^{-\infty}(Z;E)$ is connected and a general curve between two points can be smoothed and flattened at the ends.

The delooping sequence in the present context is the short exact sequence of groups

$$G^{-\infty}_{\mathrm{sus}}(Z;E) \xrightarrow{\ \iota\ } \tilde{G}^{-\infty}_{\mathrm{sus}}(Z;E) \xrightarrow{\ R_\infty\ } G^{-\infty}(Z;E) \tag{1.7}$$

where the map to the quotient group is explicitly given by (1.6) and the flatness of the paths at $+\infty$ ensures exactness in the middle.

Lemma 1. *The group $\tilde{G}^{-\infty}_{\mathrm{sus}}(Z;E)$ is contractible.*

Proof. It is only the 'flatness at infinity' of the elements of $\tilde{G}^{-\infty}_{\mathrm{sus}}(Z;E)$ that distinguishes this result from the standard contractibility, by shortening the curve, of the pointed path space of a group. To maintain this condition during the contraction, first identify $(-\infty,\infty)$ by radial compactification with the interior of $[0,1]$. Since the singularities in the compactification are swamped by the rapid vanishing of the derivatives at the end points, this gives the alternative description of the group as

$$\begin{gathered}\tilde{G}^{-\infty}_{\mathrm{sus}}(Z;E)=\\ \{a\in\mathcal{C}^\infty([0,1]_x;G^{-\infty}(Z;E));\frac{da}{dx}\in\dot{\mathcal{C}}^\infty([0,1];\Psi^{-\infty}(Z;E)),\ a(0)=\mathrm{Id}\},\end{gathered}\tag{1.8}$$

where $\dot{\mathcal{C}}^\infty([0,1];\Psi^{-\infty}(Z;E))$ is the space of smooth functions vanishing together with all their derivatives at $x=0$ and $x=1$.

Now, let $\rho:[0,1]\longrightarrow[0,1]$ be a smooth function with $\rho(0)=0$ and $\rho(x)=1$ near $x=1$ and consider the homotopy

$$\psi_t(x)=\begin{cases}2t\rho(x), & t\in[0,\frac12],\\ \rho(x)+(t-\frac12)(x-\rho(x)), & t\in[\frac12,1],\end{cases}\tag{1.9}$$

between the constant map, $\psi_0(x)$, and the identity map $\psi_1(x)=x$. Note that $\psi_t(1)=1$ for $\frac12\le t\le 1$ and ψ_t is flat at 1 for $0\le t\le\frac12$. It follows that the composite $f(\psi_t(x))$ with $f\in\mathcal{C}^\infty([0,1])$ is flat at $x=1$ for all t if f is flat at $x=1$. Thus composition

$$\Psi_t:\tilde{G}^{-\infty}_{\mathrm{sus}}(Z;E)\ni a\longmapsto a\circ\psi_t\in\tilde{G}^{-\infty}_{\mathrm{sus}}(Z;E)\tag{1.10}$$

gives the desired deformation retraction to the identity element. □

This argument is not limited to this particular group and holds in greater generality.

2. Universal Chern forms

The group $G^{-\infty}(Z;E)$, identified as an open dense subset of $\Psi^{-\infty}(Z;E)$, is an infinite dimensional manifold modelled on the Fréchet space $\mathcal{C}^\infty(Z^2;\mathrm{Hom}(E)\otimes\Omega_R)$. We shall fix the space of smooth functions on $G^{-\infty}(Z;E)$ and more generally the smooth sections of form bundles and other tensor bundles.

First, it is natural to identify the tangent space at any point with the linear space in which the group is embedded. Then the cotangent space can be identified with its dual, $\mathcal{C}^{-\infty}(Z^2;\mathrm{Hom}(E')\otimes\Omega_L)$, the space of distributional sections, where Ω_L is the left density bundle. Thus

$$T^*_aG^{-\infty}(Z;E)=\mathcal{C}^{-\infty}(Z^2;\mathrm{Hom}(E')\otimes\Omega_L)\tag{2.1}$$

with the duality between smooth tangent and cotangent fibres given by distributional pairing. This can be written formally as bundle pairing followed by integration

$$\begin{gathered}T^*_aG^{-\infty}(Z;E)\times T_aG^{-\infty}(Z;E)\ni(\alpha,B)\longrightarrow\alpha\cdot B\in\mathbb{C},\\ \alpha\cdot B=\int_{Z^2}\alpha(z,z')B(z,z').\end{gathered}\tag{2.2}$$

Having defined the tangent and cotangent fibres at each point, the fibres of the cotensor bundles are interpreted as completed tensor products. Thus

$$(T^*)^{\otimes k}_a = \mathcal{C}^{-\infty}(Z^{2k}; \bigotimes_j \pi_j^* \operatorname{Hom}(E') \otimes \Omega_{kL}) \tag{2.3}$$

where Ω_{kL} is the tensor product of the (trivial) real line bundles on each left factor of all the pairs and the homomorphism bundle is lifted from each pair of factors.

Since $G^{-\infty}(Z;E)$ is a metric space with the topology induced from $\Psi^{-\infty}(Z;E)$, continuity for functions is immediately defined. More generally, continuity for sections of any of the tensor bundles is defined by insisting that a k-cotensor field should be a continuous map from the metric space $G^{-\infty}(Z;E)$ (or indeed any subset of it) into the distributional space (2.3) in the strong sense that it should map locally into some fixed Sobolev, hence Hilbert, space

$$H^{-N}(Z^{2k}; \bigotimes_j \pi_j^* \operatorname{Hom}(E') \otimes \Omega_{kL})$$

and should be continuous for the metric topologies. The meaning of directional derivatives is then clear. For a map to be C^1, we insist that all directional derivatives exist at each point, that they are jointly defined by an element of the next higher tensor space, i.e. distribution in two more variables, and that the resulting section of this tensor bundle is also continuous. Then infinite differentiability is defined by iteration.

The form bundles are defined, as usual, as the totally antisymmetric parts of the corresponding cotensor bundles. Smoothness as a form is smoothness as a cotensor field. The deRham differential is the map from smooth k-forms to smooth $(k+1)$-forms given in the usual way by differentiation followed by antisymmetrization.

If $F : G^{-\infty}(Z;E) \longrightarrow \mathbb{C}$ is smooth and $b \in G^{-\infty}(Z;E)$ then $L(b)^*F(a) = F(ba)$ is also smooth, as is $R(b)^*F$ defined by $R(b)^*F(a) = F(ab^{-1})$. Thus $G^{-\infty}(Z;E)$ acts on its space of smooth functions, as in the setting of finite dimensional Lie groups. These actions extend to cotensor fields and hence to forms.

The universal odd Chern character is given by a slight reinterpretation of the standard finite-dimensional formula

$$\begin{aligned} &\operatorname{Ch}_{\mathrm{odd}}(a) = \\ &\quad \frac{1}{2\pi i} \operatorname{Tr}\left(\int_0^1 a^{-1} da \exp\left(\frac{t(1-t)(a^{-1}da)^2}{2\pi i} \right) dt \right) \in \mathcal{C}^{\infty}(G^{-\infty}(Z;E); \Lambda^{\mathrm{odd}}). \end{aligned} \tag{2.4}$$

Namely expanding out the exponential in formal power series and carrying out the resulting integrals reduces this to an infinite sum

$$\operatorname{Ch}_{\mathrm{odd}}(a) = \sum_{k=0}^{\infty} c_k \operatorname{Tr}((a^{-1}da)^{2k+1}),\ c_k = \frac{1}{(2\pi i)^{k+1}} \frac{k!}{(2k+1)!}. \tag{2.5}$$

Here, each da is the identification of the tangent space at a with $\Psi^{-\infty}(Z;E)$ – so can be thought of as the differential of the identity. Thus, for any $2k+1$ elements $b_j \in \Psi^{-\infty}(Z;E)$, the evaluation on $(T_a)^{\otimes(2k+1)}$ of an individual term is

$$\begin{aligned} &\operatorname{Tr}((a^{-1}da)^{2k+1})(b_1, \dots, b_{2k+1}) = \\ &\qquad \sum_{\sigma} (-1)^{\operatorname{sign}(\sigma)} \operatorname{Tr}(a^{-1}b_{\sigma(1)} a^{-1} b_{\sigma(2)} \dots a^{-1} b_{\sigma(2k+1)}). \end{aligned} \tag{2.6}$$

The trace is well defined since the product is an element of $\Psi^{-\infty}(Z;E)$. It is also defined at each point by a distribution, as required above, and the same is true of all derivatives. Namely at each point the distribution defining this form is just the total antisymmetrization (of variables in pairs) of

$$A(z'_{2k+1}, z_1)A(z'_1, z_2)A(z'_2, z_3)\dots A(z'_{2k}, z_{2k+1}) \tag{2.7}$$

where A is the Schwartz kernel of a^{-1}. Note that while this *is* smooth in the sense described above, the kernel representing the form at a given point is not a smooth function because of the presence of the identity factors in the operators. Due to the identity

$$\frac{d}{dt}a_t^{-1} = -a^{-1}\frac{da_t}{dt}a^{-1} \tag{2.8}$$

differentiation gives a similar form, but with less symmetrization, with respect to parameters. Thus (2.6) defines a form in each odd degree.

As a result of antisymmetrization the form corresponding to (2.6) for an even power is identically zero. Moreover the computation of the deRham differential, based on the identities (2.8), $d^2a = 0$ and $d(a^{-1}daa^{-1}) = 0$ yields

$$d\operatorname{Tr}\left((a^{-1}da)^{2k+1}\right) = -\operatorname{Tr}\left((a^{-1}da)^{2k+2}\right) = 0 \Longrightarrow d\operatorname{Ch}_{\mathrm{odd}} = 0 \tag{2.9}$$

globally on $G^{-\infty}(Z;E)$. The Chern character (2.4) is universal in the sense that if $f : X \longrightarrow G^{-\infty}(Z;E)$ is any smooth map from a compact manifold X, then

$$[f^*\operatorname{Ch}_{\mathrm{odd}}] = \operatorname{Ch}_{\mathrm{odd}}([f]) \in H^{\mathrm{odd}}(X;\mathbb{C}) \tag{2.10}$$

represents the odd Chern character of the K-class defined by the homotopy class $[f]$ of f.

The abelian group structure on $\mathrm{K}^1(X)$ is derived from the non-abelian group structure on $G^{-\infty}(Z;E)$ and in particular the linearity of the odd Chern character is a consequence of the following result. Here we say that a form on a product of two (infinite-dimensional) manifolds $M_1 \times M_2$ 'has no pure terms' if it vanishes when restricted to $\{p_1\} \times M_2$ or $M_1 \times \{p_2\}$ for any points $p_1 \in M_1$ or $p_2 \in M_2$.

Proposition 1. *There is a smooth form δ_{even} on $G^{-\infty}(Z;E)\times G^{-\infty}(Z;E)$ of even degree which has no pure terms, vanishes when pulled back to the 'product diagonal' $\{(a, a^{-1})\}$ and is such that in terms of pull-back under the product map and two projections:*

$$\begin{array}{ccccc} & & G^{-\infty}(Z;E) & & \\ & & \uparrow{\scriptstyle m} & & \\ & & G^{-\infty}(Z;E)\times G^{-\infty}(Z;E) & & \\ & \swarrow^{\pi_L} & & \searrow^{\pi_R} & \\ G^{-\infty}(Z;E) & & & & G^{-\infty}(Z;E) \end{array} \tag{2.11}$$

the form in (2.4) *satisfies*

$$m^*\operatorname{Ch}_{odd} = \pi_L^*\operatorname{Ch}_{odd} + \pi_R^*\operatorname{Ch}_{odd} + d\delta_{even}. \tag{2.12}$$

Proof. For any two bundles the group $G^{-\infty}(Z;E) \oplus G^{-\infty}(Z;F)$ can be identified as the diagonal subgroup of $G^{-\infty}(Z;E\oplus F)$ and the Chern form restricted to this subgroup clearly splits as the direct sum. So, to prove (2.12) we work on $E\oplus E$ and take an homotopy which connects ab acting on the left factor of E, so as $ab\oplus\mathrm{Id}$ on $E\oplus E$, with $a\oplus b$ acting on $E\oplus E$. This can be constructed in terms of a rotation between the two factors. Thus

$$M(t)=\begin{pmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{pmatrix},\ t\in[0,\pi/2] \tag{2.13}$$

is such that

$$B(t)=M^{-1}(t)\begin{pmatrix}b & 0\\ 0 & \mathrm{Id}\end{pmatrix}M(t)\text{ satisfies }B(0)=\begin{pmatrix}b & 0\\ 0 & \mathrm{Id}\end{pmatrix},\ B(\pi/2)=\begin{pmatrix}\mathrm{Id} & 0\\ 0 & b\end{pmatrix}. \tag{2.14}$$

Using this family, consider the map

$$H:[0,1]\times G^{-\infty}(Z;E)\times G^{-\infty}(Z;E)\longmapsto A(0)B(t)\in G^{-\infty}(Z;E\oplus E). \tag{2.15}$$

It follows that the form $\alpha=H^*\operatorname{Ch}_{\mathrm{odd}}$ is a closed form on the product and hence decomposing in terms of the factor $[0,1]$,

$$\alpha=dt\wedge\alpha_1(t)+\alpha_2(t) \tag{2.16}$$

where the α_i are smooth 1-parameter families of forms on $G^{-\infty}(Z;E)\times G^{-\infty}(Z;E)$,

$$d\alpha_2=0,\ d\alpha_1=\frac{\partial}{\partial t}\alpha_2 \tag{2.17}$$

where d is now the deRham differential on $G^{-\infty}(Z;E)\times G^{-\infty}(Z;E)$. Thus, setting

$$\delta_{\mathrm{even}}=-\int_0^{\pi/2}\alpha_1(t)dt, \tag{2.18}$$

(2.12) follows.

