J. DIFFERENTIAL GEOMETRY
x78 (2008) 465-473

EQUIVARIANT AND FRACTIONAL INDEX OF PROJECTIVE ELLIPTIC OPERATORS

V. Mathai, R.B. Melrose & I.M. Singer

Abstract

In this note the fractional analytic index, for a projective elliptic operator associated to an Azumaya bundle, of [**5**] is related to the equivariant index of [**1**, **6**] for an associated transversally elliptic operator.

Introduction

Recall the setup in [**5**]. Let $\mathcal{A}$ be an Azumaya bundle of rank N over a compact oriented manifold X and let $\mathcal{P}$ denote the associated principal $\mathrm{PU}(N)$-bundle of trivializations of $\mathcal{A}$. Let $\mathbb{E} = (E^+, E^-)$ denote a pair of projective vector bundles associated to $\mathcal{A}$ (or $\mathcal{P}$), which is to say a projective $\mathbb{Z}_2$ superbundle. For each such pair, we defined in [**5**] projective pseudodifferential operators $\Psi^\bullet_\epsilon(X, \mathbb{E})$ with support in an ϵ-neighborhood of the diagonal in $X \times X$. The principal symbol $\sigma(D)$ of an elliptic projective pseudodifferential operator D defines an element of the compactly-supported twisted K-theory

$$[(\tau^*(\mathbb{E}), \sigma(D))] \in K^0(T^*X, \tau^*\mathcal{A}),$$

where $\tau : T^*X \to X$ is the projection. The fractional analytic index of D, which is defined using a parametrix, gives a homomorphism

$$\mathrm{ind_a}\colon K^0(T^*X, \tau^*\mathcal{A}) \longrightarrow \mathbb{Q}. \tag{1}$$

On the other hand, the projective vector bundles $E^\pm$ can be realized as vector bundles, $\widehat{\mathbb{E}} = (\widehat{E}^+, \widehat{E}^-)$, in the ordinary sense over the total space of $\mathcal{P}$ with an action of $\widehat{G} = \mathrm{SU}(N)$ which is equivariant with respect to the action of $G = \mathrm{PU}(N)$ and in which the center, $\mathbb{Z}_N$, acts as the Nth roots of unity. Following [**1**, **6**], the $\widehat{G}$-equivariant pseudodifferential operators $\Psi^\bullet_{\widehat{G}}(P, \widehat{\mathbb{E}})$ are defined for any equivariant bundles, as is the notion of transversal ellipticity. The principal symbol $\sigma(A)$ of a transversally elliptic $\widehat{G}$-equivariant pseudodifferential operator, A, fixes an element in equivariant K-theory

$$[(\tau^*(\widehat{\mathbb{E}}), \sigma(A))] \in K^0_{\widehat{G}}(T^*_{\widehat{G}}\mathcal{P})$$

Received 11/01/2006.

and all elements arise this way. The $\widehat{G}$-equivariant index of A, which is defined using a partial parametrix for A, is a homomorphism,

$$\operatorname{ind}_{\widehat{G}} \colon K^0_{\widehat{G}}(T^*_{\widehat{G}}\mathcal{P}) \longrightarrow \mathcal{C}^{-\infty}(\widehat{G}). \tag{2}$$

The restriction on the action of the center on the lift of a projective bundle to $\mathcal{P}$, as opposed to a general $\widehat{G}$-equivariant bundle for the $\mathrm{PU}(N)$ action, gives a homomorphism,

$$\iota : K^0(T^*X, \tau^*\mathcal{A}) \longrightarrow K^0_{\widehat{G}}(T^*_{\widehat{G}}\mathcal{P}). \tag{3}$$

