CHAPTER 4

Isotropic calculus

The algebra of 'isotropic' pseudodifferential operators on $\mathbb{R}^n$ has global properties very similar to the algebra of pseudodifferential operators on a compact manifold discussed below. There are several reasons for the extensive discussion here. First it is pretty! Second it is useful in the sense that it embeds the harmonic oscillator in a broader context. Thirdly, many of the global constructions here carry over almost unchanged to the case of compact manifolds and it may help to see them in a somewhat simpler setting. Finally, it is useful in a geometric and topological sense as may become clearer below in the discussion of K-theory.

4.1. Isotropic operators

As noted in the discussion in Chapter 2, there are other sensible choices of the class of amplitudes which can be admitted in the definition of a space of pseudodifferential operators rather than the basic case of $S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ discussed there. One of the smallest such choices is the class which is completely symmetric in the variables x and ξ and consists of the symbols on $\mathbb{R}^{2n}$. Thus, $a \in S^m(\mathbb{R}^{2n}_{x,\xi})$ satisfies the estimates

$$|D^\alpha_x D^\beta_\xi a(x,\xi)| \le C_{\alpha,\beta}(1+|x|+|\xi|)^{m-|\alpha|-|\beta|} \tag{4.1}$$

for all multiindices α and β. Recall that there is a subspace of 'classical' or polyhomogeneous symbols

$$S^m_{\mathrm{ph}}(\mathbb{R}^{2n}) \subset S^m(\mathbb{R}^{2n}) \tag{4.2}$$

defined by the condition that its elements are asymptotic sums of terms $a_j \in S^m(\mathbb{R}^{2n})$ with a_j positively homogeneous of degree $m-j$ in $|(x,\xi)| \ge 1$.

If $m \le 0$, it follows that $a \in S^m_\infty(\mathbb{R}^n_x;\mathbb{R}^n_\xi)$; if $m > 0$ this is not true, however,

Lemma 4.1. *For any p and n*

$$S^m(\mathbb{R}^{p+n}) \subset \begin{cases} \bigcap\limits_{0\ge r\ge m} (1+|x|^2)^{r/2} S^{m-r}_\infty(\mathbb{R}^p_x;\mathbb{R}^n_\xi) & m \le 0 \\ (1+|x|^2)^{m/2} S^m_\infty(\mathbb{R}^p_x;\mathbb{R}^n_\xi), & m > 0. \end{cases} \tag{4.3}$$

Proof. This follows from (4.1) and the inequalities

$$1+|x|+|\xi| \le (1+|x|)(1+|\xi|),$$
$$1+|x|+|\xi| \ge (1+|x|)^t(1+|\xi|)^{1-t},\ 0 \le t \le 1.$$

□

In view of these estimates the following definition makes sense.

DEFINITION 4.1. *For any $m \in \mathbb{R}$ we define*

$$\Psi^m_{\text{iso}}(\mathbb{R}^n) \subset \Psi^m_{\infty-\text{iso}}(\mathbb{R}^n) \subset \langle x\rangle^{m_+}\Psi^m_\infty(\mathbb{R}^n) \tag{4.4}$$

as the subspaces determined by

$$\begin{aligned} A \in \Psi^m_{\text{iso}}(\mathbb{R}^n) &\Longleftrightarrow \sigma_L(A) \in S^m_{\text{ph}}(\mathbb{R}^{2n}) \\ A \in \Psi^m_{\infty-\text{iso}}(\mathbb{R}^n) &\Longleftrightarrow \sigma_L(A) \in S^m(\mathbb{R}^{2n}). \end{aligned} \tag{4.5}$$

Note however that the notation has been switched here. The space with the absence of any subscript corresponds to classical symbols, whereas the '$\infty -$ iso' subscript refers to the symbols with 'bounds' as in (4.1).

