GEOMETRY OF PSEUDODIFFERENTIAL ALGEBRA BUNDLES AND FOURIER INTEGRAL OPERATORS

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Dedicated to Isadore M. Singer

Abstract. We study the geometry and topology of (filtered) algebra-bundles $\mathbf{\Psi}^{\mathbb{Z}}$ over a smooth manifold X with typical fibre $\Psi^{\mathbb{Z}}(Z;V)$, the algebra of classical pseudodifferential operators of integral order on the compact manifold Z acting on smooth sections of a vector bundle V. First a theorem of Duistermaat and Singer is generalized to the assertion that the group of projective invertible Fourier integral operators $\mathrm{PG}(\mathcal{F}^{\mathbb{C}}(Z;V))$, is precisely the automorphism group, $\mathrm{Aut}(\Psi^{\mathbb{Z}}(Z;V))$, of the filtered algebra of pseudodifferential operators. We replace some of the arguments in their paper by microlocal ones, thereby removing the topological assumption. We define a natural class of connections and B-fields on the principal bundle to which $\mathbf{\Psi}^{\mathbb{Z}}$ is associated and obtain a de Rham representative of the Dixmier-Douady class, in terms of the outer derivation on the Lie algebra and the residue trace of Guillemin and Wodzicki; the resulting formula only depends on the formal symbol algebra $\mathbf{\Psi}^{\mathbb{Z}}/\mathbf{\Psi}^{-\infty}$. Some examples of pseudodifferential bundles with non-torsion Dixmier-Douady class are given; in general such a bundle is not associated to a finite dimensional fibre bundle over X.

Contents

2010 *Mathematics Subject Classification.* 19K56, 58G10, 58G12, 58J20, 58J22.

Key words and phrases. Pseudodifferential algebra bundles, Fourier integral operators, automorphism group of pseudodifferential operators, derivations of pseudodifferential operators, Dixmier-Douady invariant.

Acknowledgments. The first author thanks the Australian Research Council for partial support. The second author acknowledges the support of the National Science Foundation under grant DMS1005944.

Introduction

In this paper, we study the geometry and topology of (filtered) algebra-bundles $\mathbf{\Psi}^{\mathbb{Z}}$ over a smooth manifold X with typical fibre $\Psi^{\mathbb{Z}}(Z;V)$, the algebra of classical pseudodifferential operators on the compact manifold Z acting on smooth sections of a vector bundle V. Since we do not assume that there is an underlying geometric bundle, the transition functions of $\mathbf{\Psi}^{\mathbb{Z}}$ are general order-preserving automorphisms of the typical fibre $\Psi^{\mathbb{Z}}(Z;V)$. By an extension of a theorem of Duistermaat and Singer [13] (see Theorem 2.1), every such automorphism is given by conjugation by an invertible Fourier integral operator of possibly complex order. This means in particular that $\Psi^{\mathbb{Z}}(Z;V)$ has many nontrivial outer automorphisms. The associated principal bundle $\mathbf{F}$ to $\mathbf{\Psi}^{\mathbb{Z}}$ has fibre the Fréchet Lie group of projective invertible Fourier integral operators, $\mathrm{PG}(\mathcal{F}^{\mathbb{C}}(Z;V)) = \mathrm{G}(\mathcal{F}^{\mathbb{C}}(Z;V))/\mathbb{C}^*$ the quotient by the centre, $\mathbb{C}^*\,\mathrm{Id}$, of the group of invertible Fourier integral operators of complex order. We show that the structure group can always be reduced to $\mathrm{PG}(\mathcal{F}^0(Z;V))$, the subgroup of operators of order 0. Thus the structure group arises directly through the central extension,

$$\mathbb{C}^\star \longrightarrow \mathrm{G}(\mathcal{F}^0(Z;V)) \longrightarrow \mathrm{PG}(\mathcal{F}^0(Z;V)). \tag{1}$$

A class of connections on this principal bundle is constructed from the regularized trace of the Maurer-Cartan 1-form on $\mathrm{G}(\mathcal{F}^0(Z;V))$. The curvature of such a connection is then computed explicitly via the trace-defect formula in terms of the residue trace of Guillemin [17, 18] and Wodzicki [42] giving a differential 2-form on $\mathrm{PG}(\mathcal{F}^0(Z;V))$. The central extension (1) is then fixed up to isomorphism by an additional 1-form, cf. Lemma 6.4.

The obstruction to lifting $\mathbf{F}$ to a $\mathrm{G}(\mathcal{F}^0(Z;V))$-bundle, and hence to realizing $\mathbf{\Psi}^{\mathbb{Z}}$ as a bundle of operators, is the Dixmier-Douady invariant. This can be realized in terms of the bundle gerbe associated to $\mathbf{F}$ in the sense of Murray, as further developed by Murray and Stevenson in [35]. The central extension (1) leads to a line bundle over the fibre product $\mathbf{F}^{[2]}$ of the principal bundle with itself and the Dixmier-Douady invariant is the obstruction to obtaining this in terms of the tensor difference of the two pull-backs of a bundle over $\mathbf{F}$. The Chern class of the line bundle over $\mathbf{F}^{[2]}$ can be split in terms of a 2-form, a B-field, on $\mathbf{F}$. The choice of a connection on $\mathbf{F}$ and a Higgs field, lifting the exterior derivation on the Lie algebra, enables us to construct an explicit B-field. The differential of this is a basic differential 3-form representing the image of the Dixmier-Douady class in de Rham cohomology of the base, X. This is also a de Rham representative of the Dixmier-Douady class of the Azumaya bundle obtained by completing, in the operator norm, the subbundle $\mathbf{\Psi}^{-\infty} \subset \mathbf{\Psi}^{\mathbb{Z}}$ consisting of the smoothing operators. Note that $\mathbf{\Psi}^{\mathbb{Z}}$, is *not* in general associated to a finite dimensional fibre bundle over X. A continuous section of $\mathbf{\Psi}^{\mathbb{Z}}$ over X is called a *projective family* of pseudodifferential operators on Z although in view of the conjugation by Fourier integral operators no meaning can be assigned to the notion of a projective family of differential operators in this context.

Paycha and Rosenberg (cf. [37, 38]) and others have considered what amounts to a special case of this general notion of a bundle of pseudodifferential operators in which the structure group is required to be the group of invertible pseudodifferential operators. So in this case the Dixmier-Douady invariant is trivial and there is a bundle of Fréchet spaces on which the pseudodifferential operators act fibrewise.

In outline the content of this paper is as follows. In section 1 the Lie agebra of derivations of the filtered algebra of all $\Psi^{\mathbb{Z}}(Z;V)$ is studied. From this, in section 2 the structure of Lie agebra of derivations of the graded $\star$-algebra of formal pseudodifferential operators $\Psi^{\mathbb{Z}}(Z;V)/\Psi^{-\infty}(Z:V)$ is deduced; in both cases, there are non-trivial outer derivations but in the formal case the algebra is generally larger. Section 3 is devoted to the study of the Fréchet Lie groups of automorphisms of both the graded $\star$-algebra of formal pseudodifferential operators and the filtered algebra of all pseudodifferential operators; again there are non-trivial outer automorphisms. In Section 4 the topologies on the groups of invertible pseudodifferential and Fourier integral operator are examined with some additional constructions relegated to an appendix. The main object of study here, the notion of a filtered algebra bundle of pseudodifferential over a smooth manifold X is introduced in Section 5. In section 6, a connection on the central extension (1) is described with curvature computed in terms of the residue trace of Guillemin and Wodzicki. Section 7 contains the analysis of the principal bundle of trivializations of a filtered algebra bundle of pseudodifferential and the image in deRham cohomology of the Dixmier-Douady class is computed. Finally several examples with non-torsion Dixmier-Douady class are given in Section 8.

We dedicate this paper to our friend and collaborator, Isadore M. Singer. We would like to thank him for his important input in the initial discussions on this paper and for our earlier work [27, 28, 29, 30]. The first author thanks M. Murray and D. Stevenson for discussion concering their paper [35].

1. Derivations of pseudodifferential operators

In this section the Lie algebra of derivations on the algebra of classical pseudodifferential operators acting on sections of a complex vector bundle V over a compact manifold Z are characterized. Derivations on its 'formal' quotient $\Psi^{\mathbb{Z}}(Z;V)/\Psi^{-\infty}(Z;V)$ are considered in the next section. This analysis is closely parallel to the treatment, recalled (and refined a little) below, of Duistermaat and Singer of the group of automorphisms of $\Psi^{\mathbb{Z}}(Z;V)$ and in particular is an infinitesmal version of it. As in that case, it is unnecessary to make any *a priori* assumption of continuity. Thus a derivation is simply a filtered linear map, so for some $k \in \mathbb{Z}$,

$$D : \Psi^{\mathbb{Z}}(Z;V) \longrightarrow \Psi^{\mathbb{Z}+k}(Z;V),\ \forall\ p \in \mathbb{Z} \text{ and}$$
$$D(A \circ B) = D(A) \circ B + A \circ D(B). \tag{2}$$

Commutation with an element of the algebra is an inner derivation.

Proposition 1.1. *If* $\dim Z \geq 2$ *and* Z *is connected, the quotient of the Lie algebra of derivations of* $\Psi^{\mathbb{Z}}(Z;V)$ *by the inner derivations is one-dimensional (for the circle it has dimension two).*

Proof. The first step is to show that a derivation on the subalgebra of smoothing operators is realized by an operator on $\mathcal{C}^\infty(Z;V)$.

Lemma 1.2. *If* $D : \Psi^{-\infty}(Z;V) \longrightarrow \Psi^{-\infty}(Z;V)$ *is a linear map acting as a derivation on smoothing operators,*

$$D(A \circ B) = D(A) \circ B + A \circ D(B)\ \forall\ A,\ B \in \Psi^{-\infty}(Z;V), \tag{3}$$

then there is a continuous linear operator $L : \mathcal{C}^\infty(Z;V) \longrightarrow \mathcal{C}^\infty(Z;V)$ *such that*

$$D(A) = [L, A]\ \forall\ A \in \Psi^{-\infty}(Z;V) \tag{4}$$

and L is determined up to an additive multiple of the identity.

Proof. The smoothing operators are naturally identified with the smooth sections of the (kernel) bundle $K(V) = V \boxtimes (V' \otimes \Omega)$ of the two-point homomorphism bundle over the product with a density from the right factor. Thus the derivation may be realized as a linear map $D : \mathcal{C}^\infty(Z^2; K(V)) \longrightarrow \mathcal{C}^\infty(Z^2; K(V))$ where we assume (3) but not any continuity. Choose a global density, $0 < \nu \in \mathcal{C}^\infty(Z; \Omega)$, and hence an isomorphism $K(V) \simeq V \boxtimes V'$ over Z^2 so that $D : \mathcal{C}^\infty(Z^2; V \boxtimes V') \longrightarrow \mathcal{C}^\infty(Z^2; V \boxtimes V')$ with (3) holding for the product

$$A \circ B(z, z') = \int_Z A(z, z'')B(z'', z')\nu(z'') \text{ on } \mathcal{C}^\infty(Z^2; V \boxtimes V'). \tag{5}$$

In addition choose a smooth fibre inner product $\langle , \rangle$ on V, and hence a global inner product on $\mathcal{C}^\infty(Z; V)$,

$$(u, v) = \int_Z \langle u, v\rangle \nu, \ u, \ v \in \mathcal{C}^\infty(Z; V).$$

Each section $w \in \mathcal{C}^\infty(Z; V)$ determines a continuous linear functional

$$I(w) : \mathcal{C}^\infty(X; V) \longrightarrow \mathbb{C}, \ I(w)v = \int_Z (v, w)\nu. \tag{6}$$

Following the idea of M. Eidelheit [14] we consider smoothing operators of rank one. Any pair of sections, $u, \ w \in \mathcal{C}^\infty(Z; V)$ determines such an operator

$$u \otimes I(w) \in \Psi^{-\infty}(Z; V) \text{ where } (u \otimes I(w))(v) = I(w)(v) \cdot u. \tag{7}$$

Note that this is indeed a smoothing operator since its kernel is $(u \boxtimes w^*)\nu$ where w^* is the 'dual' section of V' given by the pointwise inner product. The resulting map

$$\mathcal{C}^\infty(Z; V) \times \mathcal{C}^\infty(Z; V) \longrightarrow \Psi^{-\infty}(Z; V) \tag{8}$$

is linear in the first, but anti-linear in the second, variable. Conversely, the range of this map consists of all smoothing operators of rank one. Composition is given by pairing of the two central elements:

$$(u \otimes I(w)) \circ (u' \otimes I(w')) = (u', w)u \otimes I(w'). \tag{9}$$

Now, consider the action of a derivation on these operators. Fix an element $w \in \mathcal{C}^\infty(Z; V)$ with norm one, i.e. $I(w)(w) = 1$. The action of D defines two linear maps

$$\begin{gathered} L, \ R : \mathcal{C}^\infty(Z; V) \longrightarrow \mathcal{C}^\infty(Z; V) \text{ by} \\ Lu = (D(u \otimes I(w)))(w), \ I(Rv) = I(w) \circ D(w \otimes I(v)); \end{gathered} \tag{10}$$

where in the second case the composite of a smoothing operator and pairing against a smooth section is necessarily given by pairing against a smooth section, so in terms of adjoints,

$$Rv = (D(w \otimes I(v)))^* w, \ I(Rv)(\phi) = (D(w \otimes I(v))\phi, w).$$

The identity

$$u \otimes I(v) = (u \otimes I(w)) \circ (w \otimes I(v))$$

combined with the derivation property shows that

$$D(u \otimes I(v)) = Lu \otimes I(v) + u \otimes I(Rv) \ \forall \ u, v \in \mathcal{C}^\infty(Z; V). \tag{11}$$

So L and R determine D.

For any four smooth sections, expanding out the composition formula (9) applied to $(u\otimes I(v))\circ(u'\otimes I(v'))$ and using (11) gives,

$$
\begin{aligned}
&(u',v)\,(Lu\otimes I(v')+u\otimes I(Rv'))\\
&=(u',v)Lu\otimes I(v')+(u',Rv)u\otimes I(v')\\
&+(Lu',v)u\otimes I(v')+(u',v)u\otimes I(Rv').
\end{aligned}\tag{12}
$$

Thus the middle two terms on the right must cancel. This gives the adjoint identity $(u',Rv)=-(Lu',v)$ for all u' and v, i.e. $R=-L^*$ and hence (11) becomes

$$
\begin{gathered}
D(u\otimes v)=L\circ(u\otimes I(v))-(u\otimes I(v))\circ L,\quad\text{i.e.}\\
D(A)=[L,A],\ \ A\text{ of rank one.}
\end{gathered}\tag{13}
$$

By linearity this extends to all operators of finite rank and more generally, if E is any smoothing operator and A, B are of finite rank then so is AEB and the derivation identity shows that

$$
\begin{aligned}
&D(AEB)=\\
&\qquad[L,A]EB+AD(E)B+AE[L,B]=[L,A]EB+A[L,E]B+AE[L,B]
\end{aligned}\tag{14}
$$

so $D(E)=[L,E]$ for all smoothing operators since any operator is determined by the collection of AEB with A and B of rank one.

To see that $L:\mathcal{C}^\infty(Z;V)\longrightarrow\mathcal{C}^\infty(Z;V)$ is continuous observe that the discussion above shows that, without assuming continuity, it has a well-defined adjoint, namely $-R$. Thus $L:\mathcal{C}^\infty(Z;V)\longrightarrow\mathcal{C}^\infty(Z;V)$ is closed, since if $u_n\to u$ and $Lu_n\to w$ then

$$
(w,\phi)=\lim(Lu_n,\phi)=-\lim(u_n,R\phi)=-(u,R\phi)=(Lu,\phi).\tag{15}
$$

As a closed linear operator on a Fréchet space, L is necessarily continuous.

The uniqueness of L up to the addition of a scalar multiple of the identity follows from the fact that these are the only operators which commute with all smoothing operators. □

Now consider a filtration-preserving derivation on $\Psi^{\mathbb{Z}}(Z;V)$, the algebra of pseudodifferential operators, so for fixed $m\in\mathbb{Z}$,

$$
D:\Psi^m(Z;V)\longrightarrow\Psi^{k+m}(Z;V)\quad\forall\ m\in\mathbb{Z}.\tag{16}
$$

It follows that it induces a derivation on $\Psi^{-\infty}(Z;V)$, being the intersection of these spaces, and Lemma 1.2 generates an operator $L:\mathcal{C}^\infty(Z;V)\longrightarrow\mathcal{C}^\infty(Z;V)$. Moreover, the identity (14) again shows that

$$
D(A)=[L,A]\ \forall\ A\in\Psi^{\mathbb{Z}}(Z;V).\tag{17}
$$

As an operator, L determines and is determined by its Schwartz' kernel, which we also denote $L\in\mathcal{C}^{-\infty}(Z^2;V\boxtimes V))$. If $A\in\Psi^{\mathbb{Z}}(Z;V)$ it acts on $\mathcal{C}^\infty(Z;V)$ and its formal adjoint is an element $A^t\in\Psi^{\mathbb{Z}}(Z;V)$. Then the identity (17) can be written in terms of the kernel

$$
(A\otimes\operatorname{Id}-\operatorname{Id}\otimes A^t)L=B,\ \ B\in\Psi^{\mathbb{Z}}(Z;V)\tag{18}
$$

also representing the Schwartz kernel of the operator $D(A)$. In general neither term on the left here is a pseudodifferential operator on Z^2. However, if A is a differential operator, say of order 1, then so is A^t and then (18) represents a differential equation on Z^2.

It follows from (18) that L itself has wavefront contained in the conormal bundle to the diagonal, since at any other point in $T^*Z^2 \setminus \{0\}$ it is possible to choose A so that $(A \otimes \mathrm{Id} - \mathrm{Id} \otimes A^t)$ is elliptic. Indeed, if $(z, \zeta, z', \zeta') \in T^*Z^2 = (T^*Z)^2$ is a non-zero vector where $z \neq z'$ then either A can be chosen to be elliptic at z and vanish near z' or conversely and then $A - A^t$ is elliptic at this point. If $z = z'$ but $\zeta \neq -\zeta'$ with one of these non-zero then A can be chosen to be elliptic at one point and characteristic at the other, making $A - A^t$ microlocally elliptic.

In particular the kernel of L is smooth away from the diagonal. Cutting it off appropriately, L can be decomposed into the sum of a smoothing operator and an operator with kernel supported in a preassigned neighbourhood of the diagonal. The smoothing term gives a derivation so it is enough to suppose that L has support near the diagonal and then it is readily analysed in local coordinates. It is enough to consider its action as a map from sections supported in a coordinate patch over which V is trivial, into sections on the same coordinate patch, for a finite covering of Z by coordinate charts. In such coordinates z, L can be written in Weyl form

$$L(z, z') = (2\pi)^{-n} \int g(\frac{z + z'}{2}, \zeta) e^{iz\cdot(z - z')} d\zeta |dz'| \tag{19}$$

where $g \in \mathcal{C}^\infty(\Omega \times \mathbb{R}^n) \otimes M(N; \mathbb{C})$ is polynomially bounded. The smoothness in the base variables follows from the restriction on the wavefront set obtained above and the smoothness in the fibre variables from the compactness of the support in the normal direction to the diagonal, i.e. in $z - z'$. Note that the function g is well-defined locally.