Now, if a is held constant, $H^*\operatorname{Ch}_{\mathrm{odd}}$ is independent of a and reduces to the Chern character for $B(t)$. It follows that the individual terms in α_1 are multiples of

$$\operatorname{Tr}\Bigg(\begin{pmatrix}\mathrm{Id} & 0\\ 0 & 0\end{pmatrix}\big(\frac{\partial M(t)}{\partial t}M^{-1}(t)((db)b^{-1})^{2k}\big) \\ -\begin{pmatrix}\mathrm{Id} & 0\\ 0 & 0\end{pmatrix}\Big(M^{-1}(t)\frac{\partial M(t)}{\partial t}(b^{-1}db)^{2k}\Big)\Bigg). \tag{2.19}$$

Since $\frac{\partial M(t)}{\partial t}M^{-1}(t)$ and $M^{-1}(t)\frac{\partial M(t)}{\partial t}$ are off-diagonal this vanishes. A similar argument applies if b is held constant, so δ_{even} in (2.18) is without pure terms.

Under inversion, $a\longmapsto a^{-1}$, $\operatorname{Ch}_{\mathrm{odd}}$ simply changes sign, so under the involution $I:(a,b)\longmapsto(b^{-1},a^{-1})$ both the left side and the two Chern terms, together, on the right change sign. Thus $\delta=\delta_{\mathrm{even}}$ can be replaced by its odd part under this involution, $\frac{1}{2}(\delta-I^*\delta)$, which ensures that it vanishes when pulled back to the submanifold left invariant by I, namely $\{b=a^{-1}\}$. It still is without pure terms so the proposition is proved. □

The discussion of the suspended group $G^{-\infty}_{\mathrm{sus}}(Z;E)$ is similar. Namely the tangent space is the space of Schwartz sections $\mathcal{S}(\mathbb{R}\times Z^2;\operatorname{Hom}(E)\otimes\Omega_R)$ which can be identified, by radial compactification of the line, with $\dot{\mathcal{C}}^\infty([-1,1]\times Z^2;\operatorname{Hom}(E)\otimes\Omega_R)\subset\mathcal{C}^\infty([-1,1]\times Z^2;\operatorname{Hom}(E)\otimes\Omega_R)$, consisting of the space of smooth sections on this manifold with boundary, vanishing to infinite order at both boundaries. The dual space is then the space of Schwartz distributions $\mathcal{S}'(\mathbb{R}\times Z^2;\operatorname{Hom}(E')\otimes\Omega_L)$, or in the compactified picture the space of extendible distributional sections. Apart from these minor alterations, the discussion proceeds as before and the even Chern forms, defined by pull-back and integration are

$$\mathrm{Ch}_{\mathrm{even}} = p_*(\mathrm{ev}^* \mathrm{Ch}_{\mathrm{odd}}) \in \mathcal{C}^\infty(G^{-\infty}_{\mathrm{sus}}(Z;E);\Lambda^{\mathrm{even}}), \tag{2.20}$$

where

$$\mathrm{ev} : \mathbb{R} \times G^{-\infty}_{\mathrm{sus}}(Z;E) \ni (s,A) \longmapsto A(s) \in G^{-\infty}(Z;E) \tag{2.21}$$

is the evaluation map and p_* is the pushforward map along the fibres of the projection $p : \mathbb{R} \times G^{-\infty}_{\mathrm{sus}}(Z;E) \longrightarrow G^{-\infty}_{\mathrm{sus}}(Z;E)$ on the right factor. So, at least formally,

$$\mathrm{Ch}_{\mathrm{even}} = \frac{1}{2\pi i}\int_{\mathbb{R}}\int_0^1 \mathrm{Tr}\left((a^{-1}da)\exp\left(\frac{t(1-t)(a^{-1}da)^2}{2\pi i}\right)\right)dt, \tag{2.22}$$

where the outer integral mean integration with respect to τ of the coefficient of $d\tau$.

The analogue of Proposition 1 for the even Chern character follows from that result. Namely if we consider the corresponding product map, pointwise in the parameter, and projections:

$$\begin{array}{ccccc} & & G^{-\infty}_{\mathrm{sus}}(Z;E) & & \\ & & \uparrow{\scriptstyle m} & & \\ & & G^{-\infty}_{\mathrm{sus}}(Z;E) \times G^{-\infty}_{\mathrm{sus}}(Z;E) & & \\ & \swarrow{\scriptstyle \pi_L} & & {\scriptstyle \pi_R}\searrow & \\ G^{-\infty}_{\mathrm{sus}}(Z;E) & & & & G^{-\infty}_{\mathrm{sus}}(Z;E) \end{array} \tag{2.23}$$

then there is a smooth form δ_{odd} on the product group such that

$$m^* \mathrm{Ch}_{\mathrm{even}} = \pi_L^* \mathrm{Ch}_{\mathrm{even}} + \pi_R^* \mathrm{Ch}_{\mathrm{even}} + d\delta_{\mathrm{odd}}. \tag{2.24}$$

This odd form can be constructed from δ_{even} using the pull-back and push-forward operations for the (product) evaluation map

$$\begin{aligned} \mathrm{Ev} : \mathbb{R} \times G^{-\infty}_{\mathrm{sus}}(Z;E) \times G^{-\infty}_{\mathrm{sus}}(Z;E) \ni (\tau,a,b) \longrightarrow & \\ (a(\tau),b(\tau)) \in G^{-\infty}(Z;E) \times G^{-\infty}(Z;E) & \end{aligned} \tag{2.25}$$

as

$$\delta_{\mathrm{odd}} = -\int_{\mathbb{R}} \delta'(\tau)d\tau,\ \mathrm{Ev}^* \delta_{\mathrm{even}} = d\tau \wedge \delta'(\tau) + \delta''(\tau). \tag{2.26}$$

Since the forms are Schwartz in the evaluation parameter, the additional term

$$\int_{\mathbb{R}} \frac{\partial}{\partial \tau}\delta''(\tau) = 0 \tag{2.27}$$

and (2.24) follows; note that it does not follow from the fact that δ_{even} has no pure terms that this is true of δ_{odd} – and it is not!

Smoothness of forms in the sense discussed above certainly implies that the pull-back of such a form to a finite dimensional manifold, under a smooth map $Y \longrightarrow G^{-\infty}(Z;E)$ is smooth on Y and closed if the form on $G^{-\infty}(Z;E)$ is closed. Thus if $f : Y \longrightarrow G^{-\infty}(Z;E)$ is a representative of $[f] \in \mathrm{K}^1(Y)$ then $f^* \mathrm{Ch}_{\mathrm{odd}}$ is a sum of closed odd-degree forms on Y. The cohomology class is constant under homotopy of the map. Indeed, an homotopy between f_0 and f_1 is a map F :

$[0,1]_r \times Y \longrightarrow G^{-\infty}(Z;E)$. The fact that the Chern form pulls back to be closed shows that $F^* \operatorname{Ch}_{\text{odd}}$ is of the form

$$\begin{gathered} A(r) + dr \wedge B(r),\ d_Y A(r) = 0,\ \frac{dA(r)}{dr} = d_Y B(r) \\ \Longrightarrow A(1) - A(0) = d\int_0^1 B(r)dr. \end{gathered} \tag{2.28}$$

Thus cohomology classes in the even case are also homotopy invariant and these universal Chern forms define a map from K-theory to cohomology. This is the Chern character. The theorem of Atiyah and Hirzebruch shows that the combined even and odd Chern characters give a multiplicative isomorphism

$$\operatorname{Ch} : \big(\mathrm{K}^0(X) \oplus \mathrm{K}^1(X)\big) \otimes \mathbb{C} \longrightarrow \mathrm{H}^*(X;\mathbb{C}).$$

3. Universal eta form

As a link between the odd and even universal Chern characters defined above on the end groups in (1.7), we consider the corresponding eta form on $\tilde{G}^{-\infty}_{\text{sus}}(Z;E)$. It has the same formal definition as the even Chern character but now lifted to the larger (and contractible) group. This consists of paths in $G^{-\infty}(Z;E)$ so there is still an evaluation map

$$\widetilde{\operatorname{Ev}} : \mathbb{R} \times \tilde{G}^{-\infty}_{\text{sus}}(Z;E) \ni (s,A) \longmapsto A(s) \in G^{-\infty}(Z;E) \tag{3.1}$$

just as in (2.21).

Definition 1. The universal eta form on $\tilde{G}^{-\infty}_{\text{sus}}(Z;E)$ is defined as in (2.20) but interpreted on the group $\widetilde{G}^{-\infty}_{\text{sus}}(Z;E)$ with the evaluation map (3.1)

$$\tilde{\eta} = \tilde{p}_*\left(\widetilde{\operatorname{Ev}}^* \operatorname{Ch}_{\text{odd}}\right) \in \mathcal{C}^\infty(\widetilde{G}^{-\infty}_{\text{sus}}(Z;E);\Lambda^{\text{even}}) \tag{3.2}$$

and with $\tilde{p}_*$ the push-forward map corresponding to integration along the fibres of the projection $\tilde{p} : \mathbb{R} \times \widetilde{G}^{-\infty}_{\text{sus}}(Z;E) \longrightarrow \widetilde{G}^{-\infty}_{\text{sus}}(Z;E)$.

Integration on the fibres here is well defined since, in the integrand – which is the contraction with $\partial/\partial\tau$ – necessarily one of the terms is differentiated with respect to the the suspension parameter, which has the effect of removing the constant term at infinity. Thus the integral in (3.2) still converges rapidly.

If X is a compact smooth manifold and if $a : X \longrightarrow \widetilde{G}^{-\infty}(Z;E)$ is a smooth map, the associated eta form is

$$\eta(a) = a^*\tilde{\eta}. \tag{3.3}$$

Now, consider the diagram analogous to (2.23) but for the extended group, and hence with an additional map corresponding to restriction to $t = \infty$ in each factor:

(3.4)
$$
\begin{array}{ccccc}
 & & \tilde{G}^{-\infty}_{\text{sus}}(Z;E) & & \\
 & & \uparrow{\scriptstyle m} & & \\
 & & \tilde{G}^{-\infty}_{\text{sus}}(Z;E) \times \tilde{G}^{-\infty}_{\text{sus}}(Z;E) & & \\
 & \swarrow^{\pi_L} & \downarrow{\scriptstyle R_\infty \times R_\infty} & \searrow^{\pi_R} & \\
\tilde{G}^{-\infty}_{\text{sus}}(Z;E) & & & & \tilde{G}^{-\infty}_{\text{sus}}(Z;E) \\
 & & G^{-\infty}(Z;E) \times G^{-\infty}(Z;E). & &
\end{array}
$$

Proposition 2. *The eta form in* (3.2) *restricts to* Ch_{even} *on* $G^{-\infty}_{\text{sus}}(Z;E)$, *satisfies the identity*

$$m^*\tilde{\eta} = \pi_L^*\tilde{\eta} + \pi_R^*\tilde{\eta} + d(\tilde{\delta}_{odd}) + (R_\infty \times R_\infty)^*\delta_{even} \tag{3.5}$$

where $\tilde{\delta}_{odd}$ *is a smooth form on* $\tilde{G}^{-\infty}_{\text{sus}}(Z;E) \times \tilde{G}^{-\infty}_{\text{sus}}(Z;E)$ *which restricts to* δ_{odd} *on* $G^{-\infty}_{\text{sus}}(Z;E) \times G^{-\infty}_{\text{sus}}(Z;E)$ *and moreover* $\tilde{\eta}$ *has basic differential*

$$d\tilde{\eta} = R_\infty^* \,\mathrm{Ch}_{odd} \tag{3.6}$$

where R_∞ *is the quotient map in* (1.7).