The diagonal action of G on $\mathcal{P}^2$ preserves the diagonal which therefore has a basis of G-invariant neighborhoods. From the $\widehat{G}$-equivariant pseudodifferential operators, with support in a sufficiently small neighborhood of the diagonal, there is a surjective pushforward map, which is a homomorphism at the level of germs,

$$\pi_* \colon \Psi^\bullet_{\widehat{G},\epsilon}(\mathcal{P}, \widehat{\mathbb{E}}) \longrightarrow \Psi^\bullet_\epsilon(X, \mathbb{E}) \tag{4}$$

to the projective pseudodifferential operators. In fact, this map preserves products provided the supports of the factors are sufficiently close to the diagonal. Moreover, pushforward sends transversally elliptic $\widehat{G}$-equivariant pseudodifferential operators to elliptic projective pseudodifferential operators and covers the homomorphism (3)

$$\iota[\sigma(\pi_*(A))] = [\sigma(A)] \text{ in } K^0_{\widehat{G}}(T^*_{\widehat{G}}\mathcal{P}) \text{ for } A \in \Psi^\bullet_{\widehat{G},\epsilon}(\mathcal{P}, \mathbb{E}) \text{ elliptic.} \tag{5}$$

Proposition 5 below relates these two pictures. Namely, if $\phi \in \mathcal{C}^\infty(\mathrm{SU}(N))$ has support sufficiently close to $e \in \mathrm{SU}(N)$ and is equal to 1 in a neighborhood of e, then the evaluation functional $\mathrm{ev}_\phi : \mathcal{C}^{-\infty}(\widehat{G}) \longrightarrow \mathbb{R}$ gives a commutative diagram involving (1), (2) and (3)

$$\begin{array}{ccc} K^0(T^*X, \tau^*\mathcal{A}) & \xrightarrow{\ \iota\ } & K^0_{\widehat{G}}(T^*_{\widehat{G}}\mathcal{P}) \\ \Big\downarrow \mathrm{ind_a} & & \Big\downarrow \operatorname{ind}_{\widehat{G}} \\ \mathbb{Q} & \xleftarrow{\ \mathrm{ev}_\phi\ } & \mathcal{C}^{-\infty}(\widehat{G}). \end{array} \tag{6}$$

Informally, one can therefore say that the fractional analytic index, as defined in [**5**], is the coefficient of the delta function at the identity in $\mathrm{SU}(N)$ of the equivariant index for transversally elliptic operators on $\mathcal{P}$. Note that there may indeed be other terms in the equivariant index with support at the identity, involving derivatives of the delta function, and there are terms supported at other points of $\mathbb{Z}_N$.[1]

---

[1]We thank M. Karoubi for calling our attention to the omission of the assumption of orientation in our reference to the Thom isomorphism in our earlier paper [**4**]. It is unfortunate that we did not explicitly reference his pioneering work on twisted K-theory and refer the reader to his interesting new paper on the Arxiv, math/0701789 and the references therein.

1. Transversally elliptic operators and the equivariant index

As in [**1**, **6**], let X be a compact $\mathcal{C}^\infty$ manifold with a smooth action of a Lie group, $G \ni g : X \longrightarrow X$. In particular, the Lie algebra $\mathfrak{g}$ of G is realized as a Lie algebra of smooth vector fields $L_a \in \mathcal{C}^\infty(X; TX)$, $a \in \mathfrak{g}$, $[L_a, L_b] = L_{[a,b]}$. Let $\Gamma \subset T^*X$ denote the annihilator of this Lie algebra, so Γ is also the intersection over G of the null spaces of the pull-back maps

$$\Gamma \cap T^*_p X = \bigcap_{g \in G} \operatorname{null}\left(g^* : T^*_p X \longrightarrow T^*_{g^{-1}(p)}X\right). \tag{1}$$

Now, suppose that $\mathbb{E} = (E_+, E_-)$ is a smooth superbundle on X which has a smooth linear equivariant graded action of G. Let $P \in \Psi^k(X; \mathbb{E})$ be a pseudodifferential operator which is invariant under the induced action of G on operators and which is transversally elliptic, that is its characteristic variety does not meet Γ :

$$\Gamma \cap \Sigma(P) = \emptyset, \quad \Sigma(P) = \{\xi \in T^*X \setminus 0; \sigma(P)(\xi) \text{ is not invertible}\}. \tag{2}$$

Under these conditions (for compact G) the equivariant index is defined in [**1**, **6**] as a distribution on G. In fact this can be done quite directly. To do so, recall that for a function of compact support $\chi \in \mathcal{C}^\infty_c(G)$, the action of the group induces a graded operator

$$T_\chi : \mathcal{C}^\infty(X; \mathbb{E}) \longrightarrow \mathcal{C}^\infty(X; \mathbb{E}), \quad T_\chi u(x) = \int_G \chi(g) g^* u dg. \tag{3}$$