As in the discussion in Chapter 2 the 'residual' algebra consists just of the intersection

$$\Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n) = \Psi^{-\infty}_{\infty-\text{iso}}(\mathbb{R}^n) = \bigcap_m \Psi^m_{\infty-\text{iso}}(\mathbb{R}^n). \tag{4.6}$$

From the discussion above, an element of either space on the left has left-reduced symbol in $S^{-\infty}(\mathbb{R}^{1n}) = \mathcal{S}(\mathbb{R}^{2n})$ so its kernel is also in $\mathcal{S}(\mathbb{R}^{2n})$ and conversely. This justifies the apparently different sense in which this notation is used in Chapter 3.

As in the discussion of the traditional algebra in Chapter 2 we show the $*$-invariance and composition properties of these spaces of operators by proving an appropriate 'reduction' theorem. However there is a small difficulty here. Namely it might be supposed that it is enough to analyse $I(a)$ for $a \in S^m(\mathbb{R}^{3n})$. This however is not the case. Indeed the definition above is in terms of left-reduced symbols. If $a \in S^m(\mathbb{R}^{2n})$ is regarded as a function on $\mathbb{R}^{3n}$ which is independent of one of the variables then it is in general *not* an element of $S^m(\mathbb{R}^{3n})$ (it is an element of $S^m_\infty(\mathbb{R}^n_y;\mathbb{R}^{2n})$ since it is constant in the first variables). For this reason we need to consider some more 'hybrid' estimates.

Consider a subdivision of $\mathbb{R}^{3n}$ into two closed regions:

$$\begin{aligned} R_1(\epsilon) &= \{(x,y,\xi) \in \mathbb{R}^{3n}; |x-y| \le \epsilon(1+|x|^2+|y|^2+|\xi|^2)^{\frac12}\} \\ R_2(\epsilon) &= \{(x,y,\xi) \in \mathbb{R}^{3n}; |x-y| \ge \epsilon(1+|x|^2+|y|^2+|\xi|^2)^{\frac12}\}. \end{aligned} \tag{4.7}$$

If $a \in \mathcal{C}^\infty(\mathbb{R}^{3n})$ consider the estimates

$$|D^\alpha_x D^\beta_y D^\gamma_\xi a(x,y,\xi)| \le C_{\alpha,\beta,\gamma} \begin{cases} \langle (x,y,\xi)\rangle^{m-|\alpha|-|\beta|-|\gamma|} & \text{in } R_1(\frac18) \\ \langle (x,y)\rangle^{m_+}\langle\xi\rangle^{m-|\gamma|} & \text{in } R_2(\frac18). \end{cases} \tag{4.8}$$

The choice $\epsilon = \frac18$ here is rather arbitrary. However if ϵ is decreased, but kept positive the same estimates continue to hold for the new subdivision, since the estimates in R_1 are stronger than those in R_2 (which is increasing at the expense of R_1 as ϵ decreases). Notice too that these estimates do in fact imply that $a \in \langle x\rangle^{m_+}\langle y\rangle^{m_+}S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ and hence they do define operators in the weighted spaces – in principle $\langle x\rangle^{2m_+}\Psi^m_\infty(\mathbb{R}^n)$ although actually $\langle x\rangle^{m_+}\Psi^m_\infty(\mathbb{R}^n)$ – that were analysed in Chapter 2.

PROPOSITION 4.1. *If $a \in \mathcal{C}^\infty(\mathbb{R}^{3n})$ satisfies the estimates* (4.8) *then $A = I(a) \in \Psi^m_{\infty-\text{iso}}(\mathbb{R}^n)$ and* (2.58) *holds for $\sigma_L(A)$.*

PROOF. We separate a into two pieces. Choose $\chi \in \mathcal{C}^\infty_c(\mathbb{R})$ with $0 \le \chi \le 1$, with support in $[-\frac18,\frac18]$ and with $\chi \equiv 1$ on $[-\frac19,\frac19]$. Then consider the cutoff function on $\mathbb{R}^{3n}$

$$\psi(x,y,\xi) = \chi\left(\frac{|x-y|}{\langle(x,y,\xi)\rangle}\right). \tag{4.9}$$