Now the conjugation condition (18) implies that

$$\begin{aligned} [L, z_j] &\in \Psi^k(\Omega'; \mathbb{C}^N), \\ [L, D_{z_j}] &\in \Psi^{k+1}(\Omega'; \mathbb{C}^N) \end{aligned} \tag{20}$$

since in both cases the differential operators z_j and D_{z_j} can be cut off very close to the boundary of the coordinate patch and then (20) holds in some slightly smaller domain Ω'. Since the test operators are local, the kernels on the right are supported in the same neighbourhood of the diagonal as the kernel of L. Thus the pseudo-differential operators can also be written locally uniquely in Weyl form (19) and it follows from this uniqueness that

$$\begin{aligned} D_{\zeta_j} g &\in S^k_{\mathrm{phg}}(\Omega' \times \mathbb{R}^n) \otimes M(N, \mathbb{C}), \\ D_{z_j} g &\in S^{k+1}_{\mathrm{phg}}(\Omega' \times \mathbb{R}^n) \otimes M(N, \mathbb{C}) \end{aligned} \tag{21}$$

where the spaces on the right consist of the (matrix-valued) classical symbols of some integral order. From the first of these it follows that

$$\zeta \cdot \partial_\zeta g = h \in S^{k+1}_{\mathrm{phg}}(\Omega' \times \mathbb{R}^n) \otimes M(N, \mathbb{C}). \tag{22}$$

This differential equation is easily solved near infinity in ζ. Namely, each of the terms which are homogeneous of non-zero degree on the right can be solved away by a multiple on the left. Taking an asympotic sum of these terms gives an element $g' \in S^{k+1}_{\mathrm{phg}}(\Omega' \times \mathbb{R}^n) \otimes M(N, \mathbb{C})$ such that

$$\zeta \cdot \partial_\zeta g' = h - h_0 - h'', \ h'' \in S^{-\infty}(\Omega' \times \mathbb{R}^n) \otimes M(N, \mathbb{C}) \tag{23}$$

and where h_0 is homogeneous of degree 0 in $|\zeta| > 1$. The rapidly decaying term can be integrated away radially to give a rapidly decaying solution of $\zeta \cdot \partial_\zeta g'' = h'' - r$

where r has support in $|\zeta| < 1$. It follows that

$$g = g' + g'' + \log|\zeta| \cdot h_0(z,\zeta) + g_0(z,\zeta) \text{ in } |\zeta| \geq 1 \tag{24}$$

where all terms are smooth and g_0 is homogeneous of degree 0 in ζ. Substituting this back into (21) – and noting that all other terms are classical – it follows that

$$D_{\zeta_j} h_0(z,\zeta) = 0, \ D_{z_j} h_0(z,\zeta) = 0. \tag{25}$$

Thus in fact h_0 is constant provided the cosphere bundle is connected, i.e. Z is connected and not the circle. Since the commutator with all constant matrices must also be classical it follows that h_0 must be a constant multiple of the identity matrix

$$g(z,\zeta) = c\,\mathrm{Id}\log|\zeta| + \tilde{g}, \ \tilde{g} \in S^{k+1}_{\mathrm{phg}}(\Omega' \times \mathbb{R}^n) \otimes M(N,\mathbb{C}) \text{ in } |\zeta| > 1. \tag{26}$$

Now, consider some positive elliptic second order differential operator with scalar principal sybmol acting on sections of V and take its complex powers, see [40], P^z. Then $\log P = dP^z/dz$ at $z = 0$ is a globally defined pseudodifferential operator which, whilst non-classical, acts as a derivation on the classical operators since conjugation by P^z maps $\Psi^m(\Omega;\mathbb{C}^N)$ to itself. Moreover the Weyl symbol of $\log P$ is precisely of the form of a classical symbol (of order 0) plus $\log|\zeta|$ in any local coordinates. It follows that everywhere locally

$$L - c\log P \in \Psi^{k+1}(Z;V) \tag{27}$$

and hence this is globally true. This completes the proof of Proposition 1.1. □

2. Derivations of formal pseudodifferential operators

In the same setting as the preceding section, with Z a compact manifold and $\Psi^m(Z;V)$ denoting the space of all classical pseudodifferential operators of order m on Z acting on sections of a complex vector bundle V over Z, the quotient $\mathcal{B}^{\mathbb{Z}} = \mathcal{B}^{\mathbb{Z}}(T^*Z;\hom V) = \Psi^{\mathbb{Z}}/\Psi^{-\infty}(Z;V)$ is the space of formal pseudodifferential operators, also called the full symbol algebra. It may be identified by a (non-canonical) choice of quantization with the space of 'Laurent' series of infinite sums of homogeneous sections, of integral degree, of $\hom(V)$ over $T^*Z\setminus 0$ with homogeneity bounded below but not above. It is then a star algebra in the sense that the product is the local bundle composition at top level of homogeneity with the second term, when the bundle is locally trivialized and Weyl quantization is chosen, given by the Poisson bracket extended to matrices. The algebra acts on itself as a Lie algebra of derivations with only multiples of the identity acting trivially.

Proposition 2.1. *If Z is compact and connected, the space $\mathrm{Out}(\mathcal{B}^{\mathbb{Z}})$ of filtered outer derivations on $\mathcal{B}^{\mathbb{Z}}$, the quotient of (algebraic) derivations by inner derivations, gives a short exact sequence*

$$0 \longrightarrow \mathrm{Out}(\Psi^{\mathbb{Z}}) \longrightarrow \mathrm{Out}(\mathcal{B}^{\mathbb{Z}}) \longrightarrow H^1(S^*Z;\mathbb{C}) \longrightarrow 0. \tag{28}$$

This sequence serves to explain the assumption made by Duistermaat and Singer [13] that $H^1(S^*Z;\mathbb{C}) = 0$.

Proof. The filtered derivations on $\Psi^{\mathbb{Z}}(Z;V)$ certainly induce such derivations on the quotient, with inner derivations mapped to inner derivations, so the first map in (28) is well-defined. It is also clearly injective from the 1-dimensional space generated by $[\log Q, \cdot]$ or from the two-dimensional case for the circle. We proceed to characterize all the filtered derivations on the formal symbol algebra.

Choosing a metric on Z, the real powers of the metric function on $T^*Z \setminus 0$ are homogeneous of any given degree. This allows the leading part of a derivation $D : \mathcal{B}^{\mathbb{Z}} \longrightarrow \mathcal{B}^{\mathbb{Z}+k}$ to be normalized to map $D_k : \mathcal{C}^\infty(S^*Z; \hom V) \longrightarrow \mathcal{C}^\infty(S^*Z; \hom V)$ which is again a derivation for the matrix product. Directly from the definition such a map is local with the value at any given point only depending on the 1-jet, i.e. is given by a linear differential operator of first order. In fact, all terms in the star product are local with dependence on increasing order of jets, so evaluated at each point the derivation is given by a linear differential operator on each term in the full symbol. To examine this operator it suffices to work in local coordinates and in terms of a local trivialization of V. Thus D_k, the leading term of the derivation, is given by a matrix of first order differential operators.

The constant term in D_k is determined by evaluating the derivation on constant sections of $\hom V = M(N, \mathbb{C})$ near the point. When either factor is constant the star product is just the matrix product. Necessarily the constant term is then a derivation on $\hom V_p = \mathbb{C}^N$ at each point and therefore is given by commutation with a matrix which is uniquely determined up to addition of a multiple of the identity. The choice of a matrix of trace zero is therefore unique. Since the derivation maps smooth sections to smooth sections this defines a smooth section of $\mathcal{C}^\infty(S^*Z; \hom V)$. Composing with the appropriate power of the metric used to normalize the leading part of the derivation above, this gives the leading term of a symbol, which therefore gives an inner derivation with the same constant term. Subtracting this from the original derivation gives a derivation with leading term D_k which annihilates the constant matrices of order 0. It follows that this leading derivation is given by a section of $T(S^*Z) \otimes \hom V$ with the first factor acting as a vector field in local coordinates. In fact it must then distribute over multiplication, at each point, by constant matrices

$$D_k(uu_0)(p) = (D_k(u))u_0(p),\ \nabla u_0(p) = 0$$

at that point. So in fact D_k is reduced to a differential operator with scalar principal symbol which vanishes on the constants, i.e. a vector field acting as a multiple of the identity.

The action of D on sections of any integral homogeneity m can be deduced by composition with the appropriate power of the metric function g. Thus on symbols of order m the leading part of the derivation of order $m+k$ is

$$\begin{aligned} D(u)_k &\equiv (D(u)g^{-m-k})_0 \\ &\equiv D(ug^{-m}) \cdot g^{-k} - mua^{-m}(g^{-k-1}Dg) \equiv D_k(ug^{-m}) - mbu \end{aligned}$$

in terms of equality of the leading parts of symbols of order 0 and with b an element of $\mathcal{C}^\infty(S^*M)$. Thus in general the leading part of D as a map from homogenous sections of degree m to homogeneous sections of degree $m+k$ is given by a scalar vector field which is homogeneous of degree k.

Next we analyse the second term in homogeneity of the derivation. Working now up to error of relative homogeneity 2 the derivation induces a second map

$$D(u) = D_k u + S_{k-1}(u)$$

where S_{k-1} increases homogeneity (of each term) by $k-1$. In terms of Weyl quantization the star product, up to second order, is

$$u * v = uv - \frac{i}{2}\{u, v\} \tag{29}$$

where $\{u, v\}$ is the Poisson bracket acting in the components of matrix multiplication. So the derivation identity on the product of two scalar elements and a constant matrix, uvE can be written

$$D_k(uv)E = D_k(u)vE + uD_k(v)E.$$

$$S_{k-1}(uvE) - S_{k-1}(u)vE - uS_{k-1}(vE) = \frac{i}{2}\big(D_k\{u, v\}E - \{D_k u, v\}E - \{u, D_k v\}E\big).$$

The left side certainly involves no more than three derivatives in total, but actually is only of second order. Taking $u(p) = v(p) = 0$ allows the principal symbol of this second order differential operator to be computed – the left side is therefore symmetric in the first derivatives of u and v whereas the right side is clearly anti-symmetric. So in fact both sides are of order 1 and since the right side annihilates constants in either u or v it must vanish identically (and S_{k-1} must itself be a derivation). The resulting identity is precisely the condition that D_k be a symplectic vector field, i.e. one which distributes over the Poisson bracket, and hence is of the form

$$\omega(\cdot, D_k) = \alpha,\ d\alpha = 0$$

where α is a closed form on $T^*Z \setminus 0$ which is homogeneous of degree k.

For any degree other than 0 the closed homogeneous forms on $T^*Z \setminus 0$ are exact. In degree 0 such a form is the sum of the pull-back from S^*Z of a closed form, plus a multiple of the closed form $g^{-1}dg$ given by the metric function. The latter is exact in the sense that it is $d \log g$ which corresponds precisely to the derivation given by $\log Q$. The exact forms arise from inner derivations.

The elements of the cohomology $H^1(S^*Z, \mathbb{C})$ do correspond to derivations on $\mathcal{B}^{\mathbb{Z}}(T^*Z; \hom V)$. This can be seen for instance by passing to the universal cover $\tilde{Z}$ of Z. Since the formal symbol algebra corresponds to localization at the diagonal it can be identified with the π_1-invariant part of the quotient of the properly supported pseudodifferential operators on $\tilde{Z}$ by the properly supported smoothing operators. On $T^*\tilde{Z} \setminus 0$ every closed form is exact and the elements of $H^1(S^*Z; \mathbb{C})$ correspond to smooth functions on $S^*\tilde{Z}$ which are π_1-invariant up to shifts by constants. These can be realized as multiplication operators, hence as properly supported pseudodifferential operators, on $\tilde{Z}$, commutation with which induces π_1-equivariant derivations on the formal symbol algebra, and hence fully π_1-invariant derivations on the invariant subalgebra, i.e. derivations on $\mathcal{B}^{\mathbb{Z}}(T^*Z; \hom V)$.

This completes the proof of (28) and hence the Proposition. □

3. Automorphisms of pseudodifferential operators

Next we consider the group of order-preserving automorphisms of the classical pseudodifferential algebra; this group was characterized by Duistermaat and Singer. We recall and somewhat extend the main theorem from [13]. If $\chi : S^*Z \longrightarrow S^*Z'$ is a contact transformation between two compact manifolds, which is to say a canonical diffeomorphism between their cosphere bundles, let $\mathcal{F}(\chi)$ denote the linear space of Fourier integral operators associated to χ of complex order s; thus each $F \in \mathcal{F}^s(\chi)$ is a linear operator $F : \mathcal{C}^\infty(Z) \longrightarrow \mathcal{C}^\infty(Z')$ which has Schwartz kernel which is a Lagrangian distribution with respect to the twisted graph of χ ([22]). For the convenience of the reader a very brief discussion of Fourier Integral operators can be found in the appendix.

Theorem 3.1. *For any two compact manifolds M_1 and M_2, and complex vector bundles V_1 and V_2 over them, every linear order-preserving algebra isomorphism $\Psi^{\mathbb{Z}}(M_1; V_1) \longrightarrow \Psi^{\mathbb{Z}}(M_2; V_2)$ is of the form*

$$\Psi^{\mathbb{Z}}(M_1; V_1) \ni A \longrightarrow FAF^{-1} \in \Psi^{\mathbb{Z}}(M_2; V_2) \tag{30}$$

where $F \in \mathcal{F}^s(\chi; V_1, V_2)$ is a classical Fourier integral operator of complex order s associated to a canonical diffeomorphism $\chi : T^*M_1 \setminus 0 \longrightarrow T^*M_2 \setminus 0$ and having inverse $F^{-1} \in \mathcal{F}^{-s}(\chi^{-1}; V_2, V_1)$; F is determined by (30) up to a non-vanishing multiple of the identity.

This is the result of [13] except the restriction that $H^1(S^*Z) = \{0\}$ is removed; for simplicity of presentation the case of non-compact manifolds is not considered here, but on the other hand the action on sections of vector bundles is included. The proof is also essentially that of [13] with some rearrangement; it is closely parallel to the discussion of derivations above. The only significant differences from [13] are the use of a microlocal regularity argument in place of some of the more constructive methods in the original and an argument using spectral theory to eliminate the 'anticanonical' possibility. Note that in general there are *no* invertible Fourier integral operators between two manifolds. For such operators to exist, the manifolds must certainly have the same dimension, the cosphere bundles must be contact-diffeomorphic and the vector bundles must also have the same rank. There is also an index obstruction, [15], [25].

We start with a more general result for the automorphisms of the algebra of smoothing operators. This is a form of the 'Eidelheit Lemma' from [13]. Since the setting is slightly different we give a proof.

Proposition 3.1. *For any two compact manifolds M_1 and M_2, and complex vector bundles V_1 and V_2 over them, every linear algebra isomorphism $\Psi^{-\infty}(M_1; V_1) \longrightarrow \Psi^{-\infty}(M_2; V_2)$ is of the form*

$$\Psi^{-\infty}(M_1; V_1) \ni A \longrightarrow GAG^{-1} \in \Psi^{-\infty}(M_2; V_2) \tag{31}$$

where $G : \mathcal{C}^\infty(M_1; V_1) \longrightarrow \mathcal{C}^\infty(M_2; V_2)$ is a topological isomorphism (with respect to the standard Fréchet topology) with formal transpose $G^t : \mathcal{C}^\infty(M_2; V_2 \otimes \Omega) \longrightarrow \mathcal{C}^\infty(M_1; V_1 \otimes \Omega)$ which is also a topological isomorphism in the same sense.

Proof. Consider such an isomorphism between the algebras of smoothing operators

$$\begin{gathered} L : \Psi^{-\infty}(M_1; V_1) \xrightarrow{\simeq} \Psi^{-\infty}(M_2; V_2) \\ L(A \circ B) = L(A) \circ L(B). \end{gathered} \tag{32}$$

First, note (this is essentially Eidelheit's argument from [14]) that the elements, R, of rank 1 in $\Psi^{-\infty}(M; V)$ are characterized algebraically by the condition that for any other element A the composite $(AR)^2 = c(AR)$ for some constant c. In one direction this is just the observation that $RAR = cR$. Conversely if R has rank two or greater it is straightforward to construct a finite rank smoothing operator A which does not have this property.

Thus L must map the elements of rank 1 in the domain onto the corresponding set in the range space. As in §1, choose Hermitian inner products on the bundles

V_1 and V_2 and positive smooth densities, ν_i, on each of the manifolds. The inner products induce conjugate linear isomorphisms with the duals, $V_i' \longrightarrow V_i$, and this allows the smoothing operators to be identified with smooth sections of the homomorphism bundle but acting through the antilinear isomorphism (fixed by the inner products) from V_i to V_i' which will be denoted by replacing $\phi \in \mathcal{C}^\infty(M_i; V_i)$ by $\phi' \in \mathcal{C}^\infty(M_i, V_i')$:

$$\begin{gathered}\Psi^{-\infty}(M_i, V_i) \equiv \mathcal{C}^\infty(M_i^2; V_i \boxtimes V_i),\\ A\phi(x) = \int_{M_i} \langle a(x,\cdot), \phi'(\cdot)\rangle_{V_i} \nu_i.\end{gathered} \tag{33}$$

Now choose a non-zero element $v \in \mathcal{C}^\infty(M_1; V_1)$ with $\int_{M_1} \langle v, v\rangle_{V_1} \nu_1 = 1$. This defines a projection of rank 1, π_v with kernel under the identification (33)

$$p(x,y) = v(x)v(y) = (v \otimes v)(x,y) \in \mathcal{C}^\infty(M_1^2; V_1 \otimes V_1). \tag{34}$$

The image of this projection under L is also a projection of rank one. Choose an element $w \in \mathcal{C}^\infty(M_2; V_2)$ in the range of $L(\pi_v)$; it follows that there exists some element $h \in \mathcal{C}^\infty(M_2; V_2)$ with $\int_{M_2} \langle w, h\rangle_{V_2} \nu_2 = 1$ such that

$$L(\pi_v) \text{ has kernel } w \otimes h \in \mathcal{C}^\infty(M_2^2; V_2 \otimes V_2). \tag{35}$$

Now, consider the subset of $\Psi^{-\infty}(M_1; V_1)$ consisting of the rank one elements, R, such that $R\pi_v = R$. These have kernels of the form

$$r = \phi \otimes v \tag{36}$$

and $L(R)$ has rank one and satisfies $L(R)L(\pi_v) = L(R)$ so has kernel of the form

$$\psi \otimes h \tag{37}$$

where $\psi \in \mathcal{C}^\infty(M_2; V_2)$ is uniquely, and hence linearly, determined by ϕ. This fixes a linear isomorphism

$$G : \mathcal{C}^\infty(M_1; V_1) \longrightarrow \mathcal{C}^\infty(M_2; V_2) \text{ s.t. } L(\phi \otimes v) = (G\phi) \otimes h \tag{38}$$

in terms of kernels.