Proposition 1. *A transversally elliptic pseudodifferential operator, P, has a parametrix Q, microlocally in a neighborhood of Γ and then for any $\chi \in \mathcal{C}^\infty_c(G)$,*

$$T_\chi \circ (PQ - \mathrm{Id}_-) \in \Psi^{-\infty}(X; \mathbb{E}_-) \text{ and } T_\chi \circ (QP - \mathrm{Id}_+) \in \Psi^{-\infty}(X; \mathbb{E}_+) \tag{4}$$

are smoothing operators and

$$\operatorname{ind}_G(P)(\chi) = \operatorname{Tr}(T_\chi(PQ - \mathrm{Id}_-)) - \operatorname{Tr}(T_\chi(QP - \mathrm{Id}_+)) \tag{5}$$

defines a distribution on G which is independent of the choice of Q.

Proof. The construction of parametrices is microlocal in any region where the operator is elliptic, so Q exists with the following constraint on the operator wavefront set,

$$\left(\mathrm{WF}'(PQ - \mathrm{Id}_-) \cup \mathrm{WF}'(QP - \mathrm{Id}_+)\right) \cap \Gamma = \emptyset. \tag{6}$$

Thus WF' is the wavefront set of the Schwartz kernel of a pseudodifferential operator, as a subset of the conormal bundle to the diagonal which is then identified with the cotangent bundle of the manifold. Then Q is unique microlocally in the sense that any other such parametrix Q' satisfies $\mathrm{WF}'(Q' - Q) \cap \Gamma = \emptyset$. The definition of Γ means that for any pseudodifferential operator A with $\mathrm{WF}'(A) \cap \Gamma = \emptyset$, $T_\chi A$ is smoothing and depends continuously on χ. Thus (4) holds and by the continuity of the dependence on χ defines a distribution on G.

To see the independence of the choice of parametrix, suppose that Q_i, $i = 0, 1$ are two choices. Then $Q_t = (1-t)Q_0 + tQ_1$ is a homotopy of parametrices for $t \in [0,1]$ which defines a linear family of distributions with derivative

$$\begin{aligned} &\frac{d}{dt}\{\mathrm{Tr}(T_\chi(PQ_t - \mathrm{Id}_-)) - \mathrm{Tr}(T_\chi(Q_tP - \mathrm{Id}_+))\} \\ &= \mathrm{Tr}(T_\chi(P(Q_1 - Q_0))) - \mathrm{Tr}(T_\chi((Q_1 - Q_0)P)) \\ &= \mathrm{Tr}(P(T_\chi(Q_1 - Q_0)) - \mathrm{Tr}((T_\chi(Q_1 - Q_0))P) = 0. \end{aligned}$$

Here we use the fact that P is invariant under the action of G and so commutes with T_χ. The microlocal uniqueness of parametrices implies that $T_\chi(Q_1 - Q_0)$ is a smoothing operator so the final line follows from the vanishing of the trace on commutators where one factor is pseudo-differential and the other is smoothing. q.e.d.

Proposition 2. *The distribution in* (5) *reduces to the equivariant index of* [**1**, **6**].

Proof. The Atiyah-Singer equivariant index for a transversally elliptic operator P is equal to $\mathrm{Tr}(T_\chi(\Pi_0)) - \mathrm{Tr}(T_\chi(\Pi_1))$ where Π_j, $j = 0, 1$ are the orthogonal projections onto the nullspaces of P and P^* respectively. The desired equality therefore involves only an interchange of integrals, over G and X. Namely, if one chooses (by averaging) a G-invariant parametrix Q for P, then the index in (5),

$$\mathrm{ind}_G(P)(\chi) = \mathrm{Tr}(T_\chi(PQ - I)) - \mathrm{Tr}(T_\chi(QP - I)),$$

