A REMARK ON DISTRIBUTIONS AND THE DE RHAM THEOREM

RICHARD MELROSE

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Abstract. We show that the de Rham theorem, interpreted as the isomorphism between distributional de Rham cohomology and simplicial homology in the dual dimension for a simplicial decomposition of a compact oriented manifold, is a straightforward consequence of elementary properties of currents. The explicit construction of this isomorphism extends to other cases, such as relative and absolute cohomology spaces of manifolds with corners.

The de Rham theorem ([2, 3]) is generally interpreted as the isomorphism, for a compact oriented manifold X, between the cohomology of the de Rham complex of smooth forms

$$0 \longrightarrow \mathcal{C}^\infty(X) \longrightarrow \mathcal{C}^\infty(X;\Lambda^1) \longrightarrow \cdots \longrightarrow \mathcal{C}^\infty(X;\Lambda^n) \longrightarrow 0, \tag{1}$$

where $\dim X = n$, and the simplicial, or more usually the Čech, cohomology of X. This isomorphism is constructed using a double complex; for proofs of various stripes see [5], [4] or [6].

The distributional de Rham cohomology, the cohomology of the complex (1) with distributional coefficients (currents in the terminology of de Rham),

$$0 \longrightarrow \mathcal{C}^{-\infty}(X) \longrightarrow \mathcal{C}^{-\infty}(X;\Lambda^1) \longrightarrow \cdots \longrightarrow \mathcal{C}^{-\infty}(X;\Lambda^n) \longrightarrow 0, \tag{2}$$

is naturally Poincaré dual to the smooth de Rham cohomology using the integration map $(\alpha,\beta) \longmapsto \int_X \alpha\wedge\beta$.

Here we show that there is a relatively simple retraction argument which shows that the homology of (2) is isomorphic to the simplicial homology, for any simplicial decomposition, in the dual dimension. The map from simplicial to distributional de Rham cohomology takes a simplex to its Poincaré dual (see for example [1]). There are many possible variants of the proof below and in particular it is likely to apply to intersection type homology theories on compact manifolds with corners.

1. Distributions and currents

We use some results from distribution theory which are well known. These are mainly to the effect that a simplex is 'regular' as a support of distributions.

Lemma 1. *Any extendible distribution on the interior of an n-simplex $S \subset \mathbb{R}^n$, i.e. an element of the dual of $\dot{\mathcal{C}}^\infty(S;\Omega) = \{u \in \mathcal{C}^\infty(\mathbb{R}^n); \operatorname{supp}(u) \subset S\}$, is the restriction of a distribution on $\mathbb{R}^n$ with support in S.*

Lemma 2. *Any current with support in a plane $\mathbb{R}^k_y \times \{0\}_z \subset \mathbb{R}^n$ is of the form*

$$\sum_{\alpha,I} \delta^{(\alpha)}(z)dz^I u_{\alpha,I}(y) \tag{3}$$

where the $u_{\alpha,I}$ are currents on $\mathbb{R}^k$ and the $\delta^{(\alpha)} = \partial^\alpha \delta(z)/\partial z^\alpha$ are derivatives of the Dirac delta function.

Proposition 1. *If X is a manifold with a simplicial decomposition then any distribution or current with support on the p-skeleton is the sum of distributions with supports on the individual p-simplexes.*

2. Poincaré Lemmas

Lemma 3. *In $\mathbb{R}^n$ the complex of distributional forms, i.e. currents, with support at the origin has homology which is one-dimensional and is in dimension n.*

Proof. We must show that a closed k-current of this type is always of the form dv for a $k-1$ current supported at the origin, unless $k = n$ in which case

$$u = c\delta(x)dx_1 \wedge \cdots \wedge dx_n + dv. \tag{4}$$

The key to this is simply the representation (3) in this case, decomposing currents supported at the origin as finite sums

$$u = \sum_{\alpha,I} c_{\alpha,I}\delta^{(\alpha)}(x)dx^I, \tag{5}$$

where $\delta^{(\alpha)}(x) = \partial_x^\alpha \delta(x)$. Each of these terms is homogeneous of degree $-\alpha_i$ or $1-\alpha_i$ under the homotheity R_i^t where $R_i^t(x) = (x_1, \cdots, x_{i-1}, tx_i, x_{i+1}, \cdots, x_n)$. Since d itself is invariant under these transformations it follows that if u is closed, so are each of the terms of fixed homogeneity in each variable. Consider the identity for currents

$$t\frac{d}{dt}(R_i^t)^*u = (R_i^t)^*(d\mathcal{L}_i + \mathcal{L}_i d)u \tag{6}$$

where $\mathcal{L}_i$ is contraction with the radial vector field, $x_i\partial_{x_i}$ in ith coordinate. Then if u is closed and homogeneous of degree a_i it follows that $\frac{d}{dt}(R_i^t)^*u|_{t=1} = a_i u = dv$, $v = \mathcal{L}_i u$. Thus all closed currents of non-zero multi-homogeneity are exact. The only currents which are homogeneous of degree zero are the multiples of $\delta(x)dx_1 \wedge \cdots \wedge dx_n$ so we have proved (5).