Clearly, ψ has support in $R_1(\frac{1}{8})$ and $\psi \in S^0_\infty(\mathbb{R}^{3n})$. It follows then that $a' = \psi a \in S^m_{\text{iso}}(\mathbb{R}^{3n})$. On the other hand, $a'' = (1-\psi)a$ has support in $R_2(\frac{1}{9})$. In this region $|x-y|$, $\langle(x,y)\rangle$ and $\langle(x,y,\xi)\rangle$ are bounded by constant multiples of each other. Thus a'' satisfies the estimates

$$\begin{aligned} |D^\alpha_x D^\beta_y D^\gamma_\xi a''(x,y,\xi)| &\le C_{\alpha,\beta,\gamma}|x-y|^{m_+}\langle\xi\rangle^{m-|\gamma|} \\ &\le C'_{\alpha,\beta,\gamma}\langle(x,y,\xi)\rangle^{m_+}\langle\xi\rangle^{m-|\gamma|},\ \operatorname{supp}(a'') \subset R_2(\frac{1}{9}). \end{aligned} \tag{4.10}$$

First we check that $I(a'') \in \mathcal{S}(\mathbb{R}^{2n})$. On $R_2(\frac{1}{9})$ it is certainly the case that $|x-y| \ge \frac{1}{9}\langle(x,y)\rangle$ and by integration by parts

$$|x-y|^{2p}D^\alpha_x D^\beta_y I(a'') = I(|D_\xi|^{2p}D^\alpha_x D^\beta_y a'').$$

For all sufficiently large p it follows from (4.10) that this is the product of $\langle(x,y)\rangle^{m_+}$ and a bounded continuous function. Thus, $I(a'') \in \mathcal{S}(\mathbb{R}^{2n})$ is the kernel of an operator in $\Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$.

So it remains only to show that $A' = I(a') \in \Psi^m_{\infty-\text{iso}}(\mathbb{R}^n)$. Certainly this is an element of $\langle x\rangle^{m_+}\Psi^m_\infty(\mathbb{R}^n)$. The left-reduced symbol of A' has an asymptotic expansion, as $\xi \to \infty$, given by the usual formula, namely (2.58). Each of the terms in this expansion

$$a_L(A') \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} D^\alpha_y D^\alpha_\xi a(x,x,\xi)$$

is in the space $S^{m-2|\alpha|}(\mathbb{R}^{2n})$. Thus we can actually choose an asymptotic sum in the stronger sense that

$$b' \in S^m(\mathbb{R}^{2n}),\ b_N = b' - \sum_{|\alpha|<N} \frac{i^{|\alpha|}}{\alpha!} D^\alpha_x D^\alpha_\xi a(x,\xi) \in S^{m-2N}(\mathbb{R}^{2n})\ \forall\ N.$$

Consider the remainder term in (2.47), given by (2.44) and (2.45). Integrating by parts in ξ to remove the factors of $(x-y)^\alpha$ the remainder, R_N, can be written as a pseudodifferential operator with amplitude

$$r_N(x,y,\xi) = \sum_{|\alpha|=N} \frac{i^{|\alpha|}}{\alpha!}\int_0^1 dt(1-t)^N(D^\alpha_\xi D^\alpha_y a)((1-t)x+ty,\xi).$$

This satisfies the estimates (4.8) with m replaced by $m-2N$. Indeed from the symbol estimates on a' the integrand satisfies the bounds

$$\begin{aligned} &|D^\beta_x D^\gamma_y D^\delta_\xi D^\alpha_\xi D^\alpha_y a'((1-t)x+ty,\xi)| \\ &\qquad\le C(1+|(x+t(x-y)|+|\xi|)^{m-2N-|\beta|-|\gamma|-|\delta|}. \end{aligned}$$

In $R_1(\frac{1}{8})$, $|x-y| \le \frac{1}{8}\langle(x,y,\xi)\rangle$ so $|x+t(x-y)|+|\xi| \ge \frac{1}{2}\langle(x,y,\xi)\rangle$ and these estimates imply the full symbol estimates there. On R_2 we immediately get the weaker estimates in (4.7).