is equal to $\mathrm{Tr}(T_\chi(\Pi_0)) - \mathrm{Tr}(T_\chi(\Pi_1))$. Let $K_\chi(x, y)$ denote the Schwartz kernel of the operator T_χ, so $K_\chi(x, y) = \int_G \delta_{gx}(y)\chi(g)dg$. Thus

$$\begin{aligned} \mathrm{Tr}(T_\chi \circ \Pi_j) &= \int_{x\in X}\int_{y\in X} K_\chi(x,y)\,\mathrm{tr}(\Pi_j(y,x))dydx \\ &= \int_{x\in X}\int_{y\in X}\int_G \delta_{gx}(y)\chi(g)dg\,\mathrm{tr}(\Pi_j(y,x))dydx \\ &= \int_G \chi(g)dg \int_{x\in X}\int_{y\in X} \delta_{gx}(y)\,\mathrm{tr}(\Pi_j(y,x))dydx \\ &= \int_G \chi(g)dg \int_{x\in X} \mathrm{tr}(\Pi_j(gx,x))dx \\ &= \int_G \chi(g)\,\mathrm{char}(\Pi_j)(g)dg, \end{aligned}$$

which shows that

$$\mathrm{Tr}(T_\chi(\Pi_0)) - \mathrm{Tr}(T_\chi(\Pi_1)) = \int_G \mathrm{ind}_G(P)(g)\chi(g)dg.$$

q.e.d.

Proposition 3. *Consider the subgroup of G defined by*

$$G_f = \{g \in G; gx = x \text{ for some } x \in X\}; \tag{7}$$

then

$$\mathrm{supp}(\mathrm{ind}_G(P)) \subset G_f. \tag{8}$$

Proof. If $G_f = G$, then there is nothing to prove. Suppose that $G_f \neq G$. Then for $g \in G \setminus G_f$, the set $\{(gx, x); x \in X\}$ is disjoint from the diagonal. It follows that if $\chi \in \mathcal{C}^\infty_c(G)$ has support sufficiently close to g and both P and its parametrix Q are chosen to have Schwartz kernels with supports sufficiently close to the diagonal (which is always possible), then the supports of the Schwartz kernels of all the terms in (4) are disjoint from the diagonal. It follows that $\mathrm{ind}_G(P)(\chi) = 0$ for such χ so $g \notin \mathrm{supp}(\mathrm{ind}_G(P))$. q.e.d.

2. Fractional and equivariant index

The finite central extension

$$\mathbb{Z}_N \longrightarrow \mathrm{SU}(N) \longrightarrow \mathrm{PU}(N) \tag{1}$$

is at the heart of the relation between the fractional and equivariant index. From an Azumaya bundle over a compact, oriented smooth manifold X we construct the principal $\mathrm{PU}(N)$-bundle $\mathcal{P}$ of trivializations. We will assume that the projective vector bundles in this section come equipped with a fixed hermitian structure.

Lemma 1. *A projective vector bundle E associated to an Azumaya bundle $\mathcal{A}$ over X lifts to a vector bundle $\widehat{E}$ over $\mathcal{P}$ with an action of $\mathrm{SU}(N)$ which is equivariant with respect to the $\mathrm{PU}(N)$ action on $\mathcal{P}$ and in which the center $\mathbb{Z}_N$ acts as the Nth roots of unity.*

Now, if $\mathbb{E}$ is a super projective vector bundle over X, it lifts to a super vector bundle $\widehat{\mathbb{E}}$ over $\mathcal{P}$ with $\mathrm{SU}(N)$ action. Consider the vector bundle $\hom(\widehat{\mathbb{E}})$ over $\mathcal{P}$ of homomorphisms from $\widehat{E}^+$ to $\widehat{E}^-$. Since the action of $\mathrm{SU}(N)$ on $\hom(\widehat{\mathbb{E}})$ is by conjugation, it descends to an action of $\mathrm{PU}(N)$, and hence $\hom(\widehat{\mathbb{E}})$ descends to a vector bundle $\hom(\mathbb{E})$ on X.