The lemma now follows from the fact that these forms are not themselves exact. This again uses the same type of homogeneity argument. If $\delta(x)dx_1 \wedge \cdots \wedge dx_n = dv$ with v supported at the origin, then v may be replaced by its homogeneous part of degree 0. Since there are no currents of form degree $n-1$ which are homogeneous of degree 0 it follows that no such v can exist. □

Next we compute the extendible distributional de Rham cohomology of the interior of the standard n-simplex in $\mathbb{R}^n$. This is also a form of the Poincaré lemma.

Lemma 4. *If u is a closed extendible k-current on the interior of $S_n = \{x \in \mathbb{R}^n; 0 \le x_i \le 1, 0 \le x_1 + \cdots + x_n \le 1\}$ then $u = dv$ with v an extendible $(k-1)$-current unless $k = 0$ in which case u is a constant.*

Proof. If u is a 0-current, i.e. a distribution, then $du = 0$ implies that u is constant. Thus we may assume that $k > 0$.

We proceed by induction over the condition that there exists a current v_j such that $\mathcal{L}_i(u - dv_j) = 0$ for all $i \le j$. For the first step we may write

$$u = u' + dx_1 \wedge u'' \tag{7}$$

where $\mathcal{L}_1 u' = \mathcal{L}_1 u'' = 0$ are respectively a k and a $k-1$ current. Then the assumption that u is closed implies that $\partial u'/\partial x_1 - d'u'' = 0$. Taking $v_1 = u'$ it follows that $u_1 = u - dv_1$ satisfies $\mathcal{L}_1(u_1) = 0$. Now we may proceed by induction since $du_1 = 0$ and $\mathcal{L}_1 u_1 = 0$ implies that u_1 is completely independent of x_1, so this condition may be maintained in subsequent steps. When $j > n-k$ it follows that u is exact. □

Lemma 3 is actually the zero dimensional case, and Lemma 4 essentially the n-dimensional case of the following proposition in which we consider the standard p-simplex in $\mathbb{R}^n$:

(8)
$$S_p = \{x \in \mathbb{R}^n; x_1 = \cdots = x_{n-p} = 0,\ x_j \geq 0,\ j > n-p,\ x_{n-p+1} + \cdots + x_n \leq 1\}.$$

The basic current we associate with S_p is

$$D(S_p) = \chi(S_p)\delta(x_1)\cdots\delta(x_{n-p})dx_1 \wedge \cdots dx_{n-p}. \tag{9}$$

Here $\chi(S_p)$ is the characteristic function of S_p in the variables x_j, $j > n-p$.

Proposition 2. *If u is a k-current on $\mathbb{R}^n$ with support contained in S_p and $du = 0$ in $\mathbb{R}^n \setminus \partial S_p$ then there is a $(k-1)$-current v with support in S_p such that*

$$u = \begin{cases} dv + u' & \text{if } k \neq n-p, \\ dv + u' + cD(S_p) & \text{if } k = n-p \end{cases} \quad \text{with } \operatorname{supp}(u') \subset \partial S_p. \tag{10}$$

Proof. Let us write the first $n-p$ variables as y and the second p variables as z. The decomposition analogous to (5) for a closed current in this case is

$$u = \sum_{\alpha, I} \delta^{(\alpha)}(y) dy^I \wedge u_{\alpha,I}(z) \tag{11}$$

where now the $u_{\alpha,I}$ are $(k-|I|)$-currents on $\mathbb{R}^p$ with support in $S_p \subset \mathbb{R}^p$.

The homogeneity argument of Lemma 3 may now be followed. Thus, u may be decomposed into its multi-homogeneous parts under separate scaling in each of the variables in y; since d again preserves such homogeneity, each term is closed if u is. As before, the terms of non-zero homogeneity, in any of the y variables, is exact near the interior of S_p. Thus for some v' with support in S_p $u' = u - dv$ has the same closedness property and is of the form

$$u' = dy^1 \wedge \cdots dy^{n-p} u'' \tag{12}$$

where u'' is a $k-n+p$ form on $S_p \subset \mathbb{R}^p$; in particular if $k < n-p$ then $u - dv$ has support in ∂S_p.

It follows from (12) that u'' is closed in the interior of S_p as a $k-n+p$ form on $\mathbb{R}^p$. Thus Lemma 4 may be applied. This extendible current constructed there, so that (unless $k = n-p$) $u'' = dv''$ in the interior of S_p may be extended to a current w with support in S_p such that $u - dw$ has support in ∂S_p. This yields the desired result. □

3. De Rham theorem

Observe that the current $D(S_p)$ associated with the standard p-simplex is invariant under oriented diffeomorphism of a neighbourhood of it in $\mathbb{R}^n$. Thus it is well defined for any oriented simplex in an oriented manifold. In fact only the relative orientation, i.e. orientation of the normal bundle, is important.