This argument can be repeated for operators of rank one, $S \in \Psi^{-\infty}(M_1; V_1)$ such that $\pi_v S = S$. This induces a second linear (algebraic) isomorphism $H : \mathcal{C}^\infty(M_1; V_1) \longrightarrow \mathcal{C}^\infty(M_2; V_2)$ such that

$$L(v \otimes \psi) = w \otimes (H\psi). \tag{39}$$

Now, the composites of these two classes of operators are multiples of the projections:

$$SR = c\pi_v,\ S = v \otimes \eta,\ R = \phi \otimes v \Longrightarrow c = \int_{M_1} \langle \eta, \phi\rangle_{V_1}. \tag{40}$$

Since $L(SR) = L(S)L(R)$ it follows that

$$\begin{gathered}\int_{M_1} \langle \eta, \phi\rangle_{V_1} = \int_{M_2} \langle H\eta, G\phi\rangle_{V_2},\\ \int_{M_1} \langle H^{-1}\psi, \phi\rangle_{V_1} = \int_{M_2} \langle \psi, G\phi\rangle_{V_2}\end{gathered} \tag{41}$$

for all smooth sections. This shows that G is a closed operator and hence, is continuous. Moreover, H^{-1} is the adjoint of G.

Taking the composite the other way gives a general smoothing operator of rank one:

$$RS \text{ has kernel } \phi\otimes\eta \Longrightarrow L(R) \text{ has kernel } G\phi\otimes H\eta = (G\otimes H)\phi\otimes\eta \tag{42}$$

where $G\otimes H : \mathcal{C}^\infty(M_1^2; V_1\boxtimes V_1) \longrightarrow \mathcal{C}^\infty(M_2^2; V_2\boxtimes V_2)$. This is the formula (31) on the elements of rank one, and hence its linear span. Now the formula (31) follows in general, since a smoothing operator is determined by its composites with smoothing operators of rank one and these composites are themselves of rank one, so

$$\begin{aligned} L(A\circ R) = L(A)\circ L(R) = GA\circ RG^{-1} = GAG^{-1}GRG^{-1} \\ \Longrightarrow L(A) = GAG^{-1}. \end{aligned} \tag{43}$$

□

Proof of Theorem 3.1. The assumption that the isomorphism is order-preserving implies that it induces an isomorphism between the smoothing ideals, so Proposition 3.1 applies directly and gives G for which (30) holds, since as in the proof above a pseudodifferential operator is determined by its composites with rank one smoothing operators.

Denoting this isomorphism now by F, it remains to show that it is a Fourier integral operator. The notation for kernels will be continued from the discussion above, corresponding to a choice of innner products on the bundles and smooth positive-definite densities on the manifolds.

Consider $A \in \Psi^1(M_1; V_1)$ which is invertible with inverse $A^{-1} \in \Psi^{-1}(M_1; V_1)$ and hence in particular is elliptic; such an operator always exists. Let the image be $A' = FAF^{-1} \in \Psi^1(M_2; V_2)$; it must similarly be invertible and hence elliptic. If the Schwartz kernel of F is again denoted $F \in \mathcal{C}^{-\infty}(M_2\times M_1; V_2\otimes V_1)$, then

$$FA = A'F \Longrightarrow BF = 0,\ B = A'\otimes\mathrm{Id} - \mathrm{Id}\otimes A^* \tag{44}$$

Although (44) is not a pseudodifferential equation, since B is not in general a pseudodifferential operator on $M_2\times M_1$, it behaves as though it were in terms of regularity.

Proposition 3.2. *The Schwartz kernel of F satsifies*

$$\begin{aligned} \mathrm{WF}(F)\cap((T^*M_2\setminus\{0\})\times\{0\}) = \emptyset = \mathrm{WF}(F)\cap(\{0\}\times(T^*M_1\setminus\{0\})), \\ \mathrm{WF}(F)\subset\{\sigma(A)(x,-\xi) = \sigma(A')(y,\eta)\}\subset(T^*M_2\setminus\{0\})\times(T^*M_1\setminus\{0\}). \end{aligned} \tag{45}$$

Proof. The operator B in (30) is microlocally a pseudodifferential operator away from the zero sections in each factor of the product. The second part of (45) corresponds to elliptic regularity in this region, meaning that the wavefront set of a solution F to $BF = 0$ can only have wavefront set where B is non-elliptic, i.e. outside its characteristic variety. So, it is actually the first part of (46) that is not quite obvious. Fortunately we are able to choose either A or A' to be a differential operator, say of order 2 since we know that such a globally elliptic operator, with positive diagonal symbol, does exist, indeed by adding large positive constant to it we can assume it is invertible. The corresponding transformed operator will not in general be differential but must be invertible, with inverse of order -2 and hence must be elliptic. It follows not only that the corresponding B in (46) is microlocally a pseudodifferential operator of order 2 away from the zero section of T^*M_2, and *is* a pseudodifferential operator near the zero section of T^*M_1 but also that it is elliptic near the zero section of T^*M_1. So the second half of the first part of (46) again follows by (microlocal) elliptic regularity. Reversing the roles of M_1 and M_2 the same argument applies to give microlocality near the zero section of T^*M_2. □

Note in particular that the disjointness of the wavefront set of the kernel from the zero section in either factor implies that if $A \in \Psi^{-\infty}(M_1; V_1)$ is a smoothing operator, or $A' \in \Psi^{-\infty}(M_2; V_2)$, then separately

$$FA,\ A'F \text{ have kernels in } \mathcal{C}^\infty(M_2 \times M_1; V_2 \boxtimes V_1' \otimes \Omega(M_1)) \tag{46}$$

– rather than just the difference being smoothing which follows directly from the algebra condition.

Now, the earlier parts of the argument in [13], which we briefly recall, can be followed. Since, for any manifold and vector bundle, $\Psi^0(M; V)/\Psi^{-1}(M; V) = \mathcal{C}^\infty(S^*M; \hom V)$ is an isomorphism of algebras, the isomorphism L must induce an algebra isomorphism of l of $\mathcal{C}^\infty(S^*M_1; \hom V_1)$ to $\mathcal{C}^\infty(S^*M_2; \hom V_2)$. This in turn must map maximal and prime ideals to such ideals. These ideals in $\mathcal{C}^\infty(S^*M; \hom V)$ correspond to vanishing at a point and hence l induces a bijection $\chi : S^*M \longrightarrow S^*M'$ and l must itself be pull-back with respect to the inverse of l. Thus χ must be a diffeomorphism such that the second part of (45) is refined to

$$\mathrm{WF}(F) \subset \mathrm{graph}'(\chi), \tag{47}$$

the twisted graph. The same argument of course applies to the inverse of G and the inverse of χ.

The quotient $\Psi^1(M; V)/\Psi^0(M; V)$ is the space of smooth sections on $T^*M \setminus 0_M \longrightarrow \pi^* \hom(V)$ which are homogeneous of degree 1. It is generated, modulo the multiplicative action of $\mathcal{C}^\infty(S^*M; \hom V)$, by the symbol of one element with scalar principal positive and elliptic symbol. In [13] it is observed that the symbol of the image must be real and non-vanishing. In fact it follows easily that it must be positive, at least for compact manifolds. Indeed, $A + \tau \operatorname{Id}$ is invertible for all $\tau \in \mathbb{C} \setminus (-\infty, 0)$ so the same must be true of A'. If its symbol were negative then $A' = Q_0 - Q_1$ where Q_1 is positive (so self-adjoint). It follows that $P' - t$ has an inverse in $\Psi^{-1}(M')$ for $t > T$, so the same is true of A which contradicts the spectral theorem. Thus in fact the image of the symbol of A must be positive.

Again as in [13] it follows that the symbols of A' and A for all orders are related by a homogeneous diffeomorphism which projects to χ. The behaviour of commutators shows that this diffeomorphism must be symplectic for the difference symplectic structure, that is 'canonical'.

Lemma 2 in [13] now applies without the possibility of the 'anticanonical' map. So, there is a homogeneous canonical transformation, still denoted χ, such that under the isomorphism $\sigma(A) = \chi^*\sigma(A')$ and $\mathrm{WF}(F) \subset \mathrm{graph}'(\chi)$.

The remaining steps are now slightly changed in that we are trying to show that the given operator F is a Fourier integral operator associated to χ. We can now work with the weaker intertwining condition that for each $A \in \Psi^k(M)$ there exists $A' \in \Psi^k(M)$ such that the kernels satisfy

$$FA = A'F + \mathcal{C}^\infty(M' \times M; V \otimes V' \otimes \Omega_R), \tag{48}$$

that is, they are conjugate up to smoothing errors.

Choose a global pair of elliptic Fourier integral operators, G and H, associated to χ and χ^{-1} not necessarily invertible but essential inverses of each other. Consider $\tilde{F} = HF$ acting now from $\mathcal{C}^\infty(M_1; V_1)$ to $\mathcal{C}^\infty(M_1; V_1)$. From the calculus of wavefront sets this has operator wavefront set in the identity relation, the same as a pseudodifferential operator and we wish to show that it is one. From (48) we deduce that for each $A \in \Psi^k(M)$ there exists $I(A) \in \Psi^k(M)$, namely $I(A) = HA'G$, such that

$$\tilde{F}A = I(A)\tilde{F} + A'',\ A'' \in \Psi^{-\infty}(M),\ E(A) = I(A) - A \in \Psi^{k-1}(M). \tag{49}$$

Again we proceed essentially as in [13], see also § 1. The symbol of $E(A)$ is given by a derivation, hence a vector field, homogeneous of degree -1, applied to the symbol of A and which distributes over the Poisson bracket:

$$\sigma_{k-1}(E(A)) = V\sigma_k(A),\ V\{a, b\} = \{Va, b\} + \{a, Vb\}. \tag{50}$$

From this it follows that it is locally Hamiltonian on $T^*M \setminus 0_M$, meaning that

$$\omega(V, \cdot) = \gamma \tag{51}$$

where γ is a well-defined smooth closed 1-form which is homogeneous of degree 0 on $T^*M \setminus 0_M$. The only such 1-forms are locally (on cones in $T^*M \setminus 0_M$) the differentials of the sum of a smooth function homogeneous of degree 0 and $s \log |\xi|$ where $|\xi|$ is a real positive function homogeneous of degree 0 and s is a complex constant.

Now working microlocally, and iterating the argument over orders as in [13], we can construct an elliptic pseudodifferential operator $D \in \Psi^s(M)$ which, by formal conjugation, gives the same relation with $I(A)$ as in (49) for all A with essential support near the given point. It follows that the corresponding point on the diagonal is not in $\mathrm{WF}'(\tilde{F} - D)$. This however proves that $\tilde{F}$ is *globally* a pseudodifferential operator, since it is a well-defined operator with the correct wavefront set relation and is microlocally everywhere a pseudodifferential operator. Hence F is a Fourier integral operator as claimed. □

4. Group of invertible Fourier integral operators

Let us consider in more detail the group, $\mathrm{G}(\mathcal{F}^0(Z; V))$, of invertible Fourier integral operators of order 0 on a fixed complex vector bundle V over a compact manifold Z. The topology on the symbolic quotient of the algebra of Fourier integral operators is discussed by Adams et al in [5]. The group of Fourier integral operators is shown to be a Fréchet group by Omori in [36] and in papers cited there with Kobayashi, Maeda and Yoshioka. It is useful (although not really exploited in the present paper for which the results in [36] suffice) to have somewhat stronger results for the group of pseudodifferential so we proceed to discuss the topology in some detail.

We start by examining the group of invertible classical pseudodifferential operators of order 0, denoted $G^0(Z; V) \subset \Psi^0(Z; V)$. This is an open subset which is a Fréchet Lie group, but much more is true. Namely we show that there is a decreasing sequence of Banach algebras such that $G^0(Z; V)$ is the projective limit of the corresponding sequence of the groups of invertibles. The basis of this is the characterization of pseudodifferential operators by commutation conditions due to R. Beals, [6, 7, 8]. Beals was interested in showing that pseudodifferential operators could be characterized in terms of boundedness properties of commutators and composites with differential operators. Here we are only trying to obtain the natural topology on the (group of invertible) pseudodifferential operators, and then

Fourier integral operators, so we are free to simplify the discussion by using pseudodifferential operators in the defining properties.

Let $\mathcal{G}_0 = \mathrm{G}(L^2(Z;V))$ be the group of invertible bounded linear operators acting on the Hilbert space $L^2(Z;V)$; this is the group of invertible elements in the algebra $\mathcal{B}_0 = \mathcal{B}(L^2(Z;V))$ of bounded linear operators. We proceed to define, inductively, a sequence of linear subspaces

$$\mathcal{B}_j \subset \mathcal{B}_{j-1} \cdots \subset \mathcal{B}_0. \tag{52}$$

Namely, $A \in \mathcal{B}_j$, for $j \in \mathbb{Z}$, provided the following conditions hold.

(1) $A \in \mathcal{B}_{j-1}$.
(2) A restricts to the first Sobolev space, $A(H^1(Z;V)) \subset H^1(Z;V)$ and if $f \in \mathcal{C}^\infty(Z)$ and $D \in \Psi^1(Z;V)$ then $D[f,A]$, $[f,A]D \in \mathcal{B}_{j-1}$, by continuous extension in the first case.
(3) If $D \in \Psi^1(Z;V)$ is such that $\sigma_1(D)$ is pointwise a multiple of the identity then $[D,A] \in \mathcal{B}_{j-1}$.
(4) If $D \in \Psi^1(Z;V)$, $E \in \Psi^{-1}(Z;V)$ then DAE, $EAD \in \mathcal{B}_{j-1}$, again by continuous extension in the second case.

Then we set $\mathcal{G}_j = \mathcal{B}_j \cap \mathcal{G}_0$. Although a further refinement is needed to yield the classical pseudodifferential operators, we first analyze the properties of these $\mathcal{B}_j$ and $\mathcal{G}_j$.

Proposition 4.1. *Each $\mathcal{B}_j$ is a Banach algebra and $\mathcal{G}_j$ is a smooth Banach Lie group which is the open subset of invertible elements in $\mathcal{B}_j$ for which the product and inverse maps are smooth with respect to the Banach manifold topology.*

Proof. Certainly $\mathcal{B}_0$ is a Banach space and $\mathcal{G}_0$ is a Banach Lie group which is the open subset of invertible elements. So it suffices to proceed by induction.

By standard properties of pseudodifferential operators,

$$\Psi^0(Z;V) \subset \mathcal{B}_j \ \forall\ j. \tag{53}$$

Indeed, all the elements in the defining conditions are then in $\Psi^0(Z;V)$ so the statement follows by induction. From this it follows that each $\mathcal{B}_j$ is a right and left module over $\Psi^0(Z;V)$ which is used without further comment below.

To simplify the defining conditions of $\mathcal{B}_j$, let $Q \in \Psi^1(Z;V)$ be elliptic and invertible with diagonal principal symbol.

First we check that $\mathcal{B}_j$ defined by these conditions in terms of some $\mathcal{B}_{j-1}$ is an algebra if $\mathcal{B}_{j-1}$ is an algebra. So suppose A_1, $A_2 \in \mathcal{B}_j \subset \mathcal{B}_{j-1}$ and consider condition (2). Then $A_1A_2 \in \mathcal{B}_{j-1}$, it must restrict to map the Sobolev space $H^1(Z;V)$ into itself and

$$\begin{aligned} &D[A_1A_2, f] \\ &\qquad = DA_1[A_2,f] + D[A_1,f]A_2 = DA_1Q^{-1}Q[A_2,f] + D[A_1,f]A_2 \in \mathcal{B}_{j-1} \end{aligned} \tag{54}$$

using (2) and (3) for the factors. Similarly for $[A_1A_2,f]D$, so (2) holds for the product. Condition (3) is even simpler and (4) holds for A_1A_2 since

$$DA_1A_2E = (DA_1Q^{-1})(QA_2E),\ EA_1A_2Q = (EA_1Q)(Q^{-1}A_2D). \tag{55}$$

It follows from an inductive application of this argument that all the $\mathcal{B}_j$ are algebras.

At this stage, it is convenient to refine the second condition. Note that if f, $g \in \mathcal{C}^\infty(M)$ have disjoint supports then $DfAg = D[f,A]g$. So it follows from the second condition that

$$fAgD,\ DfAg \in \mathcal{B}_{j-1} \text{ if } f,\ g \in \mathcal{C}^\infty(M),\ \operatorname{supp}(f) \cap \operatorname{supp}(g) = \emptyset. \tag{56}$$

To exploit this, choose a finite covering of the manifold by coordinate patches, $U_a \subset M$ each diffeomorphic to a ball centred at the origin and with a fixed coordinate system $x_{k,a} \in \mathcal{C}^\infty(U_a)$, $1 \le k \le \dim M$. Then choose a partition of unity $\chi_a \in \mathcal{C}^\infty(M)$ subordinate to the U_a and a second collection of functions $\chi'_a \in \mathcal{C}^\infty_c(U_a) \subset \mathcal{C}^\infty(M)$ such that $\chi'_a \equiv 1$ in a neighourhood of $\operatorname{supp}(\chi_a)$ for each a. This allows any operator $A \in \mathcal{B}_l$ to be decomposed as

$$A = \sum_a A_a + \sum_a \chi_a A(1-\chi'_a),\ A_a = \chi_a A \chi'_a, \tag{57}$$

where all terms are in $\mathcal{B}_l$. Thus the second condition above defining $A \in \mathcal{B}_j$, where we know that $A \in \mathcal{B}_{j-1}$, certainly implies that

$$\begin{gathered} Q\chi_a A(1-\chi'_a),\ \chi_a A(1-\chi'_a)Q \in \mathcal{B}_{j-1}\ \forall\ a, \\ Q[x_{k,a}, A_a],\ [x_{k,a}, A_a]Q \in \mathcal{B}_{j-1}\ \forall\ a, k. \end{gathered} \tag{58}$$

The second collection of conditions makes sense since supports are confined to U_a.

Conversely, this finite collection of conditions implies the second condition above for all $f \in \mathcal{C}^\infty(M)$. Again we prove this by induction, which in particular will show that we may define a norm $\|\cdot\|_j$ on $\mathcal{B}_j$ by setting

$$\begin{aligned} \|A\|_j = \|A\|_{j-1} &+ \sum_a \|Q\chi_a A(1-\chi'_a)\|_{j-1} + \sum_a \|\chi_a A(1-\chi'_a)Q\|_{j-1} \\ &+ \sum_{a,k} \|Q[x_{k,a}, A_a]\|_{j-1} + \sum_{a,k} \|Q[x_{k,a}, A_a]\|_{j-1} \\ &+ \|[Q, A]\|_{j-1} + \|QAQ^{-1}\|_{j-1} + \|Q^{-1}AQ\|_{j-1}. \end{aligned} \tag{59}$$

Lemma 4.2. *The conditions* (58) *together with the requirements* $[Q, A]$, QAQ^{-1} *and* $Q^{-1}AQ \in \mathcal{B}_{j-1}$ *on* $A \in \mathcal{B}_{j-1}$ *imply that* $A \in \mathcal{B}_j$ *and* (59) *(defined inductively) is a norm on* $\mathcal{B}_j$ *with respect to which it is a Banach algebra, so* $\|A_1A_2\|_j \le C_j\|A_1\|_j\|A_2\|_j$ *for all* $A_1,\ A_2 \in \mathcal{B}_j$.