The space $\Psi^m(\mathcal{P};\widehat{\mathbb{E}})$ of pseudodifferential operators over $\mathcal{P}$ acting from sections of $\widehat{E}^+$ to $\widehat{E}^-$ may be identified with the corresponding space of kernels on $\mathcal{P}\times\mathcal{P}$ which are distributional sections of $\mathrm{Hom}(\widehat{\mathbb{E}})\otimes$

Ω_R, the 'big' homomorphism bundle over $\mathcal{P}^2$ with fiber at (p,p') consisting of the homomorphisms from $\widehat{E}^+_{p'}$ to $\widehat{E}^-_p$, tensored with the right density bundle and with conormal singularities only at the diagonal. We are interested in the $\mathrm{SU}(N)$-invariant part $\Psi^m_{\mathrm{SU}(N)}(\mathcal{P};\widehat{\mathbb{E}})$ corresponding to the kernels which are invariant under the 'diagonal' action of $\mathrm{SU}(N)$.

Proposition 4. *If $\Omega\subset\mathcal{P}^2$ is a sufficiently small neighborhood of $\mathrm{Diag}\subset\mathcal{P}^2$ invariant under the diagonal $\mathrm{PU}(N)$-action, there is a well-defined push-forward map into the projective pseudodifferential operators*

$$\left\{P\in\Psi^m_{\mathrm{SU}(N)}(\mathcal{P};\widehat{\mathbb{E}});\ \mathrm{supp}(P)\subset\Omega\right\}\ni A\longrightarrow\pi_*(A)\in\Psi^m_\epsilon(X;\mathbb{E}) \tag{2}$$

which preserves composition of elements with support in Ω' such that $\Omega'\circ\Omega'\subset\Omega$.

Proof. The push–forward map extends the averaging map in the $\mathrm{PU}(N)$-invariant case in which the action of $\mathbb{Z}_N$ is trivial. Then

$$\pi^*(\pi_*(A)\phi)=A\pi^*\phi \tag{3}$$

defines $\pi_*(A)$ unambiguously, since $\pi^*\phi$ is a $\mathrm{PU}(N)$-invariant section and hence so is $A\pi^*\phi$, so it determines a unique section of the quotient bundle. It is also immediate in this case that

$$\pi_*(AB)=\pi_*(A)\pi_*(B) \tag{4}$$

by the assumed $\mathrm{PU}(N)$-invariance of the operators. Definition (3) leads to a formula for the Schwartz kernel of $\pi_*(A)$. Namely, writing A for the Schwartz kernel of A on $\mathcal{P}^2$,

$$\pi_*A(x,x')=\int_{\pi^{-1}(x)\times\pi^{-1}(x')}A(p,p'). \tag{5}$$

Since the projection map is a fibration, to make sense of this formal integral we only need to use the fact that the bundle, of which the integrand is a section, is naturally identified with the pull-back of a bundle over the base tensored with the density bundle over the domain. The composition formula (4) then reduces to Fubini's theorem, using the invariance of the kernels under the diagonal $\mathrm{PU}(N)$ action.

In the projective case we instead start from the formula (5). As shown in [**5**], the vector bundle $\hom(\mathbb{E})$ over X lifted to the diagonal in X^2 extends to a small neighborhood Ω of the diagonal as a vector bundle $\mathrm{Hom}(\mathbb{E})$ with composition property. In terms of the vector bundle $\mathrm{Hom}(\widehat{\mathbb{E}})$ over $\mathcal{P}^2$ this can be seen from the fact that each point in $(x,x')\in\Omega$ is covered by a set of the form

$$\{(gp,g'p');g,\ g'\in\mathrm{SU}(N),\ g'g^{-1}\in B\} \tag{6}$$

for $B\subset\mathrm{SU}(N)$ some small neighborhood of the identity. Namely, if p is any lift of x then there is a lift $p'\in\mathcal{P}$ of x' which is close to p and all such lifts are of the form (6). The diagonal action on

$\mathrm{Hom}(\widehat{\mathbb{E}})$ descends to a $\mathrm{PU}(N)$ action and it follows that $\mathrm{Hom}(\widehat{\mathbb{E}})$ may be naturally identified over the set (6) with the fiber of $\mathrm{Hom}(\mathbb{E})$. Hence $\mathrm{Hom}(\widehat{\mathbb{E}})$ may be identified over a neighborhood of the diagonal in $\mathcal{P}^2$ with the pull-back of $\mathrm{Hom}(\mathbb{E})$ and this identification is consistent with the composition property.