Thus, for large N, the remainder term gives an operator in $\langle x\rangle^{\frac{m}{2}-N}\Psi^{\frac{m}{2}-N}_\infty(\mathbb{R}^n)$. The difference between A' and the operator $B' \in \Psi^m_{\infty-\text{iso}}(\mathbb{R}^n)$, which is R_N plus an operator in $\Psi^{m-2N}_{\infty-\text{iso}}(\mathbb{R}^n)$ for any N is therefore in $\Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$. Thus $A \in \Psi^m_{\text{iso}}(\mathbb{R}^n)$. □

This is a perfectly adequate replacement in this context for our previous reduction theorem, so now we can show the basic result.

THEOREM 4.1. *The spaces* $\Psi^m_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ *(resp.* $\Psi^m_{\mathrm{iso}}(\mathbb{R}^n)$*) of isotropic (resp. polyhomogeneous isotropic) pseudodifferential operators on* $\mathbb{R}^n$*, defined by* (4.5) *form an order-filtered* $*$*-algebra with residual space* $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) = \mathcal{S}(\mathbb{R}^{2n})$ *(resp. the same) as spaces of kernels.*

PROOF. The condition that a continuous linear operator A on $\mathcal{S}(\mathbb{R}^n)$ be an element of $\Psi^m_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ is that it be an element of $(1+|x|^2)^{m/2}\Psi^m_\infty(\mathbb{R}^n)$ if $m \ge 0$ or $\Psi^m_\infty(\mathbb{R}^n)$ if $m < 0$ with left-reduced symbol an element of $S^m(\mathbb{R}^{2n}_{x,\xi})$:

$$q_l : S^m_\infty(\mathbb{R}^{2n}) \longleftrightarrow \Psi^m_{\infty-\mathrm{iso}}(\mathbb{R}^n). \tag{4.11}$$

Thus A^* has right-reduced symbol in $S^m_\infty(\mathbb{R}^{2n})$. This satisfies the estimates (4.8) as a function of x, y and ξ. Thus Proposition 4.1 shows that $A^* \in \Psi^m_{\infty-\mathrm{iso}}(\mathbb{R}^n)$, since its left-reduced symbol is in $S^m(\mathbb{R}^{2n})$, proving the $*$-invariance. Moreover it also follows that any $B \in \Psi^{m'}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ has right-reduced symbol in $S^{m'}(\mathbb{R}^{2n})$. Thus if $A \in \Psi^m_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ and $B \in \Psi^{m'}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ then using this result to right-reduce B we see that the composite operator has kernel $I(a_L(x,\xi)b_R(y,\xi))$ where $a_L \in S^m_\infty(\mathbb{R}^{2n})$ and $b_R \in S^{m'}_\infty(\mathbb{R}^{2n})$. Now it again follows that this product satisfies the estimates (4.8) of order $m + m'$. Hence, again applying Proposition 4.1, we conclude that $A \circ B \in \Psi^{m+m'}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$. This proves the theorem for $\Psi^*_{\infty-\mathrm{iso}}(\mathbb{R}^n)$.

The proof for the polyhomogeneous space $\Psi^m_{\mathrm{iso}}(\mathbb{R}^n)$ follows immediately, since the symbol expansions all preserve polyhomogeneity. □

One further property of the isotropic calculus that distinguishes it strongly from the traditional calculus is that it is invariant under Fourier transformation.

PROPOSITION 4.2. *If* $A \in \Psi^m_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ *(resp.* $\Psi^m_{\mathrm{iso}}(\mathbb{R}^n)$*) then* $\hat{A} \in \Psi^m_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ *(resp.* $\Psi^m_{\mathrm{iso}}(\mathbb{R}^n)$*) where* $\widehat{\hat{A}u} = A\hat{u}$ *with* $\hat{u}$ *being the Fourier transform of* $u \in \mathcal{S}(\mathbb{R}^n)$.

The proof of this is outlined in Problem 2.20.