Proof. Proceeding by induction we may suppose the result known for $\mathcal{B}_l$ for $l \le j-1$. Expanding DfA using (57) the terms arising from the second sum are already in $\mathcal{B}_{j-1}$ by the first part of (58) and similarly for DAf. Thus it is enough to consider the commutators with each of the localized operators A_a. That is we need to check that (58) implies that

$$Q[f, A_a] \in \mathcal{B}_{j-1}\ \forall\ f \in \mathcal{C}^\infty_c(U_a). \tag{60}$$

Consider the commutator with the oscillating exponential in the local coordinate patch U_a, where the local coordinates in U_a are now denoted simply by x :

$$F(\xi) = Q[e^{ix\cdot\xi}, A_a],\ \xi \in \mathbb{R}^n,\ n = \dim M. \tag{61}$$

By the third assumption, this is a bounded operator from $H^1(Z; V)$ to $L^2(Z; V)$ and as such is clearly smooth and

$$\frac{d}{ds}F(s\xi) = Q[ix\cdot\xi e^{isx\cdot\xi}, A_a] = Qix\cdot\xi Q^{-1}F(s\xi) + \sum_k i\xi_k Q[x_k, A_a]e^{isx\cdot\xi}. \tag{62}$$

Since $F(0) = 0$ solving this differential equation gives

$$F(\xi) = e^{iQx\cdot\xi Q^{-1}} \int_0^1 \left(e^{-isQx\cdot\xi Q^{-1}} \sum_k i\xi_k Q[x_k, A_a] e^{isx\cdot\xi} \right) ds \tag{63}$$

Now, by hypothesis, each of the $Q[x_k, A_a] \in \mathcal{B}_{j-1}$ and it is straightforward to check that $e^{isx\cdot\xi}$ and $e^{-isQx\cdot\xi Q^{-1}}$ for $s \in [-1, 1]$ are continuous maps into $\mathcal{B}_{j-1}$ with norms which grow at most polynomially in $|\xi|$. Thus $\|F(\xi)\|_{j-1} \le C(1 + |\xi|)^{N(j)}$. Now, expressing $f \in \mathcal{C}_c^\infty(U_a)$ in terms of its Fourier transform, which is a rapidly decreasing function of ξ, it follows by integration that $Q[f, A_a] \in \mathcal{B}_{j-1}$. A similar argument applies to $[f, A_a]Q$ so the first part of the Lemma follows.

Completeness of $\mathcal{B}_j$ also follow inductively and the earlier argument that $\mathcal{B}_j$ is an algebra extends directly to give the product estimate on the norm. □

Note that it is always possible to rescale $\|\cdot\|_j$ by the constant C_j so that

$$\|AB\|_j \le \|A\|_j \|B\|_j, \ \forall \ A, B \in \mathcal{B}_j. \tag{64}$$

Returning to the proof of the Proposition, we now show that

$$\bigcap_j \mathcal{B}_j = \Psi^0_\infty(Z; V) \tag{65}$$

is the larger algebra of pseudodifferential operators 'with symbols relative to L^∞' but not necessarily classical. This is essentially the content of Beals' result, so we only briefly recall the argument.

So, suppose $A \in \mathcal{B}_j$ for all j. We know that the conditions (56) apply inductively to show that $Q^p f A g$ and $f A g Q^p$ are bounded on $L^2(Z; V)$ for all p if $f, g \in \mathcal{C}^\infty(Z)$ have disjoint supports. This simply means that the Schwartz kernel of A is smooth away from the diagonal. Thus it suffices to show that each of the $A_a = \chi A \chi'_a$ is a pseudodifferential operator in the local coordinate chart $x_{k,a}$ which we can denote by x_k. Since the kernel of A_a now has compact support in the coordinate patch it can be written in oscillatory integral form

$$A_a = (2\pi)^{-n} \int e^{i(x-x')\cdot\xi} e(x, \xi) d\xi$$

where e is smooth in ξ and possibly a distribution in x. The commutation conditions now show that the operator obtained by replacing $e(x, \xi)$ by

$$\xi^\gamma D_x^\alpha D_\xi^\beta e(x, \xi), \ |\gamma| \le |\beta| \tag{66}$$

is also bounded on L^2. Now following [6] this shows that A_a is a pseudodifferential operator of some fixed order in the sense that

$$|D_x^\alpha D_\xi^\beta e(x, \xi)| \le C_{\alpha,\beta}(1 + |\xi|)^{N-\beta} \tag{67}$$

but then boundedness on L^2 implies that (67) holds with $N \le 0$. This completes the argument in the case of 'pseudodifferential operators with bounds'.

To refine the argument to give *classical* pseudodifferential operators we need to add an extra condition to the iterative definition of $\mathcal{B}_j \subset \mathcal{B}_{j-1}$. Namely in the local expression (4) for A_a we need to ensure that $e(x, \xi)$ has an asymptotic expansion in terms of decreasing integral homogeneity. This follows from the same type of estimates (67) with $N = 0$ on the iteratively differentiated amplitude:-

$$|D_x^\alpha D_\xi^\beta \left(q(x,\xi)^L \left(\prod_{p=0}^{L} (\xi \cdot \partial_\xi + p) \right) e(x,\xi) \right) | \le C_{\alpha,\beta}(1 + |\xi|)^{-|\beta|} \tag{68}$$

for all L, where q is an elliptic symbol.

This can be arranged by appending the conditions

$$\mathcal{R}(A) = Q \sum_a \sum_k D_{x_{k,a}}[x_{j,a}, A_a] \in \mathcal{B}_{j-1}. \tag{69}$$

Again these conditions are independent of the partition of unity (subordinate to a coordinate cover) used.

Lemma 4.3. *For any partition of unity subordinate to a coordinate covering,* (69) *gives a map*

$$\mathcal{R} : \Psi^0(Z;V) \longmapsto \Psi^0(Z:V). \tag{70}$$

Conversely, if $A \in \Psi^0_\infty(Z;V)$ *is a pseudodifferential operator 'with bounds' then for any* $l \in \mathbb{N}$,

$$\mathcal{R}^l(A) \in \Psi^0_\infty(Z;V) \Longrightarrow A = A' + A'',\ A' \in \Psi^0(Z;V),\ A'' \in \Psi^{-l}_\infty(Z;V). \tag{71}$$

Proof. From the basic properties of pseudodifferential operators

$$\mathcal{R}(A) : \Psi^0_\infty(Z;V) \longrightarrow \Psi^1(Z:V).$$

Moreover,

$$\sigma_1(\mathcal{R}(A)) = \frac{1}{i} q(x,\xi) \sum_k \xi_k \partial_{\xi_k} \sigma_0(A) = \frac{1}{i} qR\sigma_0(A) \tag{72}$$

is given by the radial vector field on T^*Z applied to the symbol (which is well-defined even though the action is on a vector bundle since the bundle is lifted from Z). Thus, if A is classical then $\sigma_0(A)$ is represented by a function which is homogeneous of degree 0 and hence $\sigma_1(\mathcal{R}(A)) = 0$, modulo symbols of order -1, and (70) follows.

Conversely, if $A \in \Psi^0_\infty(Z;V)$ and $\mathcal{R}(A) \in \Psi^0_\infty(Z;V)$ then, since Q is elliptic, $R\sigma_0(A)$ is a symbol of order -1 and (71) holds for $l = 1$ by radial integration. Proceeding inductively, so assuming (71) for $l \le p-1$,

$$\mathcal{R}^p(A) \in \Psi^0_\infty(Z;V) \Longrightarrow \mathcal{R}(A) = B' + B'',\ B' \in \Psi^0(Z;V),\ B'' \in \Psi^{-p+1}_\infty(Z;V). \tag{73}$$

From the first part of the Lemma, $\mathcal{R}^p(B') \in \Psi^0(Z;V)$ so

$$\mathcal{R}^{p-1}(B'') \in \Psi^0_\infty(Z;V). \tag{74}$$

Now, since B'' is of order at most $-p+1$ it follows directly that $\mathcal{R}^p(B'')$ is of order 1 and its principal symbol can be computed directly in terms of the radial vector field R on T^*Z and the principal symbol q of Q :

$$i^{-p}(qR)^p b'' = i^{-k} q^p (R+p-1)(R+p-2)\cdots Rb'',\ b'' = \sigma_{-p+1}(B''). \tag{75}$$

Thus the iterative condition implies that the leading symbol of B'' is homogeneous of degree $-p+1$, modulo symbols of order $-p$, and the inductive hypothesis follows for $l = p$. This completes the proof of (71). $\square$

Finally then it follows that the classical algebra $\Psi^0(Z;V)$ is the projective limit of the Banach algebras obtained by appending (69) to the inductive definition of $\mathcal{B}_j$ and adding corresponding terms to the norm $\|\cdot\|_j$.

This completes the characterization of the Fréchet topology on $\Psi^0(Z;V)$ and shows that $G^0(Z;V)$ is indeed the intersection of the group of invertible elements. □

As noted above, this characterization of $G^0(Z;V)$ as a projective limit of smooth Banach groups is stronger than the earlier desciptions in the literature. The well-known fact that the Lie algebra of $G^0(Z;V)$ is $\Psi^0(Z;V)$ follows easily. It is also the case that the exponential map from the Lie algebra $\Psi^0(Z;V)$ to $G^0(Z;V)$ is a smooth isomorphism of a neighbourhood of 0 to a neighbourhood of the identity. This can be seen from the holomorphic functional calculus. Namely, if $B(\mathrm{Id},\frac14)$ is the ball, in terms of the L^2 operator norm, around the identity in $G^0(Z;V)$ then for each $A\in B(\mathrm{Id},\frac14)$, $\log(A)\in\mathcal{B}_0$ can be represented in terms of a contour integral around $|z-1|=\frac12$ in terms of the resolvent family. Since the latter is necessarily a map into $\Psi^0(Z;V)$ it follows directly that $\log A\in\Psi^0(Z;V)$ and that $\exp(\log A)=A$. Thus the group $G^0(Z;V)$ behaves in a manner very close to that of a finite-dimensional Lie group, in contrast to most Fréchet Lie groups.

Now, we pass to the more complicated group $\mathrm{G}(\mathcal{F}^0(Z;V))$ of invertible Fourier integral operators. First consider the subgroup $\mathrm{G}_0(\mathcal{F}^0(Z;V))$ corresponding to canonical transformations which are in the connected component of the identity, $\mathrm{Can}_0(Z)$. This subgroup gives a fibration

$$\begin{array}{ccc} G^0(Z;V) & \longrightarrow & \mathrm{G}_0(\mathcal{F}^0(Z;V)) \\ & & \downarrow \\ & & \mathrm{Can}_0(Z). \end{array} \tag{76}$$

Omori in [36], see also [33], discusses the diffeomorphism group of a compact manifold and this discussion applies to the contact group to give a 'projective (inverse) limit Hilbert' structure on $\mathrm{Can}_0(Z)$ arising from the completion of the group with respect to Sobolev topologies (of order tending to infinity).

Lemma 4.4. *The subgroup* $\mathrm{G}_0(\mathcal{F}^0(Z;V))$, *corresponding to the fibration* (76) *has a topology as a principal* $G^0(Z;V)$ *bundle as in* (76) *over* $\mathrm{Can}_0(Z)$ *with the Fréchet topology.*

Proof. As in [5] it is straightforward to construct a section of (76) near the identity in the base; this is discussed in the appendix as is the extension of the principal bundle structure to the rest of $\mathrm{G}_0(\mathcal{F}^0(Z;V))$. □

The topology then extends to other components (of $\mathrm{Can}(Z)$) in the same way. Note that in general it is not clear that the map from $\mathrm{G}(\mathcal{F}^0(Z;V))$ to $\mathrm{Can}(Z)$ is surjective, since there may be an index obstruction to the invertibility of a Fourier integral operator corresponding to a given canonical transformation and it may not, for a particular V, be possible to cancel this obstruction by composition with an elliptic pseudodifferential operator in $\Psi^0(Z;V)$ with the opposite index (if the index map for pseudodifferential operators on V is not surjective).

We are also interested in the projective quotient of this group

$$\mathrm{PG}(\mathcal{F}^{\mathbb{C}}(Z;V))=\mathrm{G}(\mathcal{F}^{\mathbb{C}}(Z;V))/\mathbb{C}^*\,\mathrm{Id}\,. \tag{77}$$

Various of its normal subgroups and their projective images will play an important role in the subsequent discussion of the index map.

For an elliptic operator the order is unambiguously determined so defines an additive homomorphism giving short exact sequences

$$
\begin{array}{ccccc}
\mathrm{G}(\mathcal{F}^0(Z;V)) & \longrightarrow & \mathrm{G}(\mathcal{F}^{\mathbb{C}}(Z;V)) & & \\
\downarrow & & \downarrow & \searrow & \\
 & & & & \mathbb{C}. \\
\downarrow & & \downarrow & \nearrow & \\
\mathrm{PG}(\mathcal{F}^0(Z;V)) & \longrightarrow & \mathrm{PG}(\mathcal{F}^{\mathbb{C}}(Z;V)) & &
\end{array}
\tag{78}
$$

Surjectivity here follows from the existence of an invertible pseudodifferential operator of order s for any $s \in \mathbb{C}$.

Since each operator is elliptic the Schwartz kernel determines the contact transformation χ, since its wavefront set must be equal to the twisted graph of the associated canonical transformation. The same is true of the projective group, since operators identified in the quotient are all elliptic. Thus there are well-defined homomorphisms with the pseudodifferential operators mapping to the identity diffeomorphism so giving sequences

$$
\begin{array}{ccccc}
\mathrm{G}(\Psi^0(Z;V)) & \longrightarrow & \mathrm{G}(\mathcal{F}^0(Z;V)) & & \\
\downarrow & & \downarrow & \searrow & \\
 & & & & \mathrm{Con}(S^*Z) \\
\downarrow & & \downarrow & \nearrow & \\
\mathrm{PG}(\Psi^0(Z;V)) & \longrightarrow & \mathrm{PG}(\mathcal{F}^0(Z;V)) & &
\end{array}
\tag{79}
$$

where $\mathrm{Con}(S^*Z)$ is the group of contact diffeomorphisms of the cosphere bundle S^*Z and there is a similar diagram for the operators of general complex order.

Lemma 4.5. *The horizontal sequences in* (79) *(and the similar ones for general complex order) are exact provided* $\Psi^0(Z;V)$ *contains an operator of index* 1.

Proof. There are certainly elliptic Fourier integral operators associated to any canonical transformation. Composition with an elliptic Fourier pseudodifferential operator of order 0 shifts the index so if there is such an operator of index 1 repeated composition with either this operator or its parametrix gives a Fourier integral operator of index 0 which can then be perturbed by a smoothing operator to be invertible. □

5. Bundles of pseudodifferential operators

The main object of study in this paper is a bundle of algebras with typical fiber the pseudodifferential operators acting on sections of a vector bundle over a fixed compact manifold.

Definition 1. A *(filtered) bundle of pseudodifferential algebras,* $\mathbf{\Psi}^{\mathbb{Z}}$, over a manifold, X, is a fiber bundle which is a Fréchet manifold with typical fibre the algebra

$\Psi^{\mathbb{Z}}(Z;V)$ for some fixed compact manifold Z and vector bundle V. That is, $\boldsymbol{\Psi}^{\mathbb{Z}}$ is equipped with a surjective smooth map

$$p : \boldsymbol{\Psi}^{\mathbb{Z}} \longrightarrow X \tag{80}$$

the fibres of which are $\mathbb{Z}$-filtered algebras and such that any point of X has an open neighbourhood U on which there is a smooth bijection

$$f_U : U \times \Psi^{\mathbb{Z}}(Z;V) \longrightarrow p^{-1}(U) \tag{81}$$

reducing p to projection onto the first factor and which is an order-preserving isomorphism of algebras at each point.

Let U_i be an open cover of X by such trivializations. It follows from the result of Duistermaat and Singer, in the form of Theorem 3.1 above, that the transition functions are smooth maps

$$g_{ij} : U_i \cap U_j \longrightarrow \mathrm{PG}(\mathcal{F}^{\mathbb{C}}(Z;V)), \tag{82}$$

satisfying the cocycle condition, where $\mathrm{PG}(\mathcal{F}^{\mathbb{C}}(Z;V))$ acts on $\Psi^{\mathbb{Z}}(Z;V)$ via the adjoint action. The principal bundle, $\mathbf{F}$, associated to $\boldsymbol{\Psi}^{\mathbb{Z}}$, is modelled on the Fréchet Lie group of projective invertible Fourier integral operators, $\mathrm{PG}(\mathcal{F}^{\mathbb{C}}(Z;V))$, formed from the algebra of Fourier integral operators of complex order on Z with coefficients in the complex vector bundle V discussed above.

Proposition 5.1. *The structure group of any filtered bundle of pseudodifferential algebras can be reduced to the Fréchet group* $\mathrm{PG}(\mathcal{F}^0(Z;V))$ *corresponding to invertible elements in the algebra of Fourier integral operators of order* 0 *rather than general complex order.*

Proof. As shown originally by Seeley, the complex powers of a positive differential operator of second order, which always exists, form an entire family of pseudodifferential operators $Q^s \in \Psi^s(Z;V)$ where Q^2 is the original differential operator.

Take a covering of the base $X = \bigcup_j U_j$ by open sets over each of which the given bundle of pseudodifferential operators $\boldsymbol{\Psi}^{\mathbb{Z}}$ is trivial, with $f_i : U_i \longrightarrow p^{-1}(U_i)$ the associated trivializations. Thus the transition maps (82) can be realized by smooth maps $U_i \cap U_j \ni x \longmapsto \tilde{g}_{ij}(x) \in \mathcal{F}^{s_{ij}(x)}(Z;V)$ where the $s_{ij} : U_i \cap U_j \longrightarrow \mathbb{C}$ are smooth maps which necessarily satisfy the (additive) cocycle condition. Such a cocycle is necessarily trivial, so there exist smooth maps $s_i : U_i \longrightarrow \mathbb{C}$ such that $s_{ij} = s_i - s_j$ on $U_i \cap U_j$; for instance using a partition of unity subordinate to the cover one can take $s_i = \sum\limits_{k \neq i} \phi_j s_{ik}$.