Thus over the fiber of the push-forward integral (5) the integrand is identified with a distributional section of the bundle lifted from the base. The properties of the push-forward, that it maps the kernels of pseudodifferential operators to pseudodifferential operators and respects products, then follow from localization, since this reduces the problem to the usual case discussed initially. q.e.d.

Lemma 2. *If* $\chi \in \mathcal{C}^\infty(\mathrm{SU}(N))$ *is equal to* 1 *in a neighborhood of* $e \in \mathrm{SU}(N)$ *and* Ω *is a sufficiently small neighborhood of the diagonal in* $\mathcal{P}^2$, *depending on* χ, *then under the push-forward map of Proposition* 4

$$\mathrm{Tr}(\pi_*(A)) = \mathrm{Tr}(T_\chi A),\ A \in \Psi^{-\infty}_{\mathrm{SU}(N)}(\mathcal{P};\widehat{\mathbb{E}}),\ \mathrm{supp}(A) \subset \Omega. \tag{7}$$

Proof. In a local trivialization of $\mathcal{P}$ the kernel of $T_\chi A$ is of the form

$$\int_{\mathrm{SU}(N)} \chi(h^{-1}g)A(x,h,x',g')dh, \tag{8}$$

so the trace is

$$\int_{\mathrm{SU}(N)\times\mathrm{SU}(N)} \chi(h^{-1}g)A(x,h,x,g)dhdgdx = \int_{\mathrm{SU}(N)} \chi(h)A(x,h,x,e)dh$$

using the invariance of A. Since the support of A is close to the diagonal and $\chi = 1$ close to the identity, $\chi = 1$ on the support, this reduces to

$$\int_{\mathrm{SU}(N)} A(x,h,x,e)dh = \mathrm{Tr}(\pi_*(A))$$

again using the $\mathrm{SU}(N)$-invariance of A. q.e.d.

Now, suppose $A \in \Psi^m_{\mathrm{SU}(N)}(\mathcal{P},\mathbb{E})$ is transversally elliptic. Then the $\mathrm{SU}(N)$-equivariant index is the distribution

$$\mathrm{ind}_{\mathrm{SU}(N)}(A)(\chi) = \mathrm{Tr}\left(T_\chi(AB - \mathrm{Id}_-)\right) - \mathrm{Tr}(T_\chi(BA - \mathrm{Id}_+)), \tag{9}$$

where $B \in \Psi^{-m}_{\mathrm{SU}(N)}(\mathcal{P},\mathbb{E})$ is a parametrix for A and $\chi \in \mathcal{C}^\infty(\mathrm{SU}(N))$. We may choose A and B to have (kernels with) supports arbitrarily close to the diagonal but maintaining the $\mathrm{SU}(N)$-invariance.

Proposition 5. *If* $\phi \in \mathcal{C}^\infty(\mathrm{SU}(N))$ *has support sufficiently close to* $e \in \mathrm{SU}(N)$ *and is equal to* 1 *in a neighborhood of* e, *then*

$$\mathrm{ind}_{\mathrm{SU}(N)}(A)(\phi) = \mathrm{ind}_a(\pi_*(A)) \tag{10}$$

for any transversally elliptic $A \in \Psi^m_{\mathrm{SU}(N)}(\mathcal{P};\widehat{\mathbb{E}})$ *with support sufficiently close to the diagonal.*

Proof. Using Lemma 2,

$$\operatorname{Tr}\left(T_\phi(AB - \mathrm{Id}_-)\right) = \operatorname{Tr}(\pi_*(A)\pi_*(B) - \pi_*(\mathrm{Id}_+)), \tag{11}$$

and similarly for the second term. Since $\pi_*(\mathrm{Id}) = \mathrm{Id}$, $\pi_*(A)\pi_*(B) - \mathrm{Id}$ is a smoothing operator. In particular $\pi_*(B)$ is a parametrix for $\pi_*(A)$, and (10) follows. q.e.d.