Proof. To compute the differential of the eta form, write the pull-back under $\widetilde{\mathrm{Ev}}$ as in (2.28):

$$\widetilde{\mathrm{Ev}}^* \,\mathrm{Ch}_{\mathrm{odd}} = A(\tau) + d\tau \wedge B(\tau) \Longrightarrow dB(\tau) = \frac{dA(\tau)}{d\tau}. \tag{3.7}$$

Since

$$\tilde{\eta} = \int_{\mathbb{R}} B(\tau),\ d\tilde{\eta} = \int_{\mathbb{R}} dB(\tau) d\tau = \int_{\mathbb{R}} \frac{dA(\tau)}{d\tau} d\tau = A(\infty) = R_\infty^* \,\mathrm{Ch}_{\mathrm{odd}}\,.$$

This proves (3.6).

Similarly, as in the proof of (2.24), pulling back the corresponding additivity formula, (2.12), for the odd Chern character gives (3.5) with the additional term arising from the integral which vanishes as in (2.27) on the suspended subgroup. □

4. Geometric Chern forms

Next we pass to a discussion of the 'geometric case'. Fix a connection on the fibration (1). That is, choose a smooth splitting

$$TM = T^H M \oplus T(M/Y) \tag{4.1}$$

where the subbundle $T^H M$ is necessarily isomorphic to ϕ^*TY. Also choose a connection ∇^E on the complex vector bundle $E \longrightarrow M$. Consider the infinite-dimensional bundle

$$\mathcal{C}^\infty(\phi;E) \longrightarrow Y \tag{4.2}$$

which has fibre $\mathcal{C}^\infty(Z_y;E_y)$, $Z_y = \phi^{-1}(y)$, $E_y = E\big|_{Z_y}$ at $y \in Y$ and space of smooth global sections written $\mathcal{C}^\infty(M/Y;E)$, which is canonically identified with

$\mathcal{C}^\infty(M;E)$. The choice of connections induces a connection on $\mathcal{C}^\infty(\phi;E)$ through the covariant differential

$$\nabla^{\phi,E}_X u = \nabla^E_{X_H}\tilde{u},\ \mathcal{C}^\infty(M/Y;E) \ni u = \tilde{u} \in \mathcal{C}^\infty(M;E), \tag{4.3}$$

where X_H is the horizontal lift of $X \in \mathcal{C}^\infty(Y;TY)$. The curvature of this connection is a 2-form on the base with values in the first-order differential operators on sections of E on the fibres

$$\omega = (\nabla^{\phi,E})^2 \in \Lambda^2 Y \otimes_{\mathcal{C}^\infty(Y)} \operatorname{Diff}^1(M/Y;E). \tag{4.4}$$

This covariant differential can be extended to the bundle $\Psi^m(\phi;E)$, for each m including $m = -\infty$, which has fibre $\Psi^m(Z_y, E_y)$ at y, and space of global smooth sections $\Psi^m(M/Y;E)$ through its action on $\mathcal{C}^\infty(M;E)$:

$$\nabla^{\phi,E}Q = [\nabla^{\phi,E}, Q],\ Q \in \Psi^m(M/Y;E). \tag{4.5}$$

The curvature of the induced connection is given by the commutator action of the curvature

$$(\nabla^{\phi,E})^2 = [\omega, \cdot]. \tag{4.6}$$

Let $\pi : G^{-\infty}(\phi;E) \longrightarrow Y$ be the infinite-dimensional bundle over Y with fibre

$$G^{-\infty}(Z_y;E_y) = \left\{ \operatorname{Id}_{E_y} + Q; Q \in \Psi^{-\infty}(Z_y;E_y),\ \operatorname{Id}_{E_y} + Q \text{ is invertible} \right\}. \tag{4.7}$$

This is naturally identified with an open subbundle of $\Psi^{-\infty}(\phi;E) \subset \Psi^m(\phi;E)$ and as such has an induced covariant differential. If $\sigma \in G^{-\infty}(M/Y;E)$ is a global section, the corresponding odd Chern character is

$$\begin{gathered}\operatorname{Ch}_{\mathrm{odd}}(\sigma, \nabla^{\phi,E}) = \frac{1}{2\pi i}\operatorname{Tr}\left(\int_0^1 (\sigma^{-1}\nabla^{\phi,E}\sigma)\exp\left(\frac{w(s,\sigma,\nabla^{\phi,E})}{2\pi i}\right)ds\right),\ \text{where} \\ w(s,\sigma,\nabla) = s(1-s)(\sigma^{-1}\nabla\sigma)(\sigma^{-1}\nabla\sigma) + (s-1)\omega - s\sigma^{-1}\omega\sigma.\end{gathered} \tag{4.8}$$

Even though the curvature ω from (4.4) is not of trace class, the term $\sigma^{-1}\nabla^{\phi,E}\sigma$ is a 1-form with values in smoothing operators, the identity being annihilated by the covariant differential, so the argument of Tr is a smoothing operator.

The form in (4.8) is the pull-back under the section σ of a 'universal' odd Chern character on the total space of the bundle. To see this, first pull the bundle back to its own total space

$$\pi^* G^{-\infty}(\phi;E) \longrightarrow G^{-\infty}(\phi;E). \tag{4.9}$$

This has a tautological section

$$a : G^{-\infty}(\phi;E) \longrightarrow \pi^* G^{-\infty}(\phi;E) \tag{4.10}$$

and carries the pulled back covariant differential $\tilde{\nabla}^{\phi,E} = \pi^*\nabla^{\phi,E}$. The geometric odd Chern character on $G^{-\infty}(\phi;E)$ is

$$\begin{gathered}\operatorname{Ch}_{\mathrm{odd}}(\tilde{\nabla}^{\phi,E}) = \frac{1}{2\pi i}\operatorname{Tr}\left(\int_0^1 a^{-1}\tilde{\nabla}^{\phi,E}a\exp\left(\frac{w(s,a,\tilde{\nabla}^{\phi,E})}{2\pi i}\right)ds\right),\ \text{where} \\ w(s,a,\tilde{\nabla}^{\phi,E}) = s(1-s)(a^{-1}\tilde{\nabla}^{\phi,E}a)(a^{-1}\tilde{\nabla}^{\phi,E}a) + (s-1)\tilde{\omega} - sa^{-1}\tilde{\omega}a;\end{gathered} \tag{4.11}$$

here $\tilde{\omega} = \pi^*\omega$ is the pull-back of the curvature. This clearly has the desired universal property for smooth sections:

$$\operatorname{Ch}_{\mathrm{odd}}(\sigma, \nabla^{\phi,E}) = \sigma^* \operatorname{Ch}_{\mathrm{odd}}(\tilde{\nabla}^{\phi,E}). \tag{4.12}$$

The basic properties of the geometric Chern character are well known and discussed, for example, in [1]. In particular of course, the forms are closed. This follows from identities for the forms $w = w(s, a, \tilde{\nabla}^{\phi,E})$ and $\theta = a^{-1}\tilde{\nabla}^{\phi,E}a$ in (4.11) which will be used below. Namely the Bianchi identity for the connection implies (cf. (3.5) in [1]) that

$$\tilde{\nabla}^{\phi,E}w = s[w,\theta] \text{ and hence}$$

$$\begin{aligned}\tilde{\nabla}^{\phi,E}\left(\theta\exp(\frac{w}{2\pi i})\right) &= -\frac{dw}{ds}\exp\left(\frac{w}{2\pi i})\right) - s[\theta\exp(\frac{w}{2\pi i}),\theta], \\ &= -2\pi i\frac{d}{ds}\exp(\frac{w}{2\pi i}) + \int_0^1 [e^{\frac{(1-r)w}{2\pi i}}, \frac{dw}{ds}e^{\frac{rw}{2\pi i}}]dr - s[\theta\exp(\frac{w}{2\pi i}),\theta].\end{aligned} \tag{4.13}$$

All the commutators have vanishing trace so

$$\begin{aligned}d\operatorname{Ch}_{\mathrm{odd}} &= \frac{1}{2\pi i}\operatorname{Tr}\tilde{\nabla}^{\phi,E}\left(\int_0^1 a^{-1}\tilde{\nabla}^{\phi,E}a\exp\left(\frac{w(s,a,\tilde{\nabla}^{\phi,E})}{2\pi i}\right)ds\right) \\ &= -\operatorname{Tr}\int_0^1\frac{d}{ds}\exp\left(\frac{w(s,a,\tilde{\nabla}^{\phi,E})}{2\pi i}\right)ds = 0\end{aligned} \tag{4.14}$$

since $\operatorname{Tr}(e^{\frac{w(0)}{2\pi i}}) = \operatorname{Tr}(e^{\frac{w(1)}{2\pi i}})$.

Lemma 2. *Under the inversion map $I : G^{-\infty}(\phi, E) \ni \sigma \longmapsto \sigma^{-1} \in G^{-\infty}(\phi; E)$ the Chern character pulls back to its negative $I^*\operatorname{Ch}_{odd} = -\operatorname{Ch}_{odd}$.*

Proof. This follows directly from (4.8) since

$$w(s,\sigma^{-1},\nabla) = \sigma w(1-s,\sigma,\nabla)\sigma^{-1},\ \sigma\nabla^{\phi,E}\sigma^{-1} = -(\nabla^{\phi,E}\sigma)\sigma^{-1} \tag{4.15}$$

and the conjugation invariance of the trace. □

Furthermore, the cohomology class defined by $\operatorname{Ch}_{\mathrm{odd}}$ is additive, in the sense that

$$\operatorname{Ch}_{\mathrm{odd}}(\sigma_1\sigma_2) = \operatorname{Ch}_{\mathrm{odd}}(\sigma_1) + \operatorname{Ch}_{\mathrm{odd}}(\sigma_2) + dF. \tag{4.16}$$

Again it is useful to give a universal version of such a multiplicativity formula.

Let $(G^{-\infty}(\phi, E))^{[2]}$ be the fibre product of $G^{-\infty}(\phi, E)$ with itself as a bundle over Y. Then there are the usual three maps

$$\begin{array}{ccccc} & & G^{-\infty}(\phi,E) & & \\ & & \uparrow{\scriptstyle m} & & \\ & & (G^{-\infty}(\phi,E))^{[2]} & & \\ & \swarrow^{\pi_S} & & \searrow^{\pi_F} & \\ G^{-\infty}(\phi,E) & & & & G^{-\infty}(\phi,E), \end{array} \tag{4.17}$$

where $\pi_F(a,b) = b$, $\pi_S(a,b) = a$ and $m(a,b) = ab$.

Proposition 3. *There is a smooth form μ on $(G^{-\infty}(\phi, E))^{[2]}$ such that*

$$m^*\operatorname{Ch}_{odd} = \pi_F^*\operatorname{Ch}_{odd} + \pi_S^*\operatorname{Ch}_{odd} + d\mu. \tag{4.18}$$

Proof. This follows from essentially the same argument as used in the proof of Proposition 1. Thus, consider the bundle of groups with E replaced by $E \oplus E$ in which the original is embedded as acting on the first copy and as the identity on the second copy. As in (2.15), this action on $E \oplus E$ is homotopic under a family of 2×2 absolute rotations, i.e. not depending on the space variables, to the action on the second copy with the identity on the first. Now, $m^* \operatorname{Ch}_{\mathrm{odd}}$ is realized through the product, i.e. diagonal action on the first factor. Applying the rotations but just in the second term of this product action, the map m is homotopic to $\pi_S \oplus \pi_F$. Since the Chern character is closed, the same argument as in Proposition 1 constructs the transgression form μ as the integral of the variation along the homotopy. □

5. Odd elliptic families

Now we turn to the consideration of a given family of self-adjoint elliptic pseudodifferential operators, $A \in \Psi^1(M/Y; E)$. In fact it is not self-adjointness that we need here, but rather the consequence that $A + it$, which is a product-type family in the space $\Psi^{1,1}_{\mathrm{ps}}(M/Y; E)$, should be fully elliptic and hence invertible for large real t, $|t| > T$. See [8] and [7] for a discussion of product-suspended pseudodifferential operators. More generally we may simply start with an elliptic family in this sense, possibly of different order, $A \in \Psi^{m,l}_{\mathrm{ps}}(M/Y; E)$. It follows from the assumed full ellipticity that for each value of the parameter $y \in Y$ the set of invertible perturbations

(5.1)
$$\mathcal{A}_y = \big\{A + it + q(t); q \in \Psi^{-\infty}_{\mathrm{sus}}(Z_y; E_y),\ (A + it + q(t))^{-1} \in \Psi^{-1}(Z_y; E_y)\ \forall\ t \in \mathbb{R}\big\} \text{ or}$$
$$\mathcal{A}_y = \big\{A(t) + q(t); q \in \Psi^{-\infty}_{\mathrm{sus}}(Z_y; E_y),\ (A(t) + q(t))^{-1} \in \Psi^{-m}(Z_y; E_y)\ \forall\ t \in \mathbb{R}\big\}$$

is non-empty. This is discussed in the proof below.