Now, replace the trivializations f_i by $f_i'(x) = f_i(x)Q^{-s_i(x)}$. The corresponding transition maps are

$$g_{ij}' = Q^{s_j(x)} f_j^{-1}(x) f_i(x) Q^{-s_i(x)} \in \mathcal{F}^0(Z;V). \tag{83}$$

This gives the desired reduction of the structure group. □

Thus a filtered bundle of pseudodifferential operators gives rise to an isomorphism class of principal bundles with structure group $\mathrm{PG}(\mathcal{F}^0(Z;V))$. We will assume from this point onwards that some choice of this principal bundle has been made, using Proposition 5.1. All the result below are independent of this choice

Then $\boldsymbol{\Psi}^{\mathbb{Z}}$ is an associated bundle, corresponding to the adjoint action and it has a well-defined smoothing subbundle $\boldsymbol{\Psi}^{-\infty}$ corresponding to the action on smoothing operators. Since the elements of $\mathcal{F}^0(Z;V)$ are bounded operators on the Hilbert space $H = L^2(Z;V)$, the principal bundle can be extended to a bundle with structure group $\mathrm{PG}(H)$, and then reduced to a bundle with structure group $\mathrm{PU}(H)$, since the quotient $\mathrm{PG}(H)/\mathrm{PU}(H)$ is contractible. Correspondingly the smoothing bundle $\mathbf{\Psi}^{-\infty}$ is then a subbundle of the associated bundle of compact operators, which is an Azumaya bundle over X in the sense of [27, 28]. The image in $H^3(X;\mathbb{C})$ of the Dixmier-Douady class, itself in $H^3(X;\mathbb{Z})$, classifying this (completed) Azumaya bundle is computed below in 'Chern-Weil' form from the bundle.

For completeness, we include the Čech definition of the Dixmier-Douady invariant here.

Definition 2. Let $g_{ij} : U_i \cap U_j \longrightarrow \mathrm{PG}(\mathcal{F}^0)$ denote the transition functions of $\mathbf{F}$ with respect to some good open cover $\{U_i\}$ of X. Let $\hat{g}_{ij} : U_i \cap U_j \longrightarrow \mathrm{G}(\mathcal{F}^0)$ denote a lift of g_{ij}. Since g_{ij} satisfies the cocycle identity,

$$\hat{g}_{ij}\hat{g}_{jk}\hat{g}_{ki} = c_{ijk}\,\mathrm{Id} : U_i \cap U_j \cap U_k \longrightarrow \mathbb{C}^*$$

is a Čech 2-cocycle on X, that is, $[c_{ijk}] \in H^2(X;\underline{\mathbb{C}^*}) \cong H^3(X;\mathbb{Z})$. This is the *Dixmier-Douady invariant* of $\mathbf{F}$ and is independent of the trivialization (and the reduction to order 0).

The Dixmier-Douady invariant measures the failure of the bundle $\mathbf{\Psi}^{\mathbb{Z}}$ to be a bundle of operators. It vanishes, as shown below, if and only if $\mathbf{F}$ lifts to a principal $\mathrm{G}(\mathcal{F}^0(Z;V))$ bundle $\widehat{\mathbf{F}}$. In this case there is an associated bundle of Fréchet spaces over X,

$$\mathcal{C}^\infty = \widehat{\mathbf{F}} \times_{\mathrm{G}(\mathcal{F}^0(Z;V))} \mathcal{C}^\infty(Z;V) \longrightarrow X,$$

and $\mathbf{\Psi}^{\mathbb{Z}}$ is a bundle of operators on $\mathcal{C}^\infty$. In fact the associated bundle of smoothing operators can then be realized as the smooth completion of the tensor product

$$\mathbf{\Psi}^{-\infty} = \widehat{\mathbf{F}} \times_{\mathrm{G}(\mathcal{F}^0(Z;V))} \mathcal{C}^\infty(Z^2; V \boxtimes V' \otimes \Omega_L) \longrightarrow X,$$

of $\mathcal{C}^\infty$ and the analogous bundle of sections of the dual bundle tensored with the density bundle, to which the adjoint action of $\mathbf{F}$ extends.

The symbol of a Fourier integral operator of order 0 (associated to a canonical diffeomorphism) can be identified with an invertible section of the pull back to the cosphere bundle of $\hom(V)$ tensored with the (pull-back of) the Maslov bundle. A non-vanishing constant multiple has the same canonical transformation so this gives a short exact sequence

$$\mathrm{PG}(\Psi^0(Z;V)) \longrightarrow \mathrm{PG}(\mathcal{F}^0(Z;V)) \xrightarrow{\sigma} \mathcal{C}^\infty(S^*Z; \pi^*\mathrm{Aut}(V) \otimes \mathcal{L}_\chi), \tag{84}$$

where the image group is discussed further below.

Composition on the left with projective pseudodifferential operators gives an action by the subgroup $\mathcal{C}^\infty(S^*Z;\mathrm{Aut}(V))$ on the group of symbols of Fourier integral operators, with

$$\mathcal{C}^\infty(S^*Z;\mathrm{Aut}(V)) \longrightarrow \mathcal{C}^\infty(S^*Z, \pi^*(\mathrm{Aut}(V) \otimes \mathcal{L}_\chi)) \xrightarrow{p} \mathrm{Can}(Z), \tag{85}$$

giving the map to the quotient Fréchet group of canonical diffeomorphisms, as in section 4.

Through this action there is an associated bundle which is a finite-dimensional manifold, the *twisted (fibre) cosphere bundle* as the associated fibre bundle,

$$\mathbf{S} = \mathbf{F} \times_{\mathrm{PG}(\mathcal{F}^\bullet(Z;V))} S^*Z \xrightarrow{\hat{\pi}} X,$$

where the action of $\mathrm{PG}(\mathcal{F}^\bullet(Z;V))$ on S^*Z is via $p\circ\chi$. The bundle $\mathbf{S}$ has typical fibre S^*Z.

6. Chern class of the Fourier integral operator central extension

Using a regularized trace on pseudodifferential operators, we describe a natural class of connections on the central extension in equation (1). The curvature is expressed in terms of the residue trace of Guillemin and Wodzicki and the exterior derivation. We compute the 1-form arising from the the central extension which together with the curvature determines the central extension up to isomorphism in the absence of torsion in degree 2 integral cohomology, see [10, 35].

Consider the Maurer-Cartan form, which is a Lie algebra valued differential 1-form

$$\Theta\in\Omega^1(\mathrm{G}(\mathcal{F}^0(Z;V));\mathfrak{g}(Z;V)),$$

where $\mathfrak{g}(Z;V)$ is the Lie algebra of $\mathrm{G}(\mathcal{F}^0(Z;V))$, consisting of the pseudodifferential operators of order 1 with the additional constraint that the principal symbol is diagonal and pure imaginary. This canonical form is determined by left-invariance and the identification of $T_{\mathrm{Id}}\,\mathrm{G}(\mathcal{F}^0(Z;V))$ with $\mathfrak{g}(Z;V)$:

$$L_g^*\Theta=\Theta,\ \Theta_{\mathrm{Id}}(v)=v. \tag{86}$$

Since $\mathcal{F}^0(Z;V)$ lies in a linear space of operators, it is directly meaningful to use the familiar notation $\Theta=F^{-1}dF$ for $F\in\mathrm{G}(\mathcal{F}^0(Z;V))$. Under the right action and the adjoint action on the Lie algebra

$$R_g^*(\Theta)=\mathrm{Ad}(g)\Theta=g\Theta g^{-1}. \tag{87}$$

Let $Q\in\Psi^1(Z;V)$ be an elliptic pseudodifferential operator which is self-adjoint and positive with respect to some inner product and density and let

$$\mathrm{Tr}_Q:\Psi^{\mathbb{Z}}(Z;V)\longrightarrow\mathbb{C}$$

denote the *regularized trace* with respect to Q. That is, $\mathrm{Tr}_Q(A)$ is the regularized value at $z=0$ of the meromorphic extension of $\mathrm{Tr}(Q^zA)$. Thus Tr_Q extends the operator trace from the ideal of trace class operators $\Psi^m(Z;V)$, $m<-\dim(Z)$, but is not itself trace.

As shown by Guillemin [18] one can use in place of Q^z any entire classical family $Q(z)\in\Psi^z(Z;W)$ with $Q(0)=\mathrm{Id}$ and entire inverse $Q(z)^{-1}\in\Psi^{-z}(Z;W)$. Then the residue trace of Guillemin and Wodzicki and the corresponding regularized trace are determined by the expansion near $z=0$

$$\mathrm{Tr}(Q(z)A)=z^{-1}\,\mathrm{Tr}_{\mathrm{R}}(A)+\mathrm{Tr}_Q(A)+zG(z). \tag{88}$$

The evaluation of the residue trace on a commutator is given by the trace-defect formula

$$\begin{gathered}\mathrm{Tr}_Q([A,B])=\mathrm{Tr}_{\mathrm{R}}([\log Q,A]B)=\mathrm{Tr}_{\mathrm{R}}((\delta_QA)B)=-\,\mathrm{Tr}_{\mathrm{R}}(A\delta_QB)\text{ where}\\ \delta_QA=[\log Q,A]=\frac{d}{dz}\Big|_{z=0}Q(z)AQ(z)^{-1}\end{gathered} \tag{89}$$

is the exterior derivation defined by $Q(z)$.

Under change of the regularizing operator Q, to Q', the regularized trace changes by

$$\mathrm{Tr}_{Q'}(A)-\mathrm{Tr}_Q(A)=\mathrm{Tr}_{\mathrm{R}}(A(\log Q'-\log Q)),\ \log Q'-\log Q\in\Psi^0(Z;V). \tag{90}$$

Lemma 6.1. *There is an entire holomorphic family $Q(z) \in \Psi^z(Z;V)$ with entire inverse $Q(-z)$ such that*

$$\operatorname{Tr}_Q(\mathrm{Id}) = 1. \tag{91}$$

Proof. Since $(Q^z)^* = Q^{\overline{z}}$ the regularized trace of the identity, which is to say the value at $z = 0$ of the zeta function, is real. If $A \in \Psi^{-\dim Z}(Z;V)$ is self-adjoint with scalar symbol then there is a self-adjoint smoothing operator $R \in \Psi^{-\infty}(Z;V)$ such that $\mathrm{Id} + \frac{1}{2}A + R$ is positive. The regularizing family $Q'(z) = (\mathrm{Id} + \frac{1}{2}A + R)^z Q^z (\mathrm{Id} + \frac{1}{2}A + R)^z \in \Psi^z(Z;V)$ corresponds to the exterior derivation

$$\log Q' = \log Q + A + 2R \tag{92}$$

and so shifts the regularized trace of the identity by $\operatorname{Tr}_{\mathrm{R}}(A)$. Since this is given by the integral of the trace of the leading symbol of A the normalization (91) can be arranged. □

As a functional on $\mathfrak{g}(Z;V) \subset \Psi^{\mathbb{Z}}(Z;V)$, the regularized trace acts on the range of the Maurer-Cartan form, so

$$A_Q = \operatorname{Tr}_Q(\Theta) \in \Omega^1(\mathrm{G}(\mathcal{F}^0(Z;V))) \tag{93}$$

is a well-defined smooth 1-form on $\mathrm{G}(\mathcal{F}^0(Z;V))$.

Lemma 6.2. *If the regularized trace is chosen to satisfy* (91) *then the 1-form A_Q in* (93) *is a connection 1-form on the central extension in equation* (1) *and if Q_1 is another such normalized regularizing family then*

$$A_Q - A_{Q_1} = -\operatorname{Tr}_{\mathrm{R}}(\Theta(\log(Q) - \log(Q_1))) \in \Omega^1(\mathrm{PG}(\mathcal{F}^0(Z;V))). \tag{94}$$

Proof. Clearly A_Q is a left-invariant 1-form on $\mathrm{G}(\mathcal{F}(Z;V))$. The central subgroup consists of the multiples of the identity, so the normalization (93) ensures that it restricts to the fibres of $\mathrm{G}(\mathcal{F}^0(Z;V))$ over $\mathrm{PG}(\mathcal{F}^0(Z;V))$ to be the Maurer-Cartan form for the centre i.e. it is a connection form. The transgression formula (94) follows and since Θ is a multiple of the identity, the difference vanishes on the fibres and hence is a smooth form on $\mathrm{PG}(\mathcal{F}^0(Z;V))$. □

Lemma 6.3. *For a normalized regularization, satisfying* (91)*, the curvature of the connection A_Q is*

$$\Omega_Q = \operatorname{Tr}_R(\delta_Q(\Theta) \wedge \Theta) \tag{95}$$

and transgresses under change of regularization by

$$\Omega_Q - \Omega_{Q_1} = -d \operatorname{Tr}_R \left(\Theta(\log(Q) - \log(Q_1))\right).$$

The first Chern class of the the central extension is therefore

$$c_1(\mathrm{G}(\mathcal{F}(Z;V))) = \left[\frac{i}{2\pi}\Omega_Q\right] \in H^2(\mathrm{PG}(\mathcal{F}(Z;V);\mathbb{Z}) \tag{96}$$

Proof. It suffices to compute the curvature as a form on $\mathrm{G}(\mathcal{F}^0(Z;V))$. For ψ_1, $\psi_2 \in \mathfrak{g}(Z;V)$ the standard formula for the differential gives

$$d \operatorname{Tr}_Q(\Theta)(\psi_1, \psi_2) = \psi_1 \operatorname{Tr}_Q(\Theta)(\psi_2) - \psi_2 \operatorname{Tr}_Q(\Theta)(\psi_1) - \operatorname{Tr}_Q(\Theta)([\psi_1, \psi_2]).$$

Since the Maurer-Cartan 1-form Θ is left-invariant the first two terms on the right side vanish. Applying the trace-defect formula

$$\operatorname{Tr}_Q(\Theta)([\psi_1, \psi_2]) = \operatorname{Tr}_Q([\psi_1, \psi_2]) = \operatorname{Tr}_{\mathrm{R}}(\delta_Q(\psi_1)\psi_2) \tag{97}$$

gives the formula (95) in general by left-invariance.

The transgression formula follows immediately from Lemma 6.2 above. □

Note that the curvature can be expanded to more resemble a suspended Chern form in which δ_Q plays the role of the push-forward. Namely

$$\Omega_Q(F) = -\operatorname{Tr}_R(F^{-1}(\delta_Q F)F^{-1}dF \wedge F^{-1}dF) + d\operatorname{Tr}_R(F^{-1}(\delta_Q F)F^{-1}dF). \tag{98}$$

The circle bundle given by a central extension of a group does not in general characterise the central extension, cf. [10, 35]. The multiplicative structure, i.e. the primitivity of the bundle, gives as a further invariant, namely a 1-form on the product of the base group with itself.

For a Lie group (possibly infinite-dimensional as in this case) G and a form β on G^p consider the form on the product G^{p+1} defined by

$$\begin{gathered}\delta\beta = \sum_{j=1}^{p+1}(-1)^{j-1}d_i^*\beta,\ d_i : G^{p+1} \longrightarrow G^p\\ d_1(g_1,\dots,g_{p+1}) = (g_2,\dots,g_{p+1}),\\ d_i(g_1,\dots,g_{p+1}) = (g_1,\dots,g_{i-1}g_i,\dots,g_{p+1}),\ 1 < i \le p\\ d_{p+1}(g_1,\dots,g_{p+1}) = (g_1,\dots,g_p).\end{gathered}\tag{99}$$

Then,

$$d\delta = \delta d,\ \delta^2 = 0. \tag{100}$$

Lemma 6.4. *The 1-form*

$$\alpha_Q(F_1,F_2) = -\operatorname{Tr}_R\left((\delta_Q F_2)F_2^{-1}F_1^{-1}dF_1)\right) \tag{101}$$

is well-defined on $(\mathrm{PG}(\mathcal{F}^0(Z;V)))^2$ *and satisfies*

$$d\alpha_Q = \delta\Omega_Q,\ \delta\alpha_Q = 0.$$

In the absence of torsion such a pair (Ω_Q,α_Q) determines the central extension up to isomorphism, see [10, 35].

Proof. The operator in (99) is well-defined on any smooth group, let $\hat\delta$ denote the corresponding operator on the full group $\mathrm{G}(\mathcal{F}^0(Z;V))$. Let Θ_i, for $i = 1, 2$ be the pull-backs to $(\mathrm{G}(\mathcal{F}^0(Z;V))^2$ of the Maurer-Cartan form from the two factors. Then

$$\begin{aligned}(\hat\delta A_Q)(F_1,F_2) &= \operatorname{Tr}_Q(\Theta_2) - \operatorname{Tr}_Q(F_2^{-1}F_1^{-1}d(F_1F_2)) + \operatorname{Tr}_Q(\Theta_1)\\ &= -\operatorname{Tr}_Q([F_2^{-1},\Theta_1F_2])\\ &= -\operatorname{Tr_R}((\delta_Q F_2)F_2^{-1}F_1^{-1}dF_1).\end{aligned}\tag{102}$$

This is just α_Q in (101).

Restricting to a fibre of (1) in the first factor, corresponds to fixing F_1 up to a scalar multiple, so Θ_1 is a scalar (1-form) multiple of the identity and α_Q is therefore a multiple of $\operatorname{Tr_R}((\delta_Q F)F^{-1})$ pulled-back from the second factor. By the trace-defect formula this is equal to $\operatorname{Tr}_Q([F^{-1},F])$ and so vanishes. Similarly, restricted to the fibre in the first factor $(\delta_Q F)F^{-1}$ is constant, so in fact α_Q descends to a form on $(\mathrm{PG}(\mathcal{F}^0(Z;V)))^2$ as claimed. In view of (100) this form satisfies the desired identities. □

7. The Dixmier-Douady class of a projective FIO bundle

Next we proceed to compute the image in deRham cohomology of the Dixmier-Douady class of a principal $\mathrm{PG}(\mathcal{F}^0(Z;V))$-bundle over X. This is the obstruction to lifting the structure group to $\mathrm{G}(\mathcal{F}^0(Z;V))$ and to compute it we use the approach of Murray, via the associated bundle gerbe, as elaborated in [35] for the case of loop groups.