Remark 1. Every compact, oriented, Riemannian manifold X of dimension $2n$, has projective vector bundles of half spinors, which are realized as $\mathrm{SU}(N)$-equivariant vector bundles $\widehat{\mathbb{S}} = (\widehat{S}^+, \widehat{S}^-)$, $N = 2^n$, over the principal $\mathrm{PU}(N)$-bundle $\mathcal{P}$ over X that is associated to the oriented orthonormal frame bundle of X, cf. §3 in [**5**]. Explicitly, $\widehat{\mathbb{S}}$ is the $\mathbb{Z}_2$-graded $\mathrm{SU}(N)$-equivariant vector bundle of spinors associated to the conormal bundle to the fibers, $T^*_{\mathrm{SU}(N)}\mathcal{P}$. On $\mathcal{P}$ there is a transversally elliptic, $\mathrm{SU}(N)$-equivariant Dirac operator $\eth^+$, defined as follows. The Levi-Civita connection on X determines in an obvious way partial spin connections $\nabla^\pm$ on $\widehat{S}^\pm$. That is, $\nabla^+ : \mathcal{C}^\infty(\mathcal{P}, \widehat{S}^+) \to \mathcal{C}^\infty(\mathcal{P}, T^*_{\mathrm{SU}(N)}\mathcal{P} \otimes \widehat{S}^+)$. If $C : \mathcal{C}^\infty(\mathcal{P}, T^*_{\mathrm{SU}(N)}\mathcal{P} \otimes \widehat{S}^+) \to \mathcal{C}^\infty(\mathcal{P}, \widehat{S}^-)$ denotes contraction given by Clifford multiplication, then $\eth^+ : \mathcal{C}^\infty(\mathcal{P}, \widehat{S}^+) \to \mathcal{C}^\infty(\mathcal{P}, \widehat{S}^-)$ is defined as the composition, $C \circ \nabla^+$.

Then $\pi_*(\eth^+)$ is just the projective Dirac operator of [**5**], and Proposition 5 relates the indices in these two senses.

Remark 2. Once the pushforward map $\pi_* \colon \Psi^\bullet_{\widehat{G}}(\mathcal{P}, \widehat{\mathbb{E}}) \longrightarrow \Psi^\bullet_\epsilon(X, \mathbb{E})$ is defined, Proposition 5 can also be deduced from the index theorem of [**5**] and the explicit topological expression for the equivariant transversal index as in [**2**, **3**] simply by comparing the formulæ.

References

[1] M.F. Atiyah, *Elliptic operators and compact groups*, Lecture Notes in Mathematics, **401**, Springer-Verlag, Berlin-New York, 1974, MR 0482866, Zbl 0297.58009.

[2] N. Berline & M. Vergne, *The Chern character of a transversally elliptic symbol and the equivariant index*, Invent. Math. **124**(**1**–**3**) (1996) 11–49, MR 1369410, Zbl 0847.46037.

[3] ———, *L'indice équivariant des opérateurs transversalement elliptiques*, Invent. Math. **124**(**1**–**3**) (1996) 51–101, MR 1369411, Zbl 0883.58037.

[4] V. Mathai, R.B. Melrose, & I.M. Singer, *The index of projective families of elliptic operators*, Geom. Topol. **9** (2005) 341–373, math.DG/0206002, MR 2140985, Zbl 1083.58021.

[5] ———, *Fractional Analytic Index*, J. Differential Geom. **74**(**2**) (2006) 265–292, math.DG/0402329, MR 2258800.

[6] I.M. Singer, *Recent applications of index theory for elliptic operators*, Partial differential equations (Proc. Sympos. Pure Math., Vol. XXIII, Univ. California, Berkeley, Calif., 1971), 11–31, Amer. Math. Soc., Providence, RI, 1973, MR 0341538, Zbl 0293.58013.

Department of Mathematics
University of Adelaide
Adelaide 5005, Australia
E-mail address: mathai.varghese@adelaide.edu.au

Department of Mathematics
Massachusetts Institute of Technology
Cambridge, MA 02139
E-mail address: rbm@math.mit.edu

Department of Mathematics
Massachusetts Institute of Technology
Cambridge, MA 02139
E-mail address: ims@math.mit.edu