Proposition 4. *If $A \in \Psi^1(M/Y; E)$ is an elliptic and self-adjoint family or $A \in \Psi^{m,l}_{\mathrm{ps}}(M/Y; E)$ is a fully elliptic product-type family then* (5.1) *defines a smooth (infinite dimensional) Fréchet subbundle $\mathcal{A}(\phi) \subset \Psi^{m,l}_{\mathrm{ps}}(\phi, E)$ (where $m = l = 1$ in the standard case) over Y with fibres which are principal spaces for the action of the bundle of groups $G^{-\infty}_{\mathrm{sus}}(\phi; E)$.*

Proof. The non-emptiness of the fibre at any point follows from standard results for the even index. Namely at each point in the base, the family $A_y(t)$ is elliptic and invertible for large $|t|$ as a consequence of the assumed full ellipticity. Thus the index of this family is an element of $\mathrm{K}^0_{\mathrm{c}}(\mathbb{R})$ and hence vanishes. For such a family there is a compactly supported, in the parameter, family $q(t)$ of smoothing operators on the fibre which is such that $A(t) + q(t)$ is invertible for all $t \in \mathbb{R}$.

Now the fact that the fibre is a principal space for the group $G^{-\infty}_{\mathrm{sus}}(Z_y; E_y)$ follows directly, since for two such perturbations q_i, $i = 1, 2$,

$$A(t) + q_1(t) = (\mathrm{Id}_{E_y} + q_{12}(t))(A(t) + q_2(t)),\ q_{12} \in \Psi^{-\infty}_{\mathrm{sus}}(Z_y; E_y) \tag{5.2}$$

and conversely. The local triviality of this bundle follows from the fact that invertibility persists under small perturbations. □

It is this bundle, $\mathcal{A}(\phi)$, which we think of as *the* index bundle since the existence of a global section is equivalent to the vanishing, in odd K-theory, of the index of the original family. Note that since we permit the orders m and l of a fully elliptic family in $\Psi^{m,l}_{\rm ps}(\phi, E)$ to take values other than 1, the index is additive in the sense that even the index bundles of two elliptic families (acting on the same bundle) compose in the obvious way.

6. Eta forms for an odd family

To define the geometric eta form, recall that it is shown above that a covariant derivative is induced on the bundle $\Psi^m(\phi; E) \longrightarrow Y$ from the connection on ϕ and the connection on E. Consider the subbundle of elliptic and invertible pseudodifferential operators, $G^m(\phi; E)$. Since product-suspended operators can be seen as one-parameter families of pseudodifferential operators, there is an evaluation map

$$\text{ev} : \mathbb{R}_\tau \times \mathcal{A}(\phi) \ni (\tau, a) \longmapsto a(\tau) \in G^m(\phi; E) \tag{6.1}$$

compatible with the bundle structure. On $G^m(\phi; E)$, consider as before the tautological bundle

$$\pi^* G^m(\phi; E) \longrightarrow G^m(\phi; E) \tag{6.2}$$

obtained by pulling back the bundle to its own total space. With $a : G^m(\phi; E) \longrightarrow \pi^* G^m(\phi; E)$ the tautological section consider the odd form

(6.3)
$$\lambda = \frac{1}{2\pi i} a^{-1}\tilde{\nabla} a \int_0^1 \exp\left(\frac{s(1-s)(a^{-1}\tilde{\nabla}a)(a^{-1}\tilde{\nabla}a) + (s-1)\tilde{\omega} - sa^{-1}\tilde{\omega}a}{2\pi i} \right) ds$$

taking values in $\pi^*\Psi^0(\phi; E)$. Here $\tilde{\nabla} = \pi^*\nabla$ and $\tilde{\omega} = \pi^*\omega$ with ω defined in (4.4).

This is *formally* the same as the argument of the trace functional in (4.11), except of course that now the section a is no longer a perturbation of the identity by a smoothing operator, but an invertible operator of order m. Nevertheless, the identities in (4.13) still hold, since they are based on the Bianchi identity. Writing the pull-back under the evaluation map as

$$\text{ev}^*(\lambda) = \lambda^t + \lambda^n \wedge d\tau, \ \ \lambda^n = \iota_{\partial \tau} \lambda \tag{6.4}$$

both the tangential and normal parts are form-valued sections of the bundle

$$\pi^*_{\mathcal{A}} \Psi^{0,0}_{\rm ps}(\phi; E) \longrightarrow \mathcal{A} \tag{6.5}$$

obtained by pulling back $\pi : \Psi^{0,0}_{\rm ps}(\phi; E) \longrightarrow Y$ to $\mathcal{A}$.

Definition 2. On the total space of the bundle $\mathcal{A}$ the (even) *geometric eta form* is

$$\eta_{\mathcal{A}} = \text{Tr}_{\rm sus}(\lambda^n) \tag{6.6}$$

where $\text{Tr}_{\rm sus}$ is the regularized trace of [9] taken fibrewise in the fibres of the bundle (6.5).

The bundle (6.5) does not have a tautological section, but

$$\pi^*_{\mathcal{A}} G^{m,l}_{\rm ps}(\phi; E) \longrightarrow \mathcal{A} \tag{6.7}$$

does, where $G^{m,l}_{\rm ps}(\phi; E) \subset \Psi^{m,l}_{\rm ps}(\phi; E)$ is the subbundle of elliptic invertible elements, with product-type pseudodifferential inverses. Denote this section by $\alpha_{\mathcal{A}} : \mathcal{A} \longrightarrow \pi^*_{\mathcal{A}} G^{m,l}_{\rm ps}(\phi; E)$. Then consider also the odd form

(6.8)
$$\tilde{\gamma}_{\mathcal{A}} = \frac{1}{2\pi i} \widetilde{\text{Tr}} \left[\int_0^1 \alpha_{\mathcal{A}}^{-1} \hat{\nabla} \alpha_{\mathcal{A}} \exp\left(\frac{s(1-s)(\alpha_{\mathcal{A}}^{-1}\hat{\nabla}\alpha_{\mathcal{A}})^2 + (s-1)\hat{\omega} - s\alpha_{\mathcal{A}}^{-1}\hat{\omega}\alpha_{\mathcal{A}}}{2\pi i} \right) ds \right]$$

where $\widetilde{\mathrm{Tr}}$ is the *formal trace* from [9] and $\hat{\nabla} = \pi_{\mathcal{A}}^* \nabla$, $\hat{\omega} = \pi_{\mathcal{A}}^* \omega$.

Proposition 5. *For an odd elliptic family of first order (so either self-adjoint or directly of product type), the exterior derivative of the geometric eta form*

$$d\eta_{\mathcal{A}} = \tilde{\gamma}_{\mathcal{A}} = \pi_{\mathcal{A}}^* \gamma_A \tag{6.9}$$

is the pull-back of a closed form on the base $\gamma_A \in \mathcal{C}^\infty(Y; \Lambda^{odd})$.

Proof. Recall first that the regularized trace is defined in [9] by taking the constant term in the asymptotic expansion of

$$\int_{-\tau}^{\tau} \int_0^{\tau_p} \cdots \int_0^{\tau_1} \mathrm{Tr}\left(\frac{\partial^p}{\partial r^p} \lambda^t(r)\right) dr d\tau_1 \cdots d\tau_p \tag{6.10}$$

as $\tau \to +\infty$. Here $p \in \mathbb{N}$ is chosen large enough so that $\frac{\partial^p}{\partial \tau^p} \lambda^t(\tau)$ is of trace class – the product-suspended property implies that high τ derivatives are of correspondingly low order in both senses. This is a trace, i.e. vanishes on commutators, but is not exact in the sense that $\widetilde{\mathrm{Tr}}(A) = \mathrm{Tr}_{\mathrm{sus}}(\frac{\partial A}{\partial \tau})$ does not necessarily vanish, but *is* determined by the asymptotic expansions of $A(\tau)$ as $\pm\tau \to \infty$ since it does vanish for smoothing Schwartz perturbations of A.

As noted above, the identities (4.13) hold for λ. For the pull-back under the evaluation map this means that modulo commutators

$$\tilde{\nabla} \lambda^n \equiv \frac{\partial}{\partial \tau} \lambda^t. \tag{6.11}$$

Now, taking p further derivatives with respect to τ gives the same identity modulo commutators where the sum of the orders of the terms becomes low as p increase. So, applying the trace functional the commutator terms vanish and it follows that

$$\mathrm{Tr}\left(\frac{\partial^p}{\partial \tau^p} \tilde{\nabla} \lambda^n(\tau)\right) \wedge d\tau = \mathrm{Tr}\left(\frac{\partial^p}{\partial \tau^p} \frac{\partial}{\partial \tau} \lambda^t(r)\right) \wedge d\tau. \tag{6.12}$$

Therefore,

$$\begin{aligned} d\eta_{\mathcal{A}} &= \mathrm{Tr}_{\mathrm{sus}}(\tilde{\nabla} \lambda^n) = \mathrm{Tr}_{\mathrm{sus}}\left(\frac{\partial}{\partial \tau} \lambda^t\right) \\ &= \widetilde{\mathrm{Tr}}(\lambda^t) = \tilde{\gamma}_{\mathcal{A}} \end{aligned} \tag{6.13}$$

by definition of the formal trace.

As already noted, the formal trace vanishes on low order perturbations so $\tilde{\gamma}_{\mathcal{A}}$ is basic, i.e. is actually the pull-back of a well-defined form γ_A on Y depending only on the initial family A. □

The index formula (4) therefore amounts to showing that the form γ_A represents the (odd) Chern character of the index of the family A in cohomology. This is difficult to approach directly, computationally, so instead we show how the index bundle $\mathcal{A}(\phi)$ can be 'trivialized' by extending the bundle of structure groups.

If $A \in \Psi_{\mathrm{ps}}^{m,l}(M/Y; E)$ is a fully elliptic family then it has an 'inverse' family which, whilst not completely well-defined, is determined up to smoothing terms. Namely the bundle $\mathcal{A}$ is locally trivial over Y and in particular has local sections. Taking a partition of unity ψ_i on Y subordinate to a cover by open set U_i over each of which there is a section A_i the inverse family can be taken to be

$$\sum_i \psi(y) A_i^{-1} \in \Psi_{\mathrm{ps}}^{-m,-l}(M/Y; E). \tag{6.14}$$

It is fully elliptic and, essentially by definition, the corresponding bundle of invertible perturbations is naturally identified with the bundle $\mathcal{A}^{-1}(\phi) \subset \Psi^{-m,-l}_{\mathrm{ps}}(\phi; E)$ consisting of the inverses of the elements of $\mathcal{A}(\phi)$.

Lemma 3. *Under the inversion map $\mathcal{A}(\phi) \longrightarrow \mathcal{A}^{-1}(\phi)$ the eta form $\eta_{\mathcal{A}^{-1}}$ on $\mathcal{A}^{-1}(\phi)$ associated with the inverse family* (6.14) *pulls back to $-\eta_{\mathcal{A}}$.*

Proof. The proof is similar to the one of Lemma 2, namely since the regularized trace vanishes on commutators, the result follows from the analog of (4.15) for the bundles $\mathcal{A}(\phi)$ and $\mathcal{A}^{-1}(\phi)$. □

We also need a variant of the multiplicative formula (4.18). For this, consider the fibre product $\mathcal{A}^{[2]}(\phi)$ of two copies of the bundle $\mathcal{A}(\phi)$ and the product map given by inversion in the second map

$$\tilde{m} : \mathcal{A}^{[2]}(\phi) \ni (A', A) \longmapsto A'A^{-1} \in G^{-\infty}_{\mathrm{sus}}(\phi; E). \tag{6.15}$$

Proposition 6. *Under pull-back under the three maps*

$$\begin{array}{ccccc} & & G^{-\infty}_{\mathrm{sus}}(\phi, E) & & \\ & & \uparrow{\scriptstyle \tilde{m}} & & \\ & & \mathcal{A}^{[2]}(\phi) & & \\ & {\scriptstyle \pi_S}\swarrow & & \searrow{\scriptstyle \pi_F} & \\ \mathcal{A}(\phi) & & & & \mathcal{A}(\phi), \end{array} \tag{6.16}$$

$$\tilde{m}^* \operatorname{Ch}_{odd} - \pi_S^* \eta_{\mathcal{A}} + \pi_F^* \eta_{\mathcal{A}} = d\delta_{\mathcal{A}} \tag{6.17}$$

for a smooth form $\delta_{\mathcal{A}}$ on $\mathcal{A}^{[2]}(\phi)$.