The principal bundle, $\mathbf{F}$, can be seen as a 'lifting gerbe' with the central extension of its structure group written as a principal $\mathbb{C}^*$-bundle

$$\begin{array}{ccccc} \mathbb{C}^* & \longrightarrow & \mathrm{G}(\mathcal{F}^0(Z;V)) & & \\ & & \downarrow & & \\ & & \mathrm{PG}(\mathcal{F}^0(Z;V)) & \text{———} & \mathbf{F} \\ & & & & \downarrow{\scriptstyle \pi} \\ & & & & X. \end{array} \tag{103}$$

A connection on the $\mathbb{C}^*$ bundle over the group was constructed in Section 6 with curvature given by (95). The associated bundle gerbe is the fibre product $\mathbf{F}^{[2]}$ with line bundle $\mathcal{L} = \tau^* \mathrm{G}(\mathcal{F}^\bullet(Z;V))$ obtained by pulling back via the fibre shift map $\tau : \mathbf{F}^{[2]} \longrightarrow \mathrm{PG}(\mathcal{F}^0(Z;V))$, $\tau(a,b) = b^{-1}a$; there are two 'simplicial' maps back to the principal bundle

$$\begin{array}{ccccc} & & \mathcal{L} & \longrightarrow & \mathrm{G}(\mathcal{F}^\bullet(Z;V)) \\ & & \downarrow & & \downarrow \\ \mathbf{F} & \overset{\pi_1}{\underset{\pi_2}{\leftleftarrows}} & \mathbf{F}^{[2]} & \overset{\tau}{\longrightarrow} & \mathrm{PG}(\mathcal{F}^\bullet(Z;V)) \\ {\scriptstyle \pi}\downarrow & \swarrow{\scriptstyle \pi^{[2]}} & & & \\ X. & & & & \end{array} \tag{104}$$

We proceed to construct a connection on $\mathcal{L}$ as a principal $\mathbb{C}^*$ bundle over $\mathbf{F}^{[2]}$ which is *primitive* in the sense that its curvature decomposes in the form $\pi_1^*\mathbf{B} - \pi_2^*\mathbf{B}$ where $\mathbf{B}$ is a 2-form on $\mathbf{F}$, the curving or B-field. Then, $d\mathbf{B}$ is necessarily basic, hence is the pull-back of a 3-form, H, on X which represents the image of the Dixmier-Douady class in $H^3(X;\mathbb{R})$.

To get an explicit formula for H we start by modifying the connection A_Q of Section 6 to the 'middle' connection form on $\mathrm{G}(\mathcal{F}^0(Z;V))$

$$\alpha = \frac{1}{2}\operatorname{Tr}_Q(\hat{\theta}_L + \hat{\theta}_R),\ \hat{\theta}_L = g^{-1}dg,\ \hat{\theta}_R = (dg)g^{-1} = g\theta_L g^{-1} \tag{105}$$

corresponding to the choice of a normalized regularized trace Tr_Q on $\Psi^{\mathbb{Z}}(Z;V)$ and hence on the Lie algebra of $\mathrm{G}(\mathcal{F}^0(Z;V))$.

The derivation δ_Q associated to the holomorphic family Q induces a map from $\mathrm{G}(\mathcal{F}^0(Z;V))$ to its Lie algebra by

$$\zeta = g^{-1}\delta_Q g. \tag{106}$$

Since $\delta_Q \operatorname{Id} = 0$, this function descends to $\mathrm{PG}(\mathcal{F}^0(Z;V))$ and then takes values in its Lie algebra:

$$\zeta : \mathrm{PG}(\mathcal{F}^0(Z;V)) \longrightarrow \mathfrak{pg} = \mathfrak{g}(\mathrm{PG}(\mathcal{F}^0(Z;V)). \tag{107}$$

As in the relation of a connection form on $\mathbf{F}$ and the Maurer-Cartan form, we consider a Higgs field related to ζ. That is, a smooth map

$$\mathbf{\Phi} : \mathbf{F} \longrightarrow \mathfrak{pg} \tag{108}$$

with the transformation law

$$g^*\mathbf{\Phi} = \zeta(g) + g^{-1}\mathbf{\Phi}g,\ g \in \mathrm{PG}(\mathcal{F}^0(Z;V)). \tag{109}$$

In terms of a local trivialization of $\mathbf{F}$, over $U \subset X$, as a principal bundle this is equivalent to requiring

$$\mathbf{\Phi} = \zeta + g^{-1}\phi(x)g \tag{110}$$

for a local field ϕ, on U, with values in $\mathfrak{pg}$. Since this condition is preserved under convex combinations such a field $\mathbf{\Phi}$ can be constructed, as for a connection, as the sum of the local fields ζ over a partition of unity subordinate to trivializations. The difference between two Higgs fields associated to δ_Q in this sense is a section of the adjoint bundle associated to $\mathbf{F}$ over X.

There is also a somewhat more natural construction of such a Higgs field which we briefly indicate. The bundle $\mathbf{\Psi}^{\mathbb{Z}}$ of pseudodifferential operators associated to $\mathbf{F}$ (by construction) can be extended to a bundle of complex-order classical operators, $\mathbf{\Psi}^{\mathbb{C}}$, since the action by conjugation of $\mathrm{PG}(\mathcal{F}^0(Z;V))$ extends from $\Psi^{\mathbb{Z}}(Z;V)$ to $\Psi^{\mathbb{C}}(Z;V)$. Choices of normalized entire regularizing family $Q(z) \in \Psi^z(Z;V)$ as in Lemma 6.1 for a covering by trivializations of $\mathbf{F}$ can be patched through a partition of unity to give a section $\tilde{Q}(z) \in \mathcal{C}^\infty(X;\mathbf{\Psi}^z(Z;V))$ with the desired properties on each fibre, including the normalization condition (91). Then

$$\Phi(F) = \frac{d}{dz}\Big|_{z=0} F^{-1}\tilde{Q}(z)FQ(z)^{-1},\ \Phi : \mathbf{F} \longrightarrow \mathfrak{pg}(\mathcal{F}^0(Z;V)) \tag{111}$$

is a Higgs field for the derivation associated to the fixed choice $Q(z)$ of regularizing family in $\Psi^{\mathbb{C}}(Z;V)$.

Denote the pull back of ζ to $\mathbf{F}$ under τ as Υ; a choice of Higgs field associated to δ_Q provides a global splitting in the sense that

$$\Upsilon = \mathbf{\Phi}_1 - \tau^{-1}\mathbf{\Phi}_2\tau,\ \mathbf{\Phi}_i = \pi_i^*\mathbf{\Phi} \tag{112}$$

since this follows from (110) in any local trivialization. There is also a relation between Υ and the pull-back of the Maurer-Cartan form

$$d\Upsilon = \delta_Q\Theta + [\Upsilon, \Theta]. \tag{113}$$

This follows from the corresponding formula on $\mathrm{PG}(\mathcal{F}^0(Z;V))$ that

$$d\zeta = d(g^{-1}\delta_Q g) = \delta_Q(g^{-1}dg) + [\zeta, g^{-1}dg] = \delta_Q(\theta) + [\zeta, \theta]. \tag{114}$$

To capture the contribution of the geometry choose a smooth connection form $\mathbf{A}$ on $\mathbf{F}$ as a principal $\mathrm{PG}(\mathcal{F}^0(Z;V))$ bundle. Then the pull-back of the Maurer-Cartan form can be expressed in terms of the two pull-backs of the connection form

$$\Theta = \mathbf{A}_1 - \tau^{-1}\mathbf{A}_2\tau,\ \mathbf{A}_i = \pi_i^*\mathbf{A}. \tag{115}$$

Indeed this follows from the expression for a connection in terms of a local trivialization

$$\mathbf{A} = g^{-1}dg + g^{-1}\gamma(x)g. \tag{116}$$

The 'symmetric' connection form in (105) pulls back to a connection on the $\mathbb{C}^*$ bundle $\mathcal{L}$. The differential of α as a form on PG follows from the trace-defect formula

$$d\alpha = \frac{1}{2}\operatorname{Tr}_Q(-\hat{\theta}_L \wedge \hat{\theta}_L + \hat{\theta}_R \wedge \hat{\theta}_R) = \pi^* w,\ \ w = \pi^* \frac{1}{2}\operatorname{Tr}_{\mathrm{R}}(\zeta\theta \wedge \theta). \tag{117}$$

Here θ_L has been replaced by the (left) Maurer-Cartan form θ on PG which is justified since any vertical part in $\hat{\theta}_L \wedge \hat{\theta}_L$ takes values as a multiple of the identity in $\mathfrak{g}(\mathcal{F}^0(Z;V))$ and so vanishes with $\operatorname{Tr}_{\mathrm{R}}(\zeta)$; the resulting basic form is fibre constant.

The pull-back of $d\alpha$ to $\mathbf{F}^{[2]}$ may therefore be written

$$W = \tau^* w = \frac{1}{2}\operatorname{Tr}_{\mathrm{R}}(\Upsilon\Theta \wedge \Theta). \tag{118}$$

Thus the differential of the connection form $\tau^*\alpha$ on the pull-back $\mathbb{C}^*$-bundle $\mathcal{L}$ is the pull-back of $\pi^* W$ from $\mathbf{F}^{[2]}$ where $W = \tau^* w$ is the pull-back of the $d\alpha$ from $\mathrm{PG}(\mathcal{F}^0(Z;V))$. This is not a primitive connection; however

Proposition 7.1. *The connection form $\tau^*\alpha - \pi^*\mu$, where*

$$\mu = \frac{1}{2} d \operatorname{Tr}_{\mathrm{R}}(\Upsilon(\mathbf{A}_1 + \tau^{-1}\mathbf{A}_2\tau),$$

is primitive on $\mathcal{L}$ with differential the pull-back of $\mathbf{B}_1 - \mathbf{B}_2$, where $\mathbf{B}_i = \pi_i^*\mathbf{B}$, for the B-field

$$\mathbf{B} = -\frac{1}{2}\operatorname{Tr}_{\mathrm{R}}((\delta_Q \mathbf{A}) \wedge \mathbf{A}) + \operatorname{Tr}_{\mathrm{R}}(\mathbf{\Phi W}), \tag{119}$$

with $\mathbf{\Phi}$ a Higgs field on $\mathbf{F}$ for the derivation δ_Q and $\mathbf{W} = d\mathbf{A} + \mathbf{A} \wedge \mathbf{A}$ the curvature form for the connection $\mathbf{A}$ on $\mathbf{F}$.

Proof. We wish to show that

$$W - d\mu = \mathbf{B}_1 - \mathbf{B}_2 \tag{120}$$

so we start by expanding $d\mu$:

$$\begin{aligned} 2d\mu = {} & \operatorname{Tr}_{\mathrm{R}}(d\Upsilon \wedge (\mathbf{A}_1 + \tau^{-1}\mathbf{A}_2\tau)) + \operatorname{Tr}_{\mathrm{R}}(\Upsilon \wedge (\mathbf{W}_1 + \tau^{-1}\mathbf{W}_2\tau) \\ & - \operatorname{Tr}_{\mathrm{R}}(\Upsilon(\mathbf{A}_1 \wedge \mathbf{A}_1 + \tau^{-1}\mathbf{A}_2 \wedge \mathbf{A}_2\tau)) - \operatorname{Tr}_{\mathrm{R}}(\Upsilon(\Theta\tau^{-1}\mathbf{A}_2\tau + \tau^{-1}\mathbf{A}_2\tau\Theta) \end{aligned} \tag{121}$$

since $d\tau = \tau\Theta$. Now, using (113) to expand $d\Upsilon$, the first term on the right in (121), with wedge product now understood, can we written

$$\begin{aligned} & \operatorname{Tr}_{\mathrm{R}}(d\Upsilon(\mathbf{A}_1 + \tau^{-1}\mathbf{A}_2\tau)) \\ & = \operatorname{Tr}_{\mathrm{R}}(\delta_Q\Theta(\mathbf{A}_1 + \mathbf{A}_2')) + \operatorname{Tr}_{\mathrm{R}}([\Upsilon, \Theta]((\mathbf{A}_1 + \mathbf{A}_2')) \\ & = - \operatorname{Tr}_{\mathrm{R}}(\Theta(\delta_Q\mathbf{A}_1 + \tau^{-1}(\delta_Q\mathbf{A}_2)\tau)) + \operatorname{Tr}_{\mathrm{R}}(\Theta([\Upsilon, \mathbf{A}_2'] + [\Upsilon, \Theta](\mathbf{A}_1 + \mathbf{A}_2')) \\ & = \operatorname{Tr}_{\mathrm{R}}((\delta_Q\mathbf{A}_1)\mathbf{A}_1) - \operatorname{Tr}_{\mathrm{R}}((\delta_Q\mathbf{A}_2)\mathbf{A}_2) + 2\operatorname{Tr}_{\mathrm{R}}(\Upsilon\mathbf{A}_1\mathbf{A}_1) \end{aligned} \tag{122}$$

where $\mathbf{A}_2' = \tau^{-1}\mathbf{A}_2\tau$. Here the identity $\operatorname{Tr}_{\mathrm{R}} \circ \delta_Q \equiv 0$ has been used in the form

$$\operatorname{Tr}_{\mathrm{R}}((\delta_Q\mathbf{A}_1)\mathbf{A}_2' + \mathbf{A}_1 t^{-1}(\delta_Q\mathbf{A}_2)\tau) = \operatorname{Tr}_{\mathrm{R}}(\Upsilon\mathbf{A}_1\mathbf{A}_2') + \operatorname{Tr}_{\mathrm{R}}(\Upsilon\mathbf{A}_2'\mathbf{A}_1)$$
$$\operatorname{Tr}_{\mathrm{R}}((\delta_Q\Theta)(\mathbf{A}_1 + \mathbf{A}_2')) = \operatorname{Tr}_{\mathrm{R}}((\delta_Q(\mathbf{A}_1 + \mathbf{A}_2')\Theta).$$

The last term on the right in (121) expands to

$$2\,\mathrm{Tr}_{\mathrm{R}}(\Upsilon\tau^{-1}\mathbf{A}_2\mathbf{A}_2\tau) - \mathrm{Tr}_{\mathrm{R}}(\Upsilon\mathbf{A}_1\tau^{-1}\mathbf{A}_2\tau) - \mathrm{Tr}_{\mathrm{R}}(\Upsilon\tau^{-1}\mathbf{A}_2\tau\mathbf{A}_1)$$

so

$$\begin{aligned}(123)\quad 2d\mu = \mathrm{Tr}_{\mathrm{R}}((\delta_Q\mathbf{A}_1)\mathbf{A}_1) - \mathrm{Tr}_{\mathrm{R}}((\delta_Q\mathbf{A}_2)\mathbf{A}_2)&\\ + \mathrm{Tr}_{\mathrm{R}}(\Upsilon(\mathbf{W}_1 + \tau^{-1}\mathbf{W}_2\tau) + \mathrm{Tr}_{\mathrm{R}}(\Upsilon\Theta\Theta).&\end{aligned}$$

The curvature form $\mathbf{W}$ of $\mathbf{A}$ is a 2-form with values in the adjoint bundle so $\tau^{-1}\mathbf{W}_2\tau = \mathbf{W}_1$ and computing in a local trivialization where $\mathbf{W} = g^{-1}w(x)g$ for a 2-form w on the base (with values in the Lie algebra)

$$\begin{aligned}&\mathrm{Tr}_{\mathrm{R}}(\Upsilon\mathbf{W}_1)\\ &\quad= \mathrm{Tr}_{\mathrm{R}}(\zeta_1 a^{-1}w(x)a - \tau^{-1}\zeta_2\tau(a^{-1}w(x)a) = \mathrm{Tr}_{\mathrm{R}}(\zeta_1 a^{-1}w(x)a - \zeta_2 b^{-1}w(x)b)\\ &\qquad= \mathrm{Tr}_{\mathrm{R}}(\mathbf{\Phi}_1\mathbf{W}_1 - \mathbf{\Phi}_2\mathbf{W}_2)\end{aligned}$$

for any Higgs field for the derivation δ_Q, $\mathbf{\Phi} = z + p^{-1}\phi(x)p$ in the trivialization. Finally then (123) can be rewritten in the form (120)

$$\begin{aligned}W - d\mu = &\left(-\frac{1}{2}\,\mathrm{Tr}_{\mathrm{R}}((\delta_Q\mathbf{A}_1)\mathbf{A}_1) + \mathrm{Tr}_{\mathrm{R}}(\mathbf{\Phi}_1\mathbf{W}_1)\right)\\ &- \left(-\frac{1}{2}\,\mathrm{Tr}_{\mathrm{R}}((\delta_Q\mathbf{A}_2)\mathbf{A}_2) + \mathrm{Tr}_{\mathrm{R}}(\mathbf{\Phi}_2\mathbf{W}_2)\right)\end{aligned}$$

as claimed. □

With Q fixed, a change of Higgs field to $\mathbf{\Phi}+\psi$ where ψ is a section of the adjoint bundle, changes $\mathbf{B}$ in (119) by a 2-form on X :

$$\mathbf{B} + \mathrm{Tr}_{\mathrm{R}}(\psi\mathbf{W}),\ \mathrm{Tr}_{\mathrm{R}}(\psi\mathbf{W}) \in \mathcal{C}^\infty(X;\Lambda^2).$$

Changing $\mathbf{A}$ to another connection $\mathbf{A}+\lambda$, where λ is a section of the adjoint bundle with values in the pull-back of the 1-form bundle on X changes the curvature form to $\mathbf{W} + d\lambda + [\mathbf{A},\lambda] + \lambda\wedge\lambda$ and hence changes the B-field to

$$(124)\qquad \mathbf{B} - \frac{1}{2}\,\mathrm{Tr}_{\mathrm{R}}((\delta_Q\lambda)\lambda) + \mathrm{Tr}_{\mathrm{R}}(\mathbf{\Phi}(d\lambda + [\mathbf{A},\lambda] + \lambda\wedge\lambda))$$

Under change of the normalized derivation on the Lie algebra, the B-field changes to

$$(125)\qquad \mathbf{B} - \frac{i}{2\pi}\,\mathrm{Tr}_R(\tfrac{1}{2}\mathbf{A}\wedge[P,\mathbf{A}])$$

where $P = \log(Q) - \log(Q') \in \Psi^0(Z;V)$.

The bundle gerbe with primitive connection has the property that $d\mathbf{B}$, with $\mathbf{B}$ given here by (119), is necessarily the pull-back of a 3-form, H on X, which represents the deRham class of the Dixmier-Douady invariant.

Theorem 7.1. *If* $\mathbf{F}$ *is a principal* $\mathrm{PG}(Z;V)$*-bundle over* X*, the image of the Dixmier-Douady class, for the central extension* (1)*, in* $H^3(Z;V)$ *is represented by the 3-form*

$$(126)\qquad H = H_{Q,\mathbf{\Phi},\mathbf{A}} = -\frac{i}{2\pi}\,\mathrm{Tr}_{\mathrm{R}}(\mathbf{W}\wedge\nabla^Q\mathbf{\Phi}),\ \nabla^Q\mathbf{\Phi} = d\mathbf{\Phi} + [\mathbf{A},\mathbf{\Phi}] - \delta_Q\mathbf{A}$$

where Tr_{R} *is the residue trace on the Lie algebra,* $\mathbf{A}$ *is a connection on* $\mathbf{F}$ *with curvature* $\mathbf{W}$ *and* $\mathbf{\Phi}$ *is a Higgs field on* $\mathbf{F}$ *for the normalized derivation* δ_Q *on* $\mathfrak{pg}(\mathcal{F}^0(Z;V))$.