Also note that asymptotic completeness then carries over from the symbol spaces. If $B_j \in \Psi^{m-j}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ then there exists

$$B \in \Psi^m_{\infty-\mathrm{iso}}(\mathbb{R}^n),\ B \sim \sum_j B_j \text{ that is } B - \sum_{j=0}^{N-1} B_j \in \Psi^{m-N}_{\infty-\mathrm{iso}}(\mathbb{R}^n)\ \forall\ N. \tag{4.12}$$

4.2. Fredholm property

An element $A \in \Psi^m_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ is said to be elliptic (of order m in the isotropic calculus) if its left-reduced symbol is elliptic in $S^m(\mathbb{R}^{2n})$.

THEOREM 4.2. *Each elliptic element* $A \in \Psi^m_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ *has a two-sided parametrix* $B \in \Psi^{-m}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ *in the sense that*

$$A \circ B - \mathrm{Id},\ B \circ A - \mathrm{Id} \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) \tag{4.13}$$

which is unique up to an element of $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ *and it follows that any* $u \in \mathcal{S}'(\mathbb{R}^n)$ *satisfying* $Au \in \mathcal{S}(\mathbb{R}^n)$ *is an element of* $\mathcal{S}(\mathbb{R}^n)$; *if* $A \in \Psi^m_{\mathrm{iso}}(\mathbb{R}^n)$ *is elliptic then its parametrix is in* $\Psi^{-m}_{\mathrm{iso}}(\mathbb{R}^n)$.

PROOF. This is just the inductive argument used to prove Lemma 2.7. Nevertheless we repeat it here.

The ellipticity of $\sigma_m(A)$ means that it has a two-sided inverse $b \in S^{-m}(\mathbb{R}^{2n})$ modulo $S^{-\infty}(\mathbb{R}^{2n}) = \mathcal{S}(\mathbb{R}^{2n})$. This in turn means that the equation $\sigma_k(A)c = d$ always has a solution $c \in S^{-m+m'-[1]}(\mathbb{R}^{2n})$ for given $d \in S^{m'-[1]}(\mathbb{R}^{2n})$ namely $c = bd$. This in turn means that given $C_j \in \Psi^{j}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ there always exists $B_j \in \Psi^{j-m}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ such that $AB_j - C_j \in \Psi^{j-1}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$. Choosing $B_0 \in \Psi^{-m}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ to have $\sigma_{-m}(B_0) = b$ we can define $C_1 = \mathrm{Id} - AB_0 \in \Psi^{-1}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$. Then, proceeding inductively we may assume that B_j for $j < l$ have been chosen such that $A(B_0 + \dots + B_{l-1}) - \mathrm{Id} = -C_l \in \Psi^{-l}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$. Then using the solvability we may choose B_l so that $AB_l - C_l = -C_{l+1} \in \Psi^{-l-1}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ which completes the induction, since $A(B_0 + \dots + B_l) - \mathrm{Id} = AB_l - C_l = -C_{l+1}$. Finally by the asymptotic completeness we may choose $B \sim B_0 + B_1 + \dots$ which is a right parametrix.

The existence of a left parametrix follows from the ellipticity of A^* and the argument showing that a right parametrix is a two-sided parametrix is essentially the same as in Lemma 2.7. □

Combining the earlier symbolic discussion and these analytic results we can see that elliptic operators are Fredholm as an operator

$$A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n) \text{ or } A : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n). \tag{4.14}$$

PROPOSITION 4.3. *If $A \in \Psi^{m}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ is elliptic then it has a generalized inverse $B \in \Psi^{-m}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ satisfying*

$$AB - \mathrm{Id} = \Pi_1,\ BA - \mathrm{Id} = \Pi_0 \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) \tag{4.15}$$

where Π_1 and Π_0 are the finite rank orthogonal (in $L^2(\mathbb{R}^n)$) projections onto the null spaces of A^* and A.