Proof. We perform the same deformation as in Proposition 3 and its earlier variants. Taking into account Proposition 5 and Lemma 2, it follows that the same conclusion holds except that extra terms may appear from $\tau \to \pm\infty$. These give a basic form so in place of the desired identity (6.17) we find instead that

$$\tilde{m}^* \operatorname{Ch}_{\mathrm{odd}} - \pi_S^* \eta_{\mathcal{A}} + \pi_F^* \eta_{\mathcal{A}} = d\delta'_{\mathcal{A}} + \pi^* \mu \tag{6.18}$$

where $\delta'_{\mathcal{A}}$ is smooth form on $\mathcal{A}^{[2]}(\phi)$ and μ is smooth form on the base, with $\pi : \mathcal{A}^{[2]}(\phi) \longrightarrow Y$. However, under exchange of the two factors, the left side of (6.18) changes sign, while the final, basic, term is unchanged. Thus taking the odd part of (6.18) gives (6.17). □

7. Extended eta invariant

The bundle $\mathcal{A}(\phi)$ in (5.1) is a bundle of principal spaces for the action of the fibres of $G^{-\infty}_{\mathrm{sus}}(\phi; E)$. The fibres can be enlarged to give an action of the central, contractible, group in (1.7) by setting

$$\widetilde{\mathcal{A}}_y = \tilde{G}^{-\infty}(Z_y; E) \cdot \mathcal{A}_y. \tag{7.1}$$

This is not so easily characterized additively but is the image of the quotient map on the fibre product

$$p_\sim : \widetilde{G}^{-\infty}_{\mathrm{sus}}(\phi; E) \times_Y \mathcal{A}(\phi) \longrightarrow \widetilde{\mathcal{A}}(\phi),\ p_\sim(\tilde{g}, A) = \tilde{g}A. \tag{7.2}$$

In particular there is an exact and fibrewise delooping sequence coming from (1.7):

$$\mathcal{A}(\phi) \longrightarrow \widetilde{\mathcal{A}}(\phi) \xrightarrow{\tilde{R}_\infty} \widetilde{G}^{-\infty}_{\text{sus}}(\phi; E). \tag{7.3}$$

The quotient map here can be defined in the fibre $\widetilde{\mathcal{A}}_y$ by

$$\tilde{R}_\infty(A'_y) = \lim_{\tau\to\infty} A'_y A_y^{-1} \tag{7.4}$$

where $A_y \in \mathcal{A}_y$ is any point in the fibre of $\mathcal{A}(\phi)$ over the same basepoint. Clearly the result does not depend on this choice of A_y.

The construction above of the eta form on $\mathcal{A}(\phi)$ extends to $\widetilde{\mathcal{A}}(\phi)$. Thus, the same form (6.3) pulls back under the evaluation map

$$\widetilde{\text{ev}} : \mathbb{R}_\tau \times \widetilde{\mathcal{A}}(\phi) \longrightarrow G^m(\phi; E) \tag{7.5}$$

to give

$$\widetilde{\text{ev}}^*(\lambda) = \widetilde{\lambda}^t + \widetilde{\lambda}^n \wedge d\tau. \tag{7.6}$$

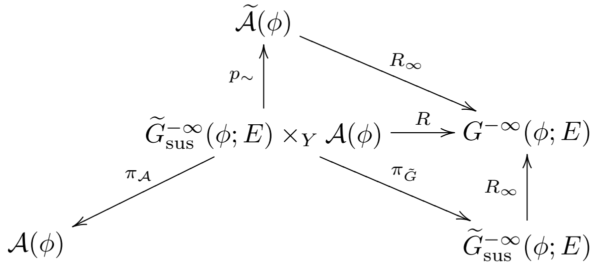
Then, extending Definition 2, set

$$\eta_{\widetilde{\mathcal{A}}} = \operatorname{Tr}_{\text{sus}}(\widetilde{\lambda}^n). \tag{7.7}$$

We use this extended bundle and eta form to analyse the invariance properties of $\eta_{\mathcal{A}}$.

Consider the fibre product with projections and quotient map

(7.8)



Proposition 7. *The diagram* (7.8) *commutes and there are smooth forms* $\tilde{\delta}_{\mathcal{A}}$ *and* $\tilde{\mu}_{\mathcal{A}}$ *respectively on the fibre product and* $G^{-\infty}(\phi; E)$ *such that the three eta forms pull back to satisfy*

$$p^*_\sim \eta_{\widetilde{\mathcal{A}}} = \pi^*_{\mathcal{A}}\eta_{\mathcal{A}} + \pi^*_{\tilde{G}}\tilde{\eta} + d\tilde{\delta}_A + R^*\tilde{\mu}_{\mathcal{A}}. \tag{7.9}$$

Proof. The commutativity of the parallelogram on the right is discussed above and defines the diagonal map, R.

The formula (7.9) is a generalization of that of Proposition 3 and the proof proceeds along the same lines. Consider the odd Chern character on $E \oplus E$. Thus, from the fibre product there are two evaluation maps, $\widetilde{\text{Ev}}$ and $\widetilde{\text{ev}}$ and we may combine these using the bundle rotation as in (2.13). This gives the two-parameter family of maps from $\widetilde{G}^{-\infty}_{\text{sus}}(\phi; E) \times_Y \mathcal{A}(\phi)$:

$$\begin{gathered}[0,1]_t \times \mathbb{R} \times \widetilde{G}^{-\infty}_{\text{sus}}(\phi; E) \times_Y \mathcal{A}(\phi) \ni (t, \tau, \tilde{g}, \widetilde{A}) \longmapsto \\ M^{-1}(t) \begin{pmatrix} \tilde{g}_y(\tau) & 0 \\ 0 & \text{Id} \end{pmatrix} M(t) \begin{pmatrix} \text{Id} & 0 \\ 0 & \widetilde{A}_y(\tau) \end{pmatrix} \in G^m(\phi, E \oplus E).\end{gathered} \tag{7.10}$$

Pulling back the form λ of (6.3) under this map and 'integrating' over $[0,1]_t \times \mathbb{R}_\tau$ gives the identity (7.9), where the τ integral is to be interpreted as part of the regularized trace. Since the form λ is closed modulo commutators, if the product decomposition of its pull-back is

$$dt \wedge d\tau \wedge \mu + dt \wedge \lambda_t + d\tau \wedge \lambda_\tau + \lambda' \tag{7.11}$$

then

$$d\mu - \frac{\partial \lambda_t}{\partial \tau} + \frac{\partial \lambda_\tau}{\partial t} \equiv 0 \tag{7.12}$$

again modulo commutators. The regularized trace and integral of the last term gives the difference of the three pulled-back eta forms and μ defines the term $\tilde{\delta}_{\mathcal{A}}$ on the fibre product.

Thus it remains to analyse the second term in (7.12). The exterior differentials in (4.11) each fall on either a factor from $\tilde{G}^{-\infty}_{\text{sus}}(\phi; E)$ or on $\mathcal{A}$. The terms involving no derivative of the first type, so the 'pure $\mathcal{A}$ part', makes no contribution, since as discussed earlier, the rotation factor $M(t)$ disappears. Thus, only terms with at least one derivative falling on the first factor, $\tilde{g}$, need be considered. This results in a smoothing operator and the regularization of the trace functional is not necessary. Then the τ integral reduces to the value of the integrand at $\tau = \infty$ which depends only on the leading term in $\mathcal{A}$ as $\tau \to \infty$, which is to say the corresponding term in A itself, and $R_\infty(\tilde{g})$. This leads to the additional term $\tilde{\mu}_A$ in (7.9). □

The left action of the groups on the fibres of the index bundle induces a contraction map

$$\begin{gathered} L : G^{-\infty}_{\text{sus}}(\phi; E) \times_Y \mathcal{A}(\phi) \longrightarrow \mathcal{A}(\phi) \\ L(s, u) = su, \ \forall\ s \in G^{-\infty}(Z_y; E_y),\ u \in \mathcal{A}_y\ \forall\ y \in Y. \end{gathered} \tag{7.13}$$

Corollary 1. *There is a smooth odd form $\tilde{\delta}$ on $G^{-\infty}_{\text{sus}}(\phi; E) \times_Y \mathcal{A}(\phi)$ such that*

$$L^* \eta_{\mathcal{A}} = \pi^*_{\mathcal{A}} \eta_{\mathcal{A}} + \pi^*_G \operatorname{Ch}_{even} + d\tilde{\delta}. \tag{7.14}$$

Proof. Restricting to the subbundle $G^{-\infty}_{\text{sus}}(\phi; E)$ in (7.8) gives a diagram which includes into it and on which R_∞ and R are trivial:

$$\begin{array}{ccccc} & & \mathcal{A}(\phi) & & \\ & & \uparrow{\scriptstyle p} & & \\ & & G^{-\infty}_{\text{sus}}(\phi; E) \times_Y \mathcal{A}(\phi) & & \\ & \swarrow{\scriptstyle \pi_{\mathcal{A}}} & & \searrow{\scriptstyle \pi_G} & \\ \mathcal{A}(\phi) & & & & G^{-\infty}_{\text{sus}}(\phi; E). \end{array} \tag{7.15}$$

Thus (7.9) restricts to this diagram with $\tilde{\mu}_A$ vanishing and $\tilde{\eta}$ reducing to $\operatorname{Ch}_{\text{even}}$ on $G^{-\infty}_{\text{sus}}(\phi; E)$. Thus (7.14) follows. □

Remark 1. If $s : U \longrightarrow G^{-\infty}_{\text{sus}}(\phi; E)$ and $\alpha : U \longrightarrow \mathcal{A}$ are sections over an open subset $U \subset Y$, then corollary 1 shows that

$$\eta(s\alpha) - \eta(\alpha) = \operatorname{Ch}_{\text{even}}(s) + d\alpha^* \tilde{\delta}. \tag{7.16}$$

8. Index formula: Proof of Theorem 1

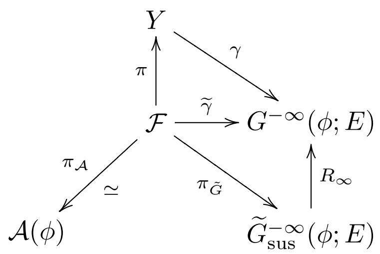
The bundle $\widetilde{\mathcal{A}}(\phi)$ has contractible fibres and hence has a global continuous section $\tilde{A} : Y \longrightarrow \widetilde{\mathcal{A}}(\phi)$; this section is easily made smooth. The inverse image of the range of this section under the vertical map, $p_\sim$, in (7.8) is a submanifold $\mathcal{F} \subset \widetilde{G}^{-\infty}_{\mathrm{sus}}(\phi; E) \times_Y \mathcal{A}(\phi)$. Indeed for each $y \in Y$ and each $B_y \in \mathcal{A}_y$ there is a unique $Q_y \in \widetilde{G}^{-\infty}_{\mathrm{sus}}(Z_y; E)$ such that

$$Q(\tau)B_y = \tilde{A}_y(\tau)$$

is the value of the section at that point. Thus, $\pi_{\mathcal{A}}$ restricts to an isomorphism from $\mathcal{F}$ to $\mathcal{A}$.

Using the section $\tilde{A} : Y \longrightarrow \tilde{\mathcal{A}}(\phi)$, we can identify $p_\sim(\mathcal{F})$ with Y so that restricting (7.8) to $\mathcal{F}$ gives the commutative diagram

(8.1)



where $\widetilde{\gamma}$ is the restriction of R to $\mathcal{F}$ and γ is the classifying map defined to make this diagram commutes.

Restricted to $\mathcal{F}$ the identity (7.9) becomes

$$\pi^*_{\mathcal{A}}\eta_{\mathcal{A}} + \pi^*_{\tilde{G}}\tilde{\eta} = \pi^*\beta_{\tilde{A}},\ \beta_{\tilde{A}} = \tilde{A}^*\widetilde{\eta}_{\tilde{A}} - \mathrm{ind}^*\,\delta_A \in \mathcal{C}^\infty(Y; \Lambda^{\mathrm{even}}). \tag{8.2}$$

From (3.6)

$$\pi^*_{\tilde{G}} d\tilde{\eta} = \widetilde{\gamma}^*(\mathrm{Ch}_{\mathrm{odd}}) \tag{8.3}$$

so pulling back to $\mathcal{A}$ under the isomorphism $\pi_{\mathcal{A}}$ gives the index formula (4):

$$d\eta_{\mathcal{A}} = -\pi^*\gamma^*(\mathrm{Ch}_{\mathrm{odd}}) + d\beta_{\tilde{A}}. \tag{8.4}$$

Since the homotopy class of the section γ represents **minus** the index class, this shows that γ_A in (6.9) represents the Chern character of the index.