Theorem 1. *Let X be an oriented compact manifold, without boundary, with a given simplicial decomposition, with (oriented) simplexes labelled $S(j) = S_p(j)$ where p is the dimension, then the chain map*

$$E : \sum_j c_j S_p(j) \longmapsto \sum_j c_j D(S_p(j)) \in \mathcal{C}^{-\infty}(X; \Lambda^{n-p}) \tag{13}$$

is a homology equivalence giving an isomorphism between the simplicial p-homology of X and its distributional $n-p$ de Rham cohomology.

Proof. By direct computation, for the standard p-simplex,

$$dD(S_p) = \sum_r D(S_{p-1}(r)) \tag{14}$$

where $S_{p-1}(r)$ are the bounding $(p-1)$-simplexes with their induced orientations. Thus the map does give a chain map:

$$d(E(c)) = E(\delta(c)) \tag{15}$$

where δ is the standard differential of simplicial homology.

To prove the theorem it suffices to show that the distributional de Rham complex can be retracted onto the simplicial subcomplex. That is,

$$\begin{gathered} u \in \mathcal{C}^{-\infty}(X; \Lambda^k),\ du = 0 \Longrightarrow u = dv + E(c), \\ dE(c) = 0 \Longrightarrow \delta(c) = 0. \end{gathered} \tag{16}$$

In fact the second of these is clear, since E is injective. We also need the corresponding statements for exact forms. That is

$$E(c) = dv,\ v \in \mathcal{C}^{-\infty}(X; \Lambda^*) \Longrightarrow c = \delta c'. \tag{17}$$

Thus suppose u is a closed k-current. Let K_j denote the j skeleton of the simplicial decomposition, i.e. the union of the j-simplexes. Proceeding step by step we first decompose u as a sum of k-currents supported on each of the n-simplexes. Each of the terms is closed in the interior of each simplex, so Proposition 2 may be applied to give a decomposition

$$u = \begin{cases} \sum_j c_j D(S_n(j)) & \text{if } k = 0 \\ dv + u_1 & \text{if } k > 0, \end{cases} \quad \text{where } \operatorname{supp}(u_1) \subset K_{n-1}. \tag{18}$$

We can ignore the case $k = 0$. Now it follows that u_1 is closed. Applying Proposition 1 to decompose u_1 as a sum over the $n-1$ skeleton and applying Proposition 2 to each part, gives a new decomposition and we may continue until we reach the $n-k$ skeleton. Thus we arrive at

$$u = \sum_j c_j D(S_{n-k}(j)) + dv + u_{k+1},\ \operatorname{supp}(u_{k+1}) \subset K_{n-k-1}. \tag{19}$$

Let $c = \sum_j c_j S_{n-k}(j)$ be the corresponding simplicial chain, so the first term in (19) is $E(c)$. Now $dE(c) = E(\delta c)$. Thus, near the interior of any $n-k-1$ simplex, $S_{n-k-1}(r)$, $du_{k+1} = -E(\delta c) = c'_r D(S_{n-k-1}(r))$. However, $D(S_{n-k-1}(r))$ is not in the range of d on currents supported on the corresponding simplex. Thus $c'_r = 0$ for all r which just gives $\delta c = 0$. Thus

$$u = E(c) + dv + u_{k+1},\ \operatorname{supp}(u_{k+1}) \subset K_{n-k-1},\ \delta c = 0,\ du_{k+1} = 0. \tag{20}$$

Now we can proceed successively, as before, and conclude that $u_{k+1} = dw$ with w supported on the $n-k-1$ skeleton.

The arguments needed for (17) are similar. Thus, it follows from $E(c) = dv$ that v is closed in the complement of the support of c. Assuming that c is an $n-p$ chain, this means that $dv = 0$ off the $n-p$ skeleton. The argument above shows that $v = E(c') + dv' + w$ where w has support on the $n-p$ skeleton. Since $dE(c') = E(\delta c')$ we conclude that $E(c - \delta c') = dw$, with w supported on the $n-p$ skeleton. As already noted, this implies that $c = \delta c'$.

This proves the de Rham theorem. □

References

[1] R. Bott and L. Tu, *Differential Forms in Algebraic Topology*, Graduate Texts in Mathematics, vol. 82, Springer, Berlin, Heidelberg, New-York, 1982.

[2] Georges de Rham, *Variétés différentiables. Formes, courants, formes harmoniques*, Hermann, Paris, 1973, Troisième édition revue et augmentée, Publications de l'Institut de Mathématique de l'Université de Nancago, III, Actualités Scientifiques et Industrielles, No. 1222b.

[3] ———, *Differentiable manifolds*, Springer-Verlag, Berlin, 1984, Forms, currents, harmonic forms, Translated from the French by F. R. Smith, With an introduction by S. S. Chern.

[4] S. T. Hu, *Differentiable manifolds*, Holt, Rinehart and Winston, Inc., New York, 1969.

[5] I. M. Singer and J. A. Thorpe, *Lecture notes on elementary topology and geometry*, Springer-Verlag, New York, 1976, Reprint of the 1967 edition, Undergraduate Texts in Mathematics.

[6] Frank W. Warner, *Foundations of differentiable manifolds and Lie groups*, Springer-Verlag, New York, 1983, Corrected reprint of the 1971 edition.

Department of Mathematics, Massachusetts Institute of Technology

E-mail address: rbm@math.mit.edu