Proof. As noted above it suffices to compute the deRham differential of the B-field in (119):

$$d\mathbf{B} = \frac{i}{2\pi}\left(\frac{1}{2}\mathrm{Tr}_{\mathrm{R}}(d\mathbf{A}\wedge\delta_Q\mathbf{A}) - \frac{1}{2}\mathrm{Tr}_{\mathrm{R}}(\mathbf{A}\wedge\delta_Q d\mathbf{A}) + \mathrm{Tr}_{\mathrm{R}}(\mathbf{\Phi}\wedge d\mathbf{F} + d\mathbf{\Phi}\wedge\mathbf{F})\right).$$

Using the trace property and Bianchi identity $d\mathbf{W} = [\mathbf{W},\mathbf{A}]$

$$d\mathbf{B} = \frac{i}{2\pi}\mathrm{Tr}_R(d\mathbf{A}\wedge\delta_Q\mathbf{A} - \mathbf{W}\wedge[\mathbf{A},\mathbf{\Phi}] - \mathbf{W}\wedge d\mathbf{\Phi}),$$

and since the residue trace of $\mathbf{A}\wedge\mathbf{A}\wedge\delta_Q\mathbf{A}$ vanishes,

$$d\mathbf{B} = \frac{i}{2\pi}\mathrm{Tr}_R(\mathbf{W}\wedge\delta_Q\mathbf{A} - \mathbf{W}\wedge[\mathbf{A},\mathbf{\Phi}] - \mathbf{W}\wedge d\mathbf{\Phi}).$$

This descends to a closed 3-form on X and so gives (126). □

The transgression formula for H_Q under change of Q is given by,

$$H_{Q,\mathbf{\Phi},\mathbf{A}} - H_{Q',\mathbf{\Phi},\mathbf{A}} = -\frac{i}{2\pi}d\,\mathrm{Tr}_R(\mathbf{A}\wedge[P,\mathbf{A}]), \tag{127}$$

where $P = \log(Q) - \log(Q')$.

The other transgression formulae are:-

$$H_{Q,\mathbf{\Phi},\mathbf{A}'} - H_{Q,\mathbf{\Phi},\mathbf{A}} = -\frac{i}{2\pi}d\,\mathrm{Tr}_R(f\wedge\nabla^Q\mathbf{\Phi}) \tag{128}$$

where $\mathbf{A}' - \mathbf{A} = f \in \Omega^1(X,\mathrm{End}(\mathbf{\Psi}^{\mathbb{Z}}))$. If $\mathbf{\Phi}'$ is another choice of Higgs field and $\mathbf{\Phi}' - \mathbf{\Phi} = \sigma$, where σ is a section of the adjoint bundle, then

$$H_{Q,\mathbf{\Phi}',\mathbf{A}} - H_{Q,\mathbf{\Phi},\mathbf{A}} = -\frac{i}{2\pi}d\,\mathrm{Tr}_R(\mathbf{W}\wedge\sigma). \tag{129}$$

Note that H is also the Dixmier-Douady class of the bundle of compact operators obtained by closing $\mathbf{\Psi}^{-\infty}$ in the topology of bounded operators on $L^2(Z;V)$.

There are higher characteristic classes for the pseudodifferential algebra bundle $\mathbf{\Psi}^{\mathbb{Z}}$ over X, which are represented by $H_Q(n) = c_n\,\mathrm{Tr}_{\mathrm{R}}(\mathbf{W}^n\wedge\nabla^Q\mathbf{\Phi})$ for appropriate constants c_n. These are closed differential forms of degree$2n+1$ on X. Explicit transgression formulae can be derived for these forms as above. The characteristic classes represented by these differential forms may be non-trivial in cohomology, since $\mathrm{G}(\mathcal{F}^0(Z;V))$ need not be contractible. Analogous results for loop groups have been obtained in [41].

8. Examples

The examples below illustrate that the rationalized Dixmier-Douady invariant computed above does not in general vanish, i.e. the Dixmier-Douady invariant itself may be non-torsion in this setting of pseudodifferential bundles.

8.1. **Finite dimensional bundle gerbes.** Here we will define the smooth Azumaya bundle and the filtered algebra bundle of pseudodifferential operators in the case of a finite dimensional bundle gerbe, thereby giving a large class of examples that satisfy the hypotheses of the main theorem in the paper.

The data we use to define a smooth Azumaya bundle is:-

- A smooth fiber bundle of compact manifolds

$$\begin{array}{ccc} Z & \longrightarrow & Y \\ & & \downarrow \phi \\ & & X. \end{array} \tag{130}$$

- A primitive line bundle J over $Y^{[2]}$, in the sense that under lifting by the three projection maps

$$Y^{[3]} \underset{\pi_F}{\overset{\pi_S}{\overset{\longrightarrow}{\underset{\longrightarrow}{\xrightarrow{\pi_C}}}}} Y^{[2]} \tag{131}$$

(corresponding respectively to the left two, the outer two and the right two factors) there is a natural isomorphism

$$\pi_S^* J \otimes \pi_F^* J = \pi_C^* J. \tag{132}$$

The data above specifies a *bundle gerbe* $(Y/X, J)$, which in turn determines an infinite rank *smooth Azumaya bundle*, $\mathcal{S} \to X$, defined in terms of its space of global sections

$$\mathcal{C}^\infty(X; \mathcal{S}) = \mathcal{C}^\infty(Y^{[2]}; J). \tag{133}$$

It has fibres isomorphic to the algebra of smoothing operators on the fibre, Z, of Y with Schwartz kernels consisting of the smooth sections of the primitive line bundle J over Z^2. The primitivity property of J ensures that $\mathcal{C}^\infty(X; \mathcal{S})$ is an algebra. The completion of this algebra of 'smoothing operators' to a bundle with fibres modelled on the compact operators has Dixmier-Douady invariant $\delta(Y/X; J) \in \mathrm{H}^3(Y; \mathbb{Z})$.

We now define the associated projective bundle of pseudodifferential operators. We do this by direct generalization of the definition of the smooth Azumaya bundle $\mathcal{S}$ above. For any $\mathbb{Z}_2$-graded bundle $\mathbb{E} = (E_+, E_-)$ over Y set the projective filtered algebra bundle of pseudodifferential operators

$$\Psi^{\mathbb{Z}}_J(Y/X; \hom(\mathbb{E})) = I^{\mathbb{Z}}(Y^{[2]}, \mathrm{Diag}; \mathrm{Hom}(\mathbb{E}) \otimes \Omega_R \otimes J) \longrightarrow X \tag{134}$$

where $\mathrm{Hom}(\mathbb{E}) = E_- \boxtimes E'_+$ over $Y^{[2]}$ and $I^\bullet$ is the space of (classical) conormal distributions. As is typical in projective index theory, the Schwartz kernel of the projective family of elliptic operators is globally defined, even though one only has local families of elliptic operators with a compatibility condition on triple overlaps given by a phase factor.

Proposition 8.1. *In the situation described above, we have the following equality of Dixmier-Douady classes,*

$$\delta(\Psi^{\mathbb{Z}}_J(Y/X; \hom(\mathbb{E}))) = \delta(Y/X; J) \in H^3(X, \mathbb{Z}).$$

Proof. Now the gerbe associated to the bundle gerbe $(Y/X; J)$ is defined as follows: let $s_i : U_i \to \phi^{-1}(U_i)$ be a local smooth section of the fibre bundle $Y \overset{\phi}{\longrightarrow} X$, where U_i is an open subset of X. Then the gerbe associated to the bundle gerbe $(Y/X, J)$ is the collection of line bundles $J_{ij} = (s_i, s_j)^* J$ on the double overlaps $U_i \cap U_j$.

The section $s_i : U_i \longrightarrow \phi^{-1}(U_i)$, induces an isomorphism of J over the open subset $V_i = \phi^{-1}(U_i) \times_{U_i} \phi^{-1}(U_i)$ of $Y^{[2]}$, with

$$J\big|_{V_i} \cong_s \operatorname{Hom}(K_i) = K_i \boxtimes K_i' \tag{135}$$

for a line bundle K_i over $\phi^{-1}(U_i) \subset Y$, where K_i' denotes the line bundle dual to K_i. Another choice of section $s_j : U_j \longrightarrow \phi^{-1}(U_j)$, determines another line bundle K_j over $\phi^{-1}(U_j) \subset Y$, satisfying

$$K_i = K_j \otimes \phi^*(J_{ij}), \tag{136}$$

where $J_{ij} = (s_i, s_j)^* J$ is the fixed local line bundle over U_{ij}.

It follows that locally, $\Psi^{\mathbb{Z}}_J$ is a family of pseudodifferential operators acting fibrewise on $\phi^{-1}(U_i)$,

$$\Psi^{\mathbb{Z}}_J(\phi^{-1}(U_i); \hom(\mathbb{E})) = \Psi^{\mathbb{Z}}(\phi^{-1}(U_i); \hom(\mathbb{E} \otimes K_i))$$

and by equation (136) that it fails to be a global family of pseudodifferential operators acting fibrewise on Y, since by (136), there is no global line bundle on Y that restricts to each of the local line bundles K_j. Therefore the gerbe corresponding to the projective bundle $\Psi^{\mathbb{Z}}_J(Y/X; \hom(\mathbb{E}))$, which is the obstruction to finding a global line bundle on Y restricting to each of the local line bundles K_j, is exactly the collection of line bundles $J_{ij} = (s_i, s_j)^* J$ on the double overlaps $U_i \cap U_j$, proving the proposition.

□

8.2. **Smooth Azumaya bundle for a sum of decomposable elements.** Here we outline an extension of the geometric setting in [28] which defines a smooth Azumaya bundle whose Dixmier-Douady invariant is a sum of decomposable classes. This is essential, since a general element in $\mathrm{H}^2(X;\mathbb{Z}) \cup \mathrm{H}^1(X;\mathbb{Z})$ is of this form.

The data we use to define a smooth Azumaya bundle is:-

- A smooth function

$$u \in \mathcal{C}^\infty(X; \mathrm{U}(1)^N) \tag{137}$$

  the homotopy class of which represents $\alpha \in \mathrm{H}^1(X, \mathbb{Z}^N)$.
  Equivalently, u defines a regular covering space

$$\begin{array}{ccc} \mathbb{Z}^N & \longrightarrow & \hat{X} \\ & & \downarrow \tau \\ & & X \end{array} \tag{138}$$

- A principal torus bundle bundle (later with connection)

$$\begin{array}{ccc} \mathrm{U}(1)^N & \longrightarrow & P \\ & & \downarrow p \\ & & X \end{array} \tag{139}$$

  with Chern class $\beta \in \mathrm{H}^2(X; \mathbb{Z}^N)$.

- A smooth fiber bundle of compact manifolds

$$\begin{array}{ccc} Z & \longrightarrow & Y \\ & & \Big\downarrow \phi \\ & & X \end{array} \tag{140}$$

  such that $\phi^*\beta = 0$ in $\mathrm{H}^2(Y;\mathbb{Z}^N)$.
- An explicit global trivialization

$$\gamma : \phi^*(P) \xrightarrow{\simeq} Y \times \mathrm{U}(1)^N. \tag{141}$$

The data (137) – (141) are shown below to determine an infinite rank *smooth Azumaya bundle*, which we denote $\mathcal{S}(\gamma)$. It has fibres isomorphic to the algebra of smoothing operators on the fibre, Z, of Y with Schwartz kernels consisting of the smooth sections of a line bundle $J(\gamma)$ over Z^2. The completion of this algebra of 'smoothing operators' to a bundle with fibres modelled on the compact operators has Dixmier-Douady invariant $\langle\alpha,\beta\rangle \in \mathrm{H}^3(Y;\mathbb{Z})$, where $\langle,\rangle$ is the pairing $\mathrm{H}^2(X;\mathbb{Z}^N) \times \mathrm{H}^1(X;\mathbb{Z}^N) \longrightarrow \mathrm{H}^3(X;\mathbb{Z})$ given by cup product and the choice of inner product $\mathbb{Z}^N \times \mathbb{Z}^N \mapsto \mathbb{Z}$ given by $(m,\ell) \mapsto m_1\ell_1 + \dots m_N\ell_N$.

An explicit trivialization of the lift, γ, as in (139) is equivalent to a global section which is the preimage under γ of the identity element of the torus $\mathrm{U}(1)^N$:

$$s' : Y \longrightarrow \phi^*(P). \tag{142}$$

Over each fiber of Y, the image is fixed so this determines a map

$$s(z_1,z_2) = s'(z_1)(s'(z_2))^{-1}$$

which is well-defined on the fiber product and is a groupoid character:

$$\begin{gathered} s : Y^{[2]} \longrightarrow \mathrm{U}(1)^N, \\ s(z_1,z_2)s(z_2,z_3) = s(z_1,z_3)\ \forall\ z_i \in Y \text{ with } \phi(z_i) = x,\ i = 1,2,3,\ \forall\ x \in X. \end{gathered} \tag{143}$$

Conversely one can start with a unitary character s of the groupoid $Y^{[2]}$ and recover the principal torus bundle P as the associated bundle

$$\begin{gathered} P = Y \times \mathrm{U}(1)^N / \simeq_s, \\ (z_1,t) \simeq_s (z_2, s(z_2,z_1)t)\ \forall\ t \in \mathrm{U}(1)^N,\ \phi(z_1) = \phi(z_2). \end{gathered} \tag{144}$$

Now, let $Q = Y^{[2]} \times_X \hat{X}$ be the fiber product of $Y^{[2]}$ and $\hat{X}$, so as a bundle over X it has typical fiber $Z^2 \times \mathbb{Z}^N$; it is also a principal $\mathbb{Z}^N$-bundle over $Y^{[2]}$. The data above determines an action of $\mathbb{Z}^N$ on the trivial bundle $Q \times \mathbb{C}$ over Q, namely

$$T_n : (z_1,z_2,\hat{x};w) \longrightarrow (z_1,z_2,\hat{x}+n, \langle\langle s(z_1,z_2),n\rangle\rangle w)\ \forall\ n \in \mathbb{Z}^N, \tag{145}$$

where $\langle\langle\cdot,\cdot\rangle\rangle\colon\ \mathrm{U}(1)^N \times \mathbb{Z}^N \longrightarrow \mathrm{U}(1)$ denotes the Pontrjagin duality pairing between the Pontrjagin dual groups $\mathrm{U}(1)^N$ and $\mathbb{Z}^N$.

Let J be the associated line bundle over $Y^{[2]}$

$$J = (Q \times \mathbb{C})/\simeq, \quad (z_1,z_2,\hat{x};w) \simeq T_n(z_1,z_2,\hat{x};w)\ \forall\ n \in \mathbb{Z}^N \tag{146}$$

The fiber of J at $(z_1,z_2) \in Y^{[2]}$ such that $\phi(z_1) = \phi(z_2) = x$ is

$$J_{z_1,z_2} = \hat{X}_x \times \mathbb{C}/\simeq, \quad (\hat{x}+n,w) \simeq (\hat{x}, \langle\langle s(z_1,z_2),-n\rangle\rangle w). \tag{147}$$

Note that this primitive line bundle *does depend* on the trivialization data in (139); we will therefore denote it $J(\gamma)$.

This line bundle is *primitive* in the sense that under lifting by the three projection maps

$$Y^{[3]} \overset{\pi_S}{\underset{\pi_F}{\overset{\pi_C}{\rightrightarrows}}} Y^{[2]} \tag{148}$$

(corresponding respectively to the left two, the outer two and the right two factors) there is a natural isomorphism

$$\pi_S^* J \otimes \pi_F^* J = \pi_C^* J. \tag{149}$$

which gives the identification

$$J_{(z,z'')} \otimes J_{(z'',z')} \simeq J_{(z,z')}. \tag{150}$$

As remarked above, $J(\gamma)$, depends on the particular global trivialization (139). Two trivializations, γ_i, $i = 1, 2$ as in (139) determine

$$\gamma_{12} : Y \longrightarrow \mathrm{U}(1)^N,\ \gamma_{12}(y)\gamma_2(y) = \gamma_1(y) \tag{151}$$

which fixes an element $[\gamma_{12}] \in \mathrm{H}^1(Y; \mathbb{Z}^N)$ and hence a line bundle K_{12} over Y with Chern class $\langle [\gamma_{12}], [\phi^*\alpha] \rangle \in \mathrm{H}^2(Y; \mathbb{Z})$. Then

$$J(\gamma_2) \simeq (K_{12}^{-1} \boxtimes K_{12}) \otimes J(\gamma_1) \tag{152}$$

with the isomorphism consistent with primitivity.

That is, we have constructed a finite dimensional bundle gerbe $(Y/X; J)$ with Dixmier-Douady invariant

$$\delta(Y/X; J) = \langle \alpha, \beta \rangle \in \mathrm{H}^3(X; \mathbb{Z}).$$

As in the previous subsection, we can also define projective bundles of pseudodifferential operators in this setup.

8.3. **A canonical example.** The following extends the discussion of an example in [28]. In particular, let $\phi : Y \to X$ be a fibre bundle of compact manifolds, with typical fiber a compact Riemann surface Σ_g of genus $g \geq 2$. Then $T(Y/X)$ is an oriented rank 2 bundle over Y. Define $\beta = \phi_*(e \cup e) \in \mathrm{H}^2(X, \mathbb{Z})$, where $e := e(T(Y/X)) \in \mathrm{H}^2(Y, \mathbb{Z})$ is the Euler class of $T(Y/X)$. By naturality of this construction, $\beta = f^*(e_1)$, where $e_1 \in \mathrm{H}^2(B\mathrm{Diff}(\Sigma_g), \mathbb{Z})$ and $f \colon X \to B\mathrm{Diff}(\Sigma_g)$ is the classifying map for $\phi : Y \to X$. e_1 is known as the universal first Mumford-Morita-Miller class, and β is the first Mumford-Morita-Miller class of $\phi : Y \to X$, cf. Chapter 4 in [34]. Therefore by Lemma 14 in [28], we have the following.

Lemma 8.2. *In the notation above, let $\phi : Y \to X$ be a fibre bundle of compact manifolds, with typical fiber a compact Riemann surface Σ_g of genus $g \geq 2$, and let $\beta \in \mathrm{H}^2(X, \mathbb{Z})$ be a multiple of the first Mumford-Morita-Miller class of $\phi : Y \to X$. Then $\phi^*(\beta) = 0$ in $\mathrm{H}^2(Y, \mathbb{Z})$.*

Let $\phi : Y \longrightarrow X$ be as above, and X be a closed Riemann surface. Then Proposition 4.11 in [34] asserts that $\langle e_1, [X] \rangle = \mathrm{Sign}(Y)$, where $\mathrm{Sign}(Y)$ is the signature of the 4-dimensional manifold Y, which is originally a result of Atiyah, cf. [34]. As a consequence, Morita is able to produce infinitely many surface bundles Y over X that have non-trivial first Mumford-Morita-Miller class.

On the other hand, given any $\beta \in \mathrm{H}^2(X, \mathbb{Z})$, we know that there is a fibre bundle of compact manifolds $\phi : Y \to X$ such that $\phi^*(\beta) = 0$ in $\mathrm{H}^2(Y, \mathbb{Z})$. In fact we can choose Y to be the total space of a principal $\mathrm{U}(n)$ bundle over X with first Chern class β. Here we can also replace $\mathrm{U}(n)$ by any compact Lie group G such that $\mathrm{H}^1(G,\mathbb{Z})$ is nontrivial and torsion-free, such as the torus $\mathbb{T}^n$.