9. Determinant of an odd elliptic family

The eta invariant, interpreted here as the constant term in the eta form (there is a factor of 2 compared to the original normalization of Atiyah, Patodi and Singer) is a normalized log-determinant. In the universal case, for the classifying spaces, $\tilde{\eta}^0$ is a well-defined function on $\tilde{G}^{-\infty}_{\mathrm{sus}}(Z; E)$ and then

$$\det(g) = \exp(2\pi i\tilde{\eta}^0) \text{ is the Freholm determinant on } G^{-\infty}(Z; E). \tag{9.1}$$

In the geometric case essentially the same result is true.

Proposition 8. *For $A \in \Psi^{m,k}_{\mathrm{ps}}(M/Y; E)$ a fully elliptic family of product-type operators on the fibres of a fibration,*

$$\tau(A) = \exp(2\pi i\eta^0_{\mathcal{A}}) \in \mathcal{C}^\infty(Y; \mathbb{C}^*), \tag{9.2}$$

where $\eta^0_{\mathcal{A}}$ *is the constant term in* (6.6)*, is a multiplicative function on fully elliptic operators on a fixed bundle,*

$$\tau(AB) = \tau(A)\tau(B),\ A \in \Psi^{m,k}_{\mathrm{ps}}(M/Y;E), B \in \Psi^{m',k'}_{\mathrm{ps}}(M/Y;E) \tag{9.3}$$

which is constant under smoothing perturbation and which represents the class associated to $\operatorname{ind}(A) \in \mathrm{K}^1(B)$ *in* $\mathrm{H}^1(Y;\mathbb{Z})$.

Proof. Theorem 1 shows that $d\eta^0_{\mathcal{A}}$ defined in principle on $\mathcal{A}$, the bundle of invertible perturbations of a given fully elliptic family A, is basic and represents the first odd Chern class of the index. For the zero form part, (7.14) implies true multiplicativity under the action of $G^{-\infty}_{\mathrm{sus}}(\phi;E)$, with the zero form part of $\mathrm{Ch}_{\mathrm{even}}$ being the numerical index. Thus indeed the tau invariant in (9.2) is a well defined function on the base which represents the first odd Chern class in integral cohomology.

Full multiplicativity follows as in [9]. □

10. Doubly suspended determinant

Let $G^{-\infty}_{\mathrm{sus}(2)}(Z;E)$ be the double flat-smooth loop group. Thus, its elements are Schwartz functions $a : \mathbb{R}^2 \longrightarrow \Psi^{-\infty}(Z;E)$ such that $\operatorname{Id} + a(t,\tau)$ is invertible for each $(t,\tau) \in \mathbb{R}^2$. Let

$$G^{-\infty}_{\mathrm{sus}(2)}(Z;E)[\epsilon/\epsilon^2] = G^{-\infty}_{\mathrm{sus}(2)}(Z;E) \oplus \mathcal{S}(\mathbb{R}^2;\Psi^{-\infty}(Z;E)) \tag{10.1}$$

be the group with the truncated $*$ (or Moyal) product obtained as in [8] by adiabatic limit from the isotropic product on $\mathcal{S}(\mathbb{R}^2;\Psi^{-\infty}(Z;E))$ and then passing to the quotient by terms of order ϵ^2. Explicitly this product is

$$(a_0+\epsilon a_1)[*](b_0+\epsilon b_1) = a_0b_0+\epsilon\left(a_0b_1 + a_1b_0 - \frac{1}{2i}\left(\frac{\partial a_0}{\partial t}\frac{\partial b_0}{\partial \tau} - \frac{\partial a_0}{\partial \tau}\frac{\partial a_0}{\partial t}\right)\right) \tag{10.2}$$

where the underlying product is in $\Psi^{-\infty}(Z;E)$.

As shown in [8], in the adiabatic limit the Fredholm determinant, for operators on $Z \times \mathbb{R}$, induces the 'adiabatic determinant'

$$\det_{\mathrm{ad}} : G^{-\infty}_{\mathrm{sus}(2)}(Z;E)[\epsilon/\epsilon^2] \longrightarrow \mathbb{C}^*,\ \det_{\mathrm{ad}}(g_1g_2) = \det_{\mathrm{ad}}(g_1)\det_{\mathrm{ad}}(g_2), \tag{10.3}$$

which, as in the unsuspended case, generates the 1-dimensional integral cohomology of $G^{-\infty}_{\mathrm{sus}(2)}(Z;E)[\epsilon/\epsilon^2]$ – which is classifying for odd K-theory. Note that there is no such multliplicative function on the leading group, without the first order (in ϵ) 'correction' terms in (10.2).

To define the adiabatic determinant, one needs to consider the adiabatic trace on $\Psi^{-\infty}_{\mathrm{sus}(2)}(Z;E)[\epsilon/\epsilon^2]$ defined by

$$\operatorname{Tr}_{\mathrm{ad}}(a) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \operatorname{Tr}_Z(a_1(t,\tau))dtd\tau,\ a = a_0 + \epsilon a_1 \in \Psi^{-\infty}_{\mathrm{sus}(2)}(Z;E)[\epsilon/\epsilon^2] \tag{10.4}$$

with $a_0, a_1 \in \Psi^{-\infty}_{\mathrm{sus}(2)}(Z;E)$. A special case of Lemma 4 below shows that this is a trace functional

$$\operatorname{Tr}_{\mathrm{ad}}(a * b - b * a) = 0,\ \forall\ a, b \in \Psi^{-\infty}_{\mathrm{sus}(2)}(Z;E)[\epsilon/\epsilon^2] \tag{10.5}$$

Consider the 1-form

$$\alpha(a) = \operatorname{Tr}_{\mathrm{ad}}(a^{-1} * da) \text{ on } G^{-\infty}(Z;E)[\epsilon/\epsilon^2], \tag{10.6}$$

where the inverse of $a \in G^{-\infty}_{\mathrm{sus}(2)}(Z;E)[\epsilon/\epsilon^2]$ is with respect to the the truncated $*$-product

$$a^{-1} = a_0^{-1} - \epsilon a_0^{-1}\left(a_1 + \frac{1}{2i}\left(\frac{\partial a_0}{\partial t}a_0^{-1}\frac{\partial a_0}{\partial \tau} - \frac{\partial a_0}{\partial \tau}a_0^{-1}\frac{\partial a_0}{\partial t}\right)\right)a_0^{-1}. \tag{10.7}$$

The adiabatic determinant is then defined by

$$\det_{\mathrm{ad}}(g) = \exp\left(\int_{[0,1]} \gamma^*\alpha\right) \tag{10.8}$$

where $\gamma;[0,1] \longrightarrow G^{-\infty}_{\mathrm{sus}(2)}(Z;E)[\epsilon/\epsilon^2]$ is any smooth path from the identity to g. Since the integral of α along a loop gives an integer multiple of $2\pi i$ (see for instance proposition 4.4 in [7]), this definition does not depend on the choice of γ. From (10.5),

$$\alpha(ab) = \mathrm{Tr}_{\mathrm{ad}}((a*b)^{-1}d(a*b)) = \mathrm{Tr}_{\mathrm{ad}}(a^{-1}*da) + \mathrm{Tr}_{\mathrm{ad}}(b^{-1}*db), \tag{10.9}$$

and hence

$$m^*\alpha = \pi_L^*\alpha + \pi_R^*\alpha \tag{10.10}$$

where

$$m : G^{-\infty}_{\mathrm{sus}(2)}(Z;E)[\epsilon/\epsilon^2] \times G^{-\infty}_{\mathrm{sus}(2)}(Z;E)[\epsilon/\epsilon^2] \ni (a,b) \longmapsto a[*]b \in G^{-\infty}_{\mathrm{sus}(2)}(Z;E)[\epsilon/\epsilon^2]. \tag{10.11}$$

is the composition given by the truncated $*$-product while π_L and π_R are the projections on the left and right factor. The multiplicativity of the adiabatic determinant follows directly from (10.10).

11. The determinant line bundle

We next describe the construction, and especially primitivity, of the determinant line bundle over a smooth classifying group for even K-theory.

Definition 3. A *primitive line bundle* over a (Fréchet-Lie) group

$$\begin{array}{c}\mathcal{L}\\ \big\downarrow\\ \mathcal{G}\end{array} \tag{11.1}$$

is a smooth, and locally trivial, line bundle equipped with an isomorphism of the line bundles

$$\begin{gathered}\pi_L^*\mathcal{L}\otimes\pi_R^*\mathcal{L} \xrightarrow{\simeq} m^*\mathcal{L} \text{ over } \mathcal{G}\times\mathcal{G},\\ \pi_L : \mathcal{G}\times\mathcal{G} \ni (a,b) \longrightarrow a \in \mathcal{G},\ \pi_R : \mathcal{G}\times\mathcal{G} \ni (a,b) \longrightarrow b \in \mathcal{G},\\ m : \mathcal{G}\times\mathcal{G} \ni (a,b) \longrightarrow ab \in \mathcal{G}\end{gathered} \tag{11.2}$$

which is associative in the sense that for any three elements, a, b, $c \in \mathcal{G}$, the two induced isomorphisms

$$
\begin{array}{ccc}
 & \mathcal{L}_{ab} \otimes \mathcal{L}_c & \\
\nearrow & & \searrow \\
\mathcal{L}_a \otimes \mathcal{L}_b \otimes \mathcal{L}_c & \dashrightarrow & \mathcal{L}_{abc} \\
\searrow & & \nearrow \\
 & \mathcal{L}_a \otimes \mathcal{L}_{bc} &
\end{array}
\tag{11.3}
$$

are the same.

For the *reduced* classifying group, $G^{-\infty}_{\mathrm{sus,ind}=0}(Z;E)$, a construction of the determinant line bundle, with this primitivity property, was given in [8], although only in the 'geometric case'. A variant of the construction there, also depending heavily on the properties of the suspended determinant but using instead the 'dressed' delooping sequence (for the loop group)

$$
G^{-\infty}_{\mathrm{sus}(2)}(Z;E)[\epsilon/\epsilon^2] \longrightarrow \tilde{G}^{-\infty}_{\mathrm{sus}(2)}(Z;E)[\epsilon/\epsilon^2] \longrightarrow G^{-\infty}_{\mathrm{sus,ind}=0}(Z;E) \tag{11.4}
$$

again constructs the determinant line bundle, with primitivity condition, over the component of the identity in the loop group. In this section, by modifying an idea from the book of Pressley and Segal, [14], we show how to extend this primitive line bundle to the whole of the classifying group.

In (11.4) the central, contractible, group is based on the half-open but smooth-flat loop group:

$$
\begin{aligned}
\tilde{G}^{-\infty}_{\mathrm{sus}(2)}(Z;E) = \Big\{ & \tilde{a} : \mathbb{R}^2_{(t,\tau)} \longrightarrow \Psi^{-\infty}(Z;E);\ \lim_{t\to-\infty} \tilde{a}(t,\tau) = 0, \\
& \frac{\partial \tilde{a}}{\partial t} \in \mathcal{S}(\mathbb{R}^2;\Psi^{-\infty}(Z;E)),\ \tilde{a}(t,\tau),\ \tilde{a}(\infty,\tau) \in G^{-\infty}(Z;E)\ \forall\ t,\tau \in \mathbb{R} \Big\}.
\end{aligned}
\tag{11.5}
$$

Note that automatically, $\lim_{\tau\to\infty} \tilde{a}(t,\tau) = 0$ for all $t \in [-\infty,\infty]$. This group has an extension with the product having the same 'correction term' given by the Poisson bracket on $\mathbb{R}^2$ as in (10.1):

$$
\tilde{G}^{-\infty}_{\mathrm{sus}(2)}(Z;E)[\epsilon/\epsilon^2] = \tilde{G}^{-\infty}_{\mathrm{sus}(2)}(Z;E) \oplus \Psi^{-\infty}_{\mathrm{sus}(2)}(Z;E) \tag{11.6}
$$

where the additional terms at level ϵ are just Schwartz functions valued in the smoothing operators without any additional invertibility. Note that the term in the product involving the Poisson bracket always leads to a Schwartz function on $\mathbb{R}^2$, since one factor is differentiated with respect to t. Thus $\tilde{G}^{-\infty}_{\mathrm{sus}(2)}(Z;E)[\epsilon/\epsilon^2]$ is again contractible, with just the addition of a lower order 'affine' term.