PROOF. As discussed above, A has a parametrix $B' \in \Psi^{-m}_{\mathrm{iso}}(\mathbb{R}^n)$ modulo $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$. Thus

$$AB' = \mathrm{Id} - E_R,\ E_R \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n),$$
$$B'A = \mathrm{Id} - E_L,\ E_L \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n).$$

Using Proposition 3.2 it follows that the null space of A is contained in the null space of $B'A = \mathrm{Id} - E_L$, hence is finite dimensional. Similarly, the range of A contains the range of $AB' = \mathrm{Id} - E_R$ so is closed with a finite codimensional complement. Defining B as the linear map which vanishes on $\mathrm{Nul}(A^*)$, and inverts A on $\mathrm{Ran}(A)$ with values in $\mathrm{Ran}(A^*) = \mathrm{Nul}(A)^{\perp}$ gives (4.15). Furthermore these identities show that $B \in \Psi^{-m}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ since applying B' gives

$$\begin{aligned} B - E_L B = B'AB = B' - B'\Pi_1,\ B - BE_R = BAB' = B' - \Pi_0 B' \Longrightarrow \\ B = B' - B'\Pi_1 + E_L B' + E_L B E_R - E_L \Pi_0 B' \in \Psi^{-m}_{\infty-\mathrm{iso}}(\mathbb{R}^n) \end{aligned} \tag{4.16}$$

where we use the fact that $EBE' \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ for any continuous linear operator B on $\mathcal{S}(\mathbb{R}^n)$ and elements $E \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$. □

COROLLARY 4.1. *If $A \in \Psi^{m}_{\mathrm{iso}}(\mathbb{R}^n)$ is elliptic then its generalized inverse lies in $\Psi^{-m}_{\mathrm{iso}}(\mathbb{R}^n)$.*

4.3. The harmonic oscillator

The harmonic oscillator is the differential operator on $\mathbb{R}^n$

$$H = \sum_{j=1}^{n}(D_j^2 + x_j^2) = \Delta + |x|^2 .$$

This is an elliptic element of $\Psi^2_{\text{iso}}(\mathbb{R}^n)$. The main immediate interest is in the spectral decomposition of H. The ellipticity of $H-\lambda$, $\lambda \in \mathbb{C}$, shows that

$$(H-\lambda)u = 0,\ u \in \mathcal{S}'(\mathbb{R}^n) \Longrightarrow u \in \mathcal{S}(\mathbb{R}^n). \tag{4.17}$$

Since H is (formally) self-adjoint, i.e., $H^* = H$, there are no non-trivial tempered solutions of $(H-\lambda)u = 0$, $\lambda \notin \mathbb{R}$. Indeed if $(H-\lambda)u = 0$,

$$0 = \langle Hu, u\rangle - \langle u, Hu\rangle = (\lambda - \overline{\lambda})\langle u, u\rangle \Longrightarrow u = 0. \tag{4.18}$$

As we shall see below in more generality, the spectrum of H is a discrete subset of $\mathbb{R}$. In this case we can compute it explicitly.

The direct computation of eigenvalues and eigenfunctions is based on the properties of the creation and annihilation operators

$$C_j = D_j + ix_j,\ C_j^* = A_j = D_j - ix_j,\ j = 1, \dots, n. \tag{4.19}$$

These satisfy the elementary identities

$$[C_j, C_k] = [A_j, A_k] = 0,\ [A_j, C_k] = 2\delta_{jk},\ j, k = 1, \dots, n \tag{4.20}$$

$$H = \sum_{j=1}^{n} C_j A_j + n,\ [C_j, H] = -2C_j,\ [A_j, H] = 2A_j . \tag{4.21}$$

Now, if λ is an eigenvalue, $Hu = \lambda u$, then

$$\begin{aligned} H(C_j u) &= C_j(Hu + 2u) = (\lambda + 2)C_j u, \\ H(A_j u) &= A_j(Hu - 2u) = (\lambda - 2)A_j u. \end{aligned} \tag{4.22}$$

Proposition 4.4. *The eigenvalues of H are*

$$\sigma(H) = \{n, n+2, n+4, \dots\}. \tag{4.23}$$

Proof. We already know that eigenvalues must be real and from the decomposition of H in (4.21) it follows that, for $u \in \mathcal{S}(\mathbb{R}^n)$,

$$\langle Hu, u\rangle = \sum_j \|A_j u\|^2 + n\|u\|^2. \tag{4.24}$$

Thus if $\lambda \in \sigma(H)$ is an eigenvalue then $