Lemma 8.3. *Let $\phi : Y \to X$ be a fibre bundle of compact manifolds with typical fiber a compact Riemann surface Σ of genus $g \geq 2$ and $\beta \in \mathrm{H}^2(X,\mathbb{Z})$. Let $\pi : P \to X$ be a principal $\mathrm{U}(n)$-bundle whose first Chern class is β. Then the fibred product $\phi \times \pi : Y \times_X P \to X$ is a fiber bundle with typical fiber $\Sigma \times \mathrm{U}(n)$, and has the property that $(\phi \times \pi)^*(\beta) = 0$ in $\mathrm{H}^2(Y \times_X P, \mathbb{Z})$.*

This follows from the obvious commutativity of the following diagram,

$$\begin{array}{ccc} Y \times_X P & \xrightarrow{pr_1} & Y \\ {\scriptstyle pr_2}\downarrow & & \downarrow{\scriptstyle \phi} \\ P & \xrightarrow{\pi} & X. \end{array} \tag{153}$$

The construction of the universal fibre bundle of Riemann surfaces which we will describe next, is well known, cf. [1, 4, 16]. Let Σ be a compact Riemann surface of genus g greater than 1, $\mathfrak{M}_{(-1)}$ the space of all hyperbolic metrics on Σ of curvature equal to -1, and $\mathrm{Diff}_+(\Sigma)$ the group of all orientation preserving diffeomorphisms of Σ. Then the quotient

$$\mathfrak{M}_{(-1)}/\mathrm{Diff}_+(\Sigma) = \mathcal{M}_g$$

is a noncompact orbifold, namely the moduli space of Riemann surfaces of genus equal to g. The fact that $\mathcal{M}_g$ has singularities can be dealt with in several ways, for instance by going to a finite smooth cover, and the noncompactness of $\mathcal{M}_g$ can be dealt with for instance by considering compact submanifolds. We will however not deal with these delicate issues in the discussion below. The group $\mathrm{Diff}_+(\Sigma)$ also acts on $\Sigma \times \mathfrak{M}_{(-1)}$ via $g(z,h) = (g(z), g^*h)$ and the resulting smooth fibre bundle,

$$\pi : Y = (\Sigma \times \mathfrak{M}_{(-1)})/\mathrm{Diff}_+(\Sigma) \longrightarrow \mathfrak{M}_{(-1)}/\mathrm{Diff}_+(\Sigma) = \mathcal{M}_g \tag{154}$$

is the *universal bundle* of genus g Riemann surfaces. The classifying map for (154) is the identity map on $\mathcal{M}_g$ so π is maximally nontrivial in a sense made precise below.

As before, let

$$e_1 = e_1(Y/\mathcal{M}_g) = \pi_*(e \cup e) \in H^2(\mathcal{M}_g;\mathbb{Z})$$

be the first Mumford-Morita-Miller class of $\pi\colon Y \to \mathcal{M}_g$.

A theorem of Harer [19, 34] asserts that:

$$H^2(\mathcal{M}_g;\mathbb{Q}) = \mathbb{Q}(e_1);$$
$$H^1(\mathcal{M}_g;\mathbb{Q}) = \{0\}.$$

Our next goal is to define a line bundle $\mathcal{L}$ over $\mathcal{M}_g$ such that $c_1(\mathcal{L}) = ke_1$ for some $k \in \mathbb{Z}$. This line bundle then automatically has the property that $\pi^*(\mathcal{L})$ is trivializable since e_1 is a characteristic class of the fibre bundle $\pi : Y \longrightarrow \mathcal{M}_g$. This is exactly the data that is needed to define a projective family of Dirac operators. The line bundle $\mathcal{L}$ turns out to be a power of the determinant line bundle of the virtual vector bundle Λ known as the Hodge bundle, which is defined using the Gysin map in K-theory.

$$\Lambda = \pi_!(T(Y/\mathcal{M}_g)) \in K^0(\mathcal{M}_g).$$

Then $\det(\Lambda)$ is actually a line bundle over $\mathcal{M}_g$. Next we need the following special Grothendieck-Riemann-Roch (GRR) calculation, cf. [28] Appendix C, Lemma 15.

Lemma 8.4. *In the notation above, one has the following identity of first Chern classes,*

$$c_1(\pi_!(T(Y/\mathcal{M}_g)) = \frac{13}{12}\pi_*(c_1(T(Y/\mathcal{M}_g))^2).$$

Observing that $c_1(T(Y/\mathcal{M}_g) = e$ and

$$c_1(\pi_!(T(Y/\mathcal{M}_g)) = c_1(\Lambda) = c_1(\det(\Lambda)),$$

the lemma above shows that $c_1(\det(\Lambda)) = \frac{13}{12}e_1$. Setting $\mathcal{L} = \det(\Lambda)^{\otimes 12}$, we obtain, cf. [28] Appendix C, Corollary 2.

Corollary 8.5. *In the notation above, $\mathcal{L}$ is a line bundle over $\mathcal{M}_g$ and one has the following identity:*

$$c_1(\mathcal{L}) = 13e_1.$$

We next construct a canonical projective family of Dirac operators on the Riemann surface Σ with fixed choice of spin structure. This family is different to the one constructed in [28] Appendix C. We enlarge the parametrizing space $\mathcal{M}_g$ by taking the product with the Jacobian variety $\mathrm{Jac}(Y)$, which is the smooth variety of all unitary characters of the fundamental group $\pi_1(Y)$ of Y.

Construct a tautological line bundle $\mathcal{P}$ over $Y \times \mathrm{Jac}(Y)$ as follows. Consider the free action of $\pi_1(Y)$,

$$\begin{aligned} \pi_1(Y) \times \widetilde{Y} \times \mathrm{Jac}(Y) \times \mathbb{C} &\to \widetilde{Y} \times \mathrm{Jac}(Y) \times \mathbb{C} \\ (\gamma, (y, \chi, z)) &\to (y.\gamma, \chi, \chi(\gamma)z), \end{aligned}$$

where $\widetilde{Y}$ is the universal covering space of Y. Then $\mathcal{P}$ is defined to be the quotient space,

$$\mathcal{P} = \left(\widetilde{Y} \times \mathrm{Jac}(Y) \times \mathbb{C}\right)/\pi_1(Y)$$

Consider now the fibre bundle $\pi \times \mathrm{Id} : Y \times \mathrm{Jac}(Y) \to \mathcal{M}_g \times \mathrm{Jac}(Y)$, with typical fibre the Riemann surface Σ. The fibre bundle is endowed with the tautological line bundle $\mathcal{P} \to Y \times \mathrm{Jac}(Y)$ over the total space of the fibre bundle. Applying the main construction in [28], we get a primitive line bundle $J \longrightarrow (Y \times \mathrm{Jac}(Y))^{[2]}$. By the construction at the end of [28] §5, we obtain a projective family of Dirac operators $\eth_{\mathcal{P}\otimes J}$ on the Riemann surface Σ, parametrized by $\mathcal{M}_g \times \mathrm{Jac}(Y)$, having analytic index,

$$\mathrm{Index}_a(\eth_{\mathcal{P}\otimes J}) \in K^0(\mathcal{M}_g \times \mathrm{Jac}(Y); e_1 \cup a),$$

where $a \in H^1(\mathrm{Jac}(Y); \mathbb{Z})$.

In the discussion above, we need to know that $\dim(\mathrm{Jac}(Y)) > 0$. We will establish this in the case when the genus $g \geq 1$. By the Leray-Serre spectral sequence for the fibre bundle $\Sigma \hookrightarrow Y \stackrel{\pi}{\to} \mathcal{M}_g$, one has the 5-term exact sequence of low degree homology groups, cf. Corollary 9.14 [11],

$$H_2(Y) \stackrel{\pi_*}{\to} H_2(\mathcal{M}_g) \stackrel{\tau}{\to} H_0(\mathcal{M}_g, H_1(\Sigma)) \to H_1(Y) \stackrel{\pi_*}{\to} H_1(\mathcal{M}_g) \to 0 \tag{155}$$

where τ denotes the transgression map.

By [26], one knows that $\mathcal{M}_g$ is simply-connected, therefore

$$H_0(\mathcal{M}_g, H_1(\Sigma)) \cong H_1(\Sigma) \cong \mathbb{Z}^{2g},$$

and by the Hurewicz theorem, $H_1(\mathcal{M}_g) = 0$. By the universal coefficient theorem and by [34], $H_2(\mathcal{M}_g) \cong H^2(\mathcal{M}_g, \mathbb{Z}) \cong \mathbb{Z}$. Therefore the 5-term exact sequence in equation (155) reduces to the 4-term exact sequence,

$$H_2(Y) \stackrel{\pi_*}{\to} H_2(\mathcal{M}_g) \stackrel{\tau}{\to} H_1(\Sigma) \to H_1(Y) \stackrel{\pi_*}{\to} 0 \tag{156}$$

Therefore, $H_1(Y) \cong H_1(\Sigma)/\mathrm{Image}(\tau)$ has rank $\geq 2g-1 > 0$ (since the rank of $\mathrm{Image}(\tau)$ is ≤ 1, therefore $\dim(\mathrm{Jac}(Y)) \geq 2g-1 > 0$, whenever $g \geq 1$.

We summarise the above as follows.

Proposition 8.6. *Let Σ be a Riemann surface of genus $g \geq 2$ and $Y \longrightarrow \mathcal{M}_g$ be the canonical family of hyperbolic metrics on Σ of curvature equal to -1. Consider the fibre bundle $\pi \times \mathrm{Id} : Y \times \mathrm{Jac}(Y) \to \mathcal{M}_g \times \mathrm{Jac}(Y)$, with typical fibre the Riemann surface Σ. The fibre bundle is endowed with the tautological line bundle $\mathcal{P} \to Y \times \mathrm{Jac}(Y)$ over the total space of the fibre bundle. Applying the main construction in* [28]*, we get a primitive line bundle $J \longrightarrow (Y \times \mathrm{Jac}(Y))^{[2]}$. By the construction at the end of* [28] *§5, we obtain a projective family of Dirac operators $\eth_{\mathcal{P}\otimes J}$ on the Riemann surface Σ, parametrized by $\mathcal{M}_g \times \mathrm{Jac}(Y)$, having analytic index,*

$$\mathrm{Index}_a(\eth_{\mathcal{P}\otimes J}) \in K^0(\mathcal{M}_g \times \mathrm{Jac}(Y); e_1 \cup a),$$

where $a \in H^1(\mathrm{Jac}(Y); \mathbb{Z})$.

Appendix: Invertible Fourier integral operators

Fourier integral operators, as introduced by Hörmander in [22], see also [21], are operators with Schwartz' kernels which are Lagrangian distributions. The distributions associated to a conic Lagrangian submanifold, $\Lambda \subset T^*M \setminus O$, are defined through local (really microlocal) parameterizations, they are then given by oscillatory integrals over the fibres of the parameterizations. The case of primary interest here, corresponds to $\Lambda = \mathrm{graph}'(\chi) \subset (T^*Z \setminus O)^2$ being the twisted graph of canonical, i.e. homogeneous symplectic, diffeomorphism. In case χ is close to the identity, in the $\mathcal{C}^\infty$ topology, it is possible to give a *global* parameterization (as for a psuedodifferential operator) which presents the kernel in terms of a single oscillatory integral. The group of canonical transformations, $\mathrm{Can}(Z)$ is naturall identified with the group of contact transformations of S^*Z and is a Fréchet Lie group, with the same $\mathcal{C}^\infty$ topology as the full group of diffeomorphisms of S^*Z. More precisely $\mathrm{Can}(Z)$ as a Fréchet manifold modelled on $\mathcal{C}^\infty(S^*Z)$, realized as the global space of smooth sections of the trivial bundle over S^*Z corresponing to functions homogeneous of degree 1 on T^*Z.

Proposition .7. *The choice of a Riemann metric on Z gives an identification of a neighbourhood of the identity in $\mathrm{Can}(Z)$ with a neighbourhood of 0 in $\mathcal{C}^\infty(S^*Z)$ which identifies the Lie algebra with $\mathcal{C}^\infty(S^*Z)$ with the normalized Poisson bracket.*

Proof. The exponential map corresponding to a choice of Riemann metric on Z gives a normal fibration, a collar neighbourhood, of the diagonal in Z^2 in the form

$$TZ \supset U \ni (z, v) \longrightarrow (\exp_z(v), z) \in U' \subset Z^2 \tag{157}$$

which is a diffeomorphism from U, an open neighbourhood of the zero section, to U', an open neighbourhood of the diagonal in Z^2. Under this map the cotangent bundle is identified with the subset of the fibre product $TZ \oplus T^*Z \oplus T^*Z$ projecting to U in the first factor

$$T^*Z \subset TZ \oplus T^*Z \oplus T^*Z,\ (\exp_z(v), z, \tau_z\xi, \eta) \longmapsto (z, v, \xi, \eta) \tag{158}$$

where $\tau_z\xi \in T^*_{\exp_z(v)}Z$ is obtained by parallel transport from z along the geodesic defining $\exp_z(v)$.

For $F \in \mathrm{Con}(Z)$ in a neighbourhood of the identity with respect to some $\mathcal{C}^2$ norm, the image of each fibre $T^*_zZ \setminus 0$ is necessarily a conic Lagrangian submanifold $\Lambda_z \subset T^*Z \setminus O$ which is close to T^*Z. Consider the reversed graph $\Gamma(F) = \{F(\sigma, \sigma); \sigma \in T^*Z\} \subset T^*Z^2$. Then in terms of (158) $\Gamma_z(F) = \Gamma(F) \cap T^*Z \times T^*_zZ$ projects diffeomorphically onto the first factor of $T^*Z \setminus 0$ and so may be written as the range of a homogeneous smooth map $\Phi_z : T^*_zZ \setminus 0 \longrightarrow T_zZ$, i.e.

$$\pi(\Gamma_z(F)) = \{(\Phi(\eta), \eta)\}. \tag{159}$$

As a homogeneous Lagrangian submanifold both the symplectic form, which is $dv \wedge d\xi$ and the canonical form $\xi \cdot dv$ vanish on $\pi(\Gamma_z(F))$ for each z. Thus if ξ are used as coordinates on $\pi(\Gamma_z(F))$ so $v = v(\xi)$ then

$$\phi(z, \xi) = v(\xi) \cdot \xi \Longrightarrow d_\xi\phi(x, \xi) = v(\xi). \tag{160}$$

Thus in fact Φ is given as the gradient of a uniquely defined smooth function $\phi(z, \xi)$ which is homogeneous of degree 1 in ξ and so projects to a smooth function $\tilde{\phi} = \phi(z, \xi)/|\xi|_g \in \mathcal{C}^\infty(S^*Z)$. This identifies a neighourhood of the identity in $\mathrm{Can}(Z)$ with a neigbourhood of 0 in $\mathcal{C}^\infty(S^*Z)$

$$\pi(\Gamma_z(F)) = \{d_\xi(\phi(z, \xi); \xi \in T^*_zZ\}. \tag{161}$$

Note that the collection of these conic Lagrangian submanifolds, of $T^*Z \times Z$ as a bundle over the second factor of Z, does determine F. Indeed, the reverse twisted graph of F can be recovered from the phase function defined over U' as

$$\psi(z, v, \eta) = v \cdot \eta - \phi(z, \eta) \tag{162}$$

where the first term is the pairing between T_zZ and T^*zZ.

Although this construction does depend on the normal fibration around the diagonal, (157), changing this induces a bundle transformation corresponding to the fact that the structure groupoid of $\mathrm{Can}(Z)$ can be reduced to the fibre-preserving local diffeomorphisms of TZ.

Since $[0, 1] \ni t \longmapsto t\phi \in \mathcal{C}^\infty(S^*Z)$ connects ϕ to 0 it follows that any canonical transformation near the identity is given as the parameter-dependent integral of the corresponding Hamilton vector fields $tH_{\tilde{\phi}}$ and hence that the Lie algebra of $\mathrm{Can}(Z)$ may be identified with the Poisson bracket projected to $\mathcal{C}^\infty(S^*Z)$,

$$[\tilde{\phi}_1, \tilde{\phi}_2] = \{\phi_1, \phi_2\}/|\xi|. \tag{163}$$

□

This parameterization of canonical transformations near the identity gives a section of the group invertible Fourier integral operators – essentially as in [5].

Proposition .8. *Given a choice of Riemann metric on a compact manifold Z and corresponding normal fibration of the diagonal as in* (158)*, the choice of a cutoff*

$\chi \in \mathcal{C}^\infty_c(U')$, equal to 1 near the diagonal and of a connection on V to define parallel transport of the identity operator on V in the oscillatory integral

$$K(\phi) = (2\pi)^{-n} \int e^{i(v\cdot\xi - \phi(z,\xi))}\chi(v,z)\,\mathrm{Id}(v,z)d\xi dg_z, \tag{164}$$

defines a section of the bundle of invertible Fourier integral operators over the neighbourhood of the identity in $\mathrm{Can}(Z)$ corresponding to $\phi|\xi|^{-1} \in \mathcal{C}^\infty(S^*Z)$ near 0 with respect to a sufficiently high $\mathcal{C}^k$.

Proof. The discussion above shows that the phase function

$$\psi(v,z,\xi) = v\cdot\xi - \phi(z,\xi) \text{ on } TZ \oplus T^*Z \tag{165}$$

as a bundle over U parameterizes the twisted graph of the canonical transformation in the sense introduced by Hörmander [21]. Thus (164) is indeed a Fourier integral operator associated, i.e. projecting to, the canonical transformation used to define ϕ. For ϕ close to zero the graph of the canonical transformation is close to the identity and (164) is therefore elliptic. The adjoints of the $K(\phi)$, with respect to some fixed choice of inner product on V and density on Z, are necessarily Fourier integral operators associated to the inverse transformations and the product

$$D(\phi) = K(\phi)^*K(\phi) \tag{166}$$

is an elliptic pseudodifferential operator. Indeed a direct application of the stationary phase lemma shows that D is a smooth map from a neighbourhood of zero in $\mathcal{C}^\infty(S^*Z)$ into the elliptic pseudodifferential operators, and hence for a possibly smaller neighbourhood into $G^0(Z;V)$. Thus $K(\phi)$ gives a section of $\mathrm{G}(\mathcal{F}^0(Z;V))$ as desired. □

If E_0 is the range of K on a neighbourhood of 0 in $\mathcal{C}^\infty(S^*Z)$ then $G^0(Z;V)\cdot E_0$ certainly has the structure of a principal $G^0(Z;V)$-bundle. To extend this to the whole of $\mathrm{G}_0(\mathcal{F}^0(Z;V))$, the part of the group corresponding to the component of the identity in $\mathrm{Can}(Z)$, it suffices to see that the local action of $\mathrm{G}_0(\mathcal{F}^0(Z;V))$ induces a smooth bundle map, but this again follows directly from the composition properties of Fourier integral operators.

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