To expand the quotient group to the whole classifying group, choose one element $s \in G^{-\infty}_{\mathrm{sus}}(Z;E)$ of index 1. Then s^j is in the component of index j so each element $a \in G^{-\infty}_{\mathrm{sus}}(Z;E)$ can be connected by a curve, and hence a flat-smooth loop, to s^j

for precisely one j. The group in (11.5) may then be further enlarged to

$$\mathcal{D}^{-\infty}_{\mathrm{sus}(2)}(Z;E) = \Big\{ \tilde{a} : \mathbb{R}^2_{(t,\tau)} \longrightarrow \Psi^{-\infty}(Z;E); \frac{\partial \tilde{a}}{\partial t} \in \mathcal{S}(\mathbb{R}^2; \Psi^{-\infty}(Z;E)), \\ \lim_{t\to -\infty} \tilde{a}(t,\tau) = s^j \text{ for some } j,\ \tilde{a}(t,\tau),\ \tilde{a}(\infty,\tau) \in G^{-\infty}(Z;E)\ \forall\ t,\tau \in \mathbb{R} \Big\}. \tag{11.7}$$

This expanded group has countably many components, labelled by j, and the restriction map to $t = \infty$ is a surjection to $G^{-\infty}_{\mathrm{sus}}(Z;E)$. Thus, after adding the same affine lower order terms, (11.4) is replaced by the new short exact sequence

$$G^{-\infty}_{\mathrm{sus}(2)}(Z;E)[\epsilon/\epsilon^2] \hookrightarrow \mathcal{D}^{-\infty}_{\mathrm{sus}(2)}(Z;E)[\epsilon/\epsilon^2] \xrightarrow{\tilde{R}_\infty} G^{-\infty}_{\mathrm{sus}}(Z;E). \tag{11.8}$$

The central group is no longer contractible, although each of its connected component is. However the 1-form α in (10.6) can be extended to give a smooth 1-form on $\mathcal{D}^{-\infty}_{\mathrm{sus}(2)}(Z;E)[\epsilon/\epsilon^2]$. Indeed, the adiabatic trace has an obvious extension to a functional on

$$\widetilde{\Psi}^{-\infty}_{\mathrm{sus}(2)}(Z;E)[\epsilon/\epsilon^2] = \widetilde{\Psi}_{\mathrm{sus}(2)}(Z;E) \oplus \Psi^{-\infty}_{\mathrm{sus}(2)}(Z;E) \tag{11.9}$$

where

$$\widetilde{\Psi}_{\mathrm{sus}(2)}(Z;E) = \Big\{ a \in \mathcal{C}^\infty(\mathbb{R}^2; \Psi^{-\infty}(Z;E); \frac{\partial a}{\partial t} \in \mathcal{S}(\mathbb{R}^2; \Psi^{-\infty}(Z;E)), \\ \lim_{t\to-\infty} a(t,\tau) = 0 \Big\}. \tag{11.10}$$

Namely

$$\widetilde{\mathrm{Tr}}_{\mathrm{ad}}(a) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \mathrm{Tr}_Z(a_1) dt d\tau,\ a = a_0 + \epsilon a_1 \in \widetilde{\Psi}^{-\infty}_{\mathrm{sus}(2)}(Z;E)[\epsilon/\epsilon^2] \tag{11.11}$$

Thus, on $\mathcal{D}^{-\infty}_{\mathrm{sus}(2)}(Z;E)[\epsilon/\epsilon^2]$, one can consider the smooth 1-form

$$\widetilde{\alpha}(a) = \frac{1}{2} \widetilde{\mathrm{Tr}}_{\mathrm{ad}}(a^{-1} * da + da * a^{-1}) \tag{11.12}$$

which restricts to α on $G^{-\infty}_{\mathrm{sus}(2)}(Z;E)[\epsilon/\epsilon^2]$.

Lemma 4. *For $a = a_0 + \epsilon a_1$ and $b = b_0 + \epsilon b_1$ in $\widetilde{\Psi}^{-\infty}_{\mathrm{sus}(2)}(Z;E)[\epsilon/\epsilon^2]$,*

$$\widetilde{\mathrm{Tr}}_{\mathrm{ad}}(a * b - b * a) = \frac{1}{2\pi i} \int_{\mathbb{R}} \mathrm{Tr}_Z \left(\frac{\partial a_0}{\partial \tau}(\infty,\tau) b_0(\infty,\tau) \right) d\tau.$$

In particular, this trace defect vanishes if $a, b \in \Psi^{-\infty}_{\mathrm{sus}(2)}(Z;E)[\epsilon/\epsilon^2]$.

Proof. By definition of the truncated $*$-product and using the trace property of Tr_Z,

$$\widetilde{\mathrm{Tr}}_{\mathrm{ad}}(a * b - b * a) = -\frac{1}{2\pi i} \int_{\mathbb{R}^2} \mathrm{Tr}_Z \left(\frac{\partial a_0}{\partial t} \frac{\partial b_0}{\partial \tau} - \frac{\partial a_0}{\partial \tau} \frac{\partial b_0}{\partial t} \right) dt d\tau. \tag{11.13}$$

Integrating by parts the first term on the right,

$$
\begin{aligned}
\int_{\mathbb{R}^2} \operatorname{Tr}_Z \left(\frac{\partial a_0}{\partial t} \frac{\partial b_0}{\partial \tau} \right) dt d\tau &= -\int_{\mathbb{R}^2} \operatorname{Tr}_Z \left(\frac{\partial^2 a_0}{\partial \tau \partial t} b_0 \right) dt d\tau \\
&= -\int_{\mathbb{R}^2} \frac{\partial}{\partial t} \left(\operatorname{Tr}_Z \left(\frac{\partial a_0}{\partial \tau} b_0 \right) \right) dt d\tau + \int_{\mathbb{R}^2} \operatorname{Tr}_Z \left(\frac{\partial a_0}{\partial \tau} \frac{\partial b_0}{\partial t} \right) dt d\tau.
\end{aligned}
\tag{11.14}
$$

Thus,

$$
\begin{aligned}
\operatorname{Tr}_{\mathrm{ad}}(a * b - b * a) &= \frac{1}{2\pi i} \int_{\mathbb{R}^2} \frac{\partial}{\partial t} \left(\operatorname{Tr}_Z \left(\frac{\partial a_0}{\partial \tau} b_0 \right) \right) dt d\tau, \\
&= \frac{1}{2\pi i} \int_{\mathbb{R}} \operatorname{Tr}_Z \left(\frac{\partial a_0}{\partial \tau}(\infty, \tau) b_0(\infty, \tau) \right) d\tau.
\end{aligned}
\tag{11.15}
$$

□

Proposition 9. *Under the maps on the product*

$$
\begin{array}{ccccc}
 & & \mathcal{D}^{-\infty}_{\mathrm{sus}(2)}(Z;E)[\epsilon/\epsilon^2] & & \\
 & & \uparrow m & & \\
 & & \mathcal{D}^{-\infty}_{\mathrm{sus}(2)}(Z;E)[\epsilon/\epsilon^2] \times \mathcal{D}^{-\infty}_{\mathrm{sus}(2)}(Z;E)[\epsilon/\epsilon^2] & & \\
 & \swarrow \pi_L & \downarrow \tilde{R}_\infty \times \tilde{R}_\infty & \searrow \pi_R & \\
\mathcal{D}^{-\infty}_{\mathrm{sus}(2)}(Z;E)[\epsilon/\epsilon^2] & & & & \mathcal{D}^{-\infty}_{\mathrm{sus}(2)}(Z;E)[\epsilon/\epsilon^2] \\
 & & (G^{-\infty}_{\mathrm{sus}}(Z,E))^2 & &
\end{array}
\tag{11.16}
$$

the 1*-form* $\tilde{\alpha}$ *in* (11.12) *satisfies*

$$
m^* \tilde{\alpha} = \pi_L^* \tilde{\alpha} + \pi_R^* \tilde{\alpha} + (\tilde{R}_\infty \times \tilde{R}_\infty)^* \delta,
\tag{11.17}
$$

with

$$
\delta(a,b) = -\frac{1}{4\pi i} \int_{\mathbb{R}} \operatorname{Tr}_Z \left(a^{-1}(da) \frac{\partial b}{\partial \tau} b^{-1} - (db) b^{-1} a^{-1} \frac{\partial a}{\partial \tau} \right) d\tau
$$

on $G^{-\infty}_{\mathrm{sus}}(Z;E) \times G^{-\infty}_{\mathrm{sus}}(Z;E)$.

Proof. If $a, b \in \mathcal{D}^{-\infty}_{\mathrm{sus}(2)}(Z;E)[\epsilon/\epsilon^2]$, the trace-defect formula of Lemma 4 gives

(11.18)
$$\begin{aligned}\widetilde{\mathrm{Tr}}_{\mathrm{ad}}((a*b)^{-1}*d(a*b)) &= \widetilde{\mathrm{Tr}}_{\mathrm{ad}}(b^{-1}*a^{-1}*da*b+b^{-1}*db)\\ &= \widetilde{\mathrm{Tr}}_{\mathrm{ad}}(a^{-1}*da)+\widetilde{\mathrm{Tr}}_{\mathrm{ad}}(b^{-1}*db)\\ &+\widetilde{\mathrm{Tr}}_{\mathrm{ad}}(b^{-1}*(a^{-1}*da*b)-(a^{-1}*da*b)*b^{-1})\\ &= widetilde\mathrm{Tr}_{\mathrm{ad}}(a^{-1}*da)+\widetilde{\mathrm{Tr}}_{\mathrm{ad}}(b^{-1}*db)\\ &+\frac{1}{2\pi i}\int_{\mathbb{R}}\mathrm{Tr}_Z\left(\frac{\partial b_0^{-1}}{\partial\tau}(\infty,\tau)(a_0^{-1}da_0b_0)(\infty,\tau)\right)d\tau\\ &= \widetilde{\mathrm{Tr}}_{\mathrm{ad}}(a^{-1}*da)+\widetilde{\mathrm{Tr}}_{\mathrm{ad}}(b^{-1}*db)\\ &-\frac{1}{2\pi i}\int_{\mathbb{R}}\mathrm{Tr}_Z\left(\frac{\partial b_0}{\partial\tau}(\infty,\tau)b_0^{-1}(\infty,\tau)a_0^{-1}(\infty,\tau)da_0(\infty,\tau)\right)d\tau.\end{aligned}$$

Similarly,

$$\begin{aligned}\widetilde{\mathrm{Tr}}_{\mathrm{ad}}(d(a*b)*(a*b)^{-1}) &= \widetilde{\mathrm{Tr}}_{\mathrm{ad}}(da*a^{-1})+\widetilde{\mathrm{Tr}}_{\mathrm{ad}}(db*b^{-1})\\ &+\frac{1}{2\pi i}\int_{\mathbb{R}}\mathrm{Tr}_Z\left(db_0(\infty,\tau)b^{-1}(\infty,\tau)a^{-1}(\infty,\tau)\frac{\partial a_0}{\partial\tau}(\infty,\tau)\right)d\tau.\end{aligned}\tag{11.19}$$

Combining these two computations, the result follows. □

Proposition 10. *The adiabatic determinant on the normal subgroup in* (11.8) *induces the determinant line bundle, $\mathcal{L}$, which is primitive over the quotient and $\tilde{\alpha}$ in* (11.12) *defines a connection ∇_{ad} on $\mathcal{L}$ with curvature form the* 2*-form part of the universal even Chern character of* (2.22).

Proof. The form $\tilde{\alpha}$ in (11.12) restricts to α in (10.6) on $G^{-\infty}_{\mathrm{sus}(2)}(Z;E)[\epsilon/\epsilon^2]$. The latter is the differential of the logarithm of $\det_{\mathrm{ad}}$. As a special case of Proposition 9 above, the first factor may be restricted to $G^{-\infty}_{\mathrm{sus}(2)}(Z;E)$, and then δ in (11.17) vanishes since $a \equiv 0$. This shows that as a connection on the trivial bundle over $\mathcal{D}^{-\infty}_{\mathrm{sus}(2)}(Z;E)[\epsilon/\epsilon^2]$, $d-\tilde{\alpha}$ is invariant under the left action of $G^{-\infty}_{\mathrm{sus}(2)}(Z;E)[\epsilon/\epsilon^2]$, acting through the adiabatic determinant on the fibres. Thus $d-\tilde{\alpha}$ projects to a connection ∇_{ad} on the determinant line bundle over $G^{-\infty}_{\mathrm{sus}}(Z;E)$ defined as the quotient by this action, i.e. as the line bundle induced by $\det_{\mathrm{ad}}$ as a representation of the structure group.

To compute the curvature we simply need to compute the differential of $\tilde{\alpha}$. Using the trace-defect formula of Lemma 4,

(11.20)
$$\begin{aligned}d\widetilde{\mathrm{Tr}}_{\mathrm{ad}}(a^{-1}*da) &= -\widetilde{\mathrm{Tr}}_{\mathrm{ad}}(a^{-1}*da*a^{-1}*da) = -\frac{1}{2}\widetilde{\mathrm{Tr}}_{\mathrm{ad}}([a^{-1}*da,a^{-1}*da]),\\ &= -\frac{1}{4\pi i}\int_{\mathbb{R}}\mathrm{Tr}_Z\left(\left(\frac{\partial}{\partial\tau}(\sigma^{-1}d\sigma)\right)\sigma^{-1}d\sigma\right)d\tau,\ \text{with } \sigma=a_0(\infty,\tau),\\ &= \frac{1}{4\pi i}\int_{\mathbb{R}}\mathrm{Tr}_Z\left(\sigma^{-1}\frac{\partial\sigma}{\partial\tau}(\sigma^{-1}d\sigma)^2-\sigma^{-1}d\left(\frac{\partial\sigma}{\partial\tau}\right)\sigma^{-1}d\sigma\right)d\tau,\\ &= \frac{1}{4\pi i}\int_{\mathbb{R}}\mathrm{Tr}_Z\left(\sigma^{-1}\frac{\partial\sigma}{\partial\tau}(\sigma^{-1}d\sigma)^2\right)d\tau - d\tilde{R}^*_\infty\beta,\end{aligned}$$

where

$$\beta = \frac{1}{4\pi i}\int_{\mathbb{R}} \operatorname{Tr}_Z\left(\sigma^{-1}\frac{\partial\sigma}{\partial\tau}\sigma^{-1}d\sigma\right)d\tau. \tag{11.21}$$

Similarly, we compute that

(11.22)

$$\begin{aligned} d\widetilde{\operatorname{Tr}}_{\mathrm{ad}}(da*a^{-1}) &= \widetilde{\operatorname{Tr}}_{\mathrm{ad}}(da*a^{-1}*da*a^{-1}) = \frac{1}{2}\widetilde{\operatorname{Tr}}_{\mathrm{ad}}([da*a^{-1},da*a^{-1}]),\\ &= \frac{1}{4\pi i}\int_{\mathbb{R}} \operatorname{Tr}_Z\left(\left(\frac{\partial}{\partial\tau}(d\sigma\sigma^{-1})\right)d\sigma\sigma^{-1}\right)d\tau,\\ &= -\frac{1}{4\pi i}\int_{\mathbb{R}} \operatorname{Tr}_Z\left(d\sigma\sigma^{-1}\frac{\partial\sigma}{\partial\tau}\sigma^{-1}d\sigma\sigma^{-1} - d\left(\frac{\partial\sigma}{\partial\tau}\right)\sigma^{-1}d\sigma\sigma^{-1}\right)d\tau,\\ &= \frac{1}{4\pi i}\int_{\mathbb{R}} \operatorname{Tr}_Z\left(\sigma^{-1}\frac{\partial\sigma}{\partial\tau}(\sigma^{-1}d\sigma)^2\right)d\tau + d\tilde{R}^*_\infty\beta. \end{aligned}$$

Recall that the 2-form part of the universal even Chern character on $G^{-\infty}_{\mathrm{sus}}(Z;E)$ is given by (cf. formula (3.7) in [6])

$$(\mathrm{Ch}_{\mathrm{even}})_{[2]} = \frac{1}{2(2\pi i)^2}\int_{\mathbb{R}} \operatorname{Tr}_Z\left(\sigma^{-1}\frac{\partial\sigma}{\partial\tau}(\sigma^{-1}d\sigma)^2\right)d\tau. \tag{11.23}$$

Thus, combining (11.20) and (11.22),

$$d\tilde{\alpha} = \frac{1}{4\pi i}\int_{\mathbb{R}} \operatorname{Tr}_Z\left(\sigma^{-1}\frac{\partial\sigma}{\partial\tau}(\sigma^{-1}d\sigma)^2\right)d\tau = 2\pi i\tilde{R}^*_\infty((\mathrm{Ch}_{\mathrm{even}})_{[2]}), \tag{11.24}$$

that is,

$$\frac{i}{2\pi}\nabla^2_{\mathrm{ad}} = \tilde{R}^*_\infty((\mathrm{Ch}_{\mathrm{even}})_{[2]}). \tag{11.25}$$

□

Next, this construction of the determinant bundle is extended to the geometric case. The sequence, (11.8), being natural, extends to give smooth bundles over the fibres of (1):

$$G^{-\infty}_{\mathrm{sus}(2)}(\phi;E)[\epsilon/\epsilon^2] \hookrightarrow \mathcal{D}^{-\infty}_{\mathrm{sus}(2)}(\phi;E)[\epsilon/\epsilon^2] \xrightarrow{\tilde{R}_\infty} G^{-\infty}_{\mathrm{sus}}(\phi;E). \tag{11.26}$$

Furthermore, using the connection chosen earlier, the form $\tilde{\alpha}$ in (11.12) can be replaced by $\tilde{\alpha}_\phi$ by substituting ∇ for d throughout. The resulting form 1-form is well-defined on $\mathcal{D}^{-\infty}_{\mathrm{sus}(2)}(\phi;E)$. Moreover the formulæ in §11 only depend on the derivation property of d so extend directly to $\tilde{\gamma}_\phi$. In particular (11.17) carries over to the fibre products. This leads directly to the following geometric version of Proposition 10.

Proposition 11. *The adiabatic determinant on the fibres of the structure bundle in* (11.8) *induces the determinant line bundle, $\mathcal{L}$, over $G^{-\infty}_{\mathrm{sus}}(\phi;E)$; the 1-form $\tilde{\alpha}_\phi$ defines a connection ∇_ϕ on $\mathcal{L}$ with curvature the* 2*-form part of the even Chern character on $G^{-\infty}_{\mathrm{sus}}(\phi;E)$.*

12. The K-theory gerbe

First we consider the universal K-theory gerbe, i.e. the gerbe over the classifying space $G^{-\infty}(Z;E)$ for odd K-theory. Recall that the delooping sequence (5) for a single manifold

$$
\begin{array}{ccc}
\mathcal{L} & & \\
\big\downarrow{\scriptstyle \pi} & & \\
G^{-\infty}_{\mathrm{sus}}(Z;E) & \longrightarrow & \tilde{G}^{-\infty}_{\mathrm{sus}}(Z;E) \\
& & \big\downarrow{\scriptstyle R_\infty} \\
& & G^{-\infty}(Z;E)
\end{array}
\tag{12.1}
$$

is a classifying sequence for K-theory, the normal subgroup is classifying for even K-theory, the central group is contractible and the quotient is classifying for odd K-theory. Moreover, in the preceding section, we have constructed the smooth primitive determinant line bundle over $G^{-\infty}_{\mathrm{sus}}(Z;E)$ with connection ∇_{ad} given in Proposition 10. This induces the K-theory gerbe over the classifying space, as a line bundle over the fibre product of two copies of the fibration R_∞ :

$$
\begin{array}{ccccc}
& & \tilde{m}^*\mathcal{L} & & \mathcal{L} \\
& & \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \pi} \\
\tilde{G}^{-\infty}_{\mathrm{sus}}(Z;E) & \overset{\pi_L}{\underset{\pi_R}{\longleftleftarrows}} & (\tilde{G}^{-\infty}_{\mathrm{sus}}(Z;E))^{[2]} & \xrightarrow{\ \tilde{m}\ } & G^{-\infty}_{\mathrm{sus}}(Z;E) \\
& {\scriptstyle R_\infty}\searrow & \big\downarrow{\scriptstyle (R_\infty)^{[2]}} & & \\
& & G^{-\infty}(Z;E). & &
\end{array}
\tag{12.2}
$$

Here $\tilde{m} : (\tilde{G}^{-\infty}_{\mathrm{sus}}(Z;E))^{[2]} \longrightarrow G^{-\infty}_{\mathrm{sus}}(Z;E)$ is the fibre-shift map $\tilde{m}(a,b) = ab^{-1}$, where $R_\infty(a) = R_\infty(b)$, by definition of the fibre product, so $\tilde{m}(a,b) \in G^{-\infty}_{\mathrm{sus}}(Z;E)$ by the exactness of (12.1), as indicated.

Theorem 2. *There is a connection* $\tilde{\nabla}_{\mathrm{ad}}$ *on* $\tilde{m}^*\mathcal{L}$ *with curvature*

$$
F_{\tilde{\nabla}_{\mathrm{ad}}} = \pi_L^*\tilde{\eta}_2 - \pi_R^*\tilde{\eta}_2 \text{ on } (\tilde{G}^{-\infty}_{\mathrm{sus}}(Z;E))^{[2]}
\tag{12.3}
$$

where the B-field, $\tilde{\eta}_2$*, is the 2-form part of the eta form in* (3.2) *which has basic differential the 3-form part of the odd Chern character on* $G^{-\infty}(Z;E)$*, as shown by* (3.6)*.*

Proof. The connection ∇_{ad} on $\mathcal{L}$ as a bundle over $G^{-\infty}_{\mathrm{sus}}(Z;E)$ given by Proposition 10 pulls back to a connection $\tilde{m}^*\nabla_{\mathrm{ad}}$ on $\tilde{m}^*\mathcal{L}$. The curvature is just the pull-back of the curvature on $G^{-\infty}_{\mathrm{sus}}(Z;E)$ and again by Proposition 10 this is the 2-form part of the Chern character. By Proposition 2, the 2-form part of the eta form on $\tilde{G}^{-\infty}_{\mathrm{sus}}(Z;E)$ pulls back under the product map as in (3.5). To apply this result here we need to invert the right factor, to change from m to $\tilde{m}$, which also has the effect of changing the sign of the eta form from that factor leading to

$$
\tilde{m}^*\tilde{\eta} = \pi_L^*\tilde{\eta} - \pi_R^*\tilde{\eta} + d(\tilde{\delta}'_{\mathrm{odd}}) + (R_\infty \times R_\infty)^*\delta'_{\mathrm{even}}
\tag{12.4}
$$

where the primes indicate that the forms are first pulled back under inversion in the second variable. Now, restricting (12.4) to the fibre diagonal gives

$$\tilde{m}^* \operatorname{Ch}_{\text{even}} = \pi_L^* \tilde{\eta} - \pi_R^* \tilde{\eta} + d(\tilde{\delta}'_{\text{odd}}) \tag{12.5}$$

since the last term now factors through the constant map to the identity. This corresponds to the middle row of (12.2). In particular, if the connection is modified by the 1-form part of δ'_{odd}

$$\tilde{\nabla}_{\text{ad}} = \tilde{m}^* \nabla_{\text{ad}} - (\tilde{\delta}'_{\text{odd}})_1 \tag{12.6}$$

then it has curvature as claimed

$$(\tilde{\nabla}_{\text{ad}})^2 = \tilde{m}^* (\operatorname{Ch}_{\text{even}})_2 = \pi_L^* \tilde{\eta}_2 - \pi_R^* \tilde{\eta}_2 \tag{12.7}$$

which is precisely the statement that $\tilde{\eta}_2$ is a B-field for the gerbe. The curving of the gerbe is then the basic form of which the differential of the B-field is the pull-back and from (3.6)

$$d\tilde{\eta}_2 = R_\infty^* (\operatorname{Ch}_{\text{odd}})_3. \tag{12.8}$$

□

13. Geometric Gerbe for an odd elliptic family

Theorem 3. *Let $A \in \Psi^1(M/Y; E)$ be a self-adjoint elliptic family as in Section 6, or a product-type fully elliptic family $A(t) \in \Psi_{\text{ps}}^{m,l}(M/Y; E)$ then the determinant bundle induces a bundle-gerbe*

$$\begin{array}{ccccc}
 & & S^*\mathcal{L} & & \mathcal{L} \\
 & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\
\mathcal{A} & \overset{\pi_L}{\underset{\pi_R}{\leftleftarrows}} & \mathcal{A}^{[2]} & \xrightarrow{S} & G_{\text{sus}}^{-\infty}(\phi; E) \\
 & {\scriptstyle p}\searrow & \downarrow{\scriptstyle p^{[2]}} & & \\
 & & Y. & &
\end{array} \tag{13.1}$$

The connection on $S^*\mathcal{L}$

$$\nabla_{\mathcal{A}} = S^* \nabla + \gamma, \ \gamma = (\tilde{\delta}_{\mathcal{A}})_1, \tag{13.2}$$

given by the 1-form part of the form in (6.17)*, is primitive in the sense that the curvature on $\mathcal{A}^{[2]}$ splits*

$$(\nabla_{\mathcal{A}})^2 = \pi_L^* \eta_{\mathcal{A},2} - \pi_R^* \eta_{\mathcal{A},2} \tag{13.3}$$

showing that $\eta_{\mathcal{A},2}$ is a B-field and that the gerbe has curving 3-form

$$d\eta_{\mathcal{A},2} = p^* \operatorname{Ch}_{\mathcal{A},3} \tag{13.4}$$

the 3-form part of the Chern character of the index bundle the family.

Proof. This result follows by an argument parallel to the preceding one, given (6.17), Proposition 10, Proposition 11 and the 3-form part of (8.4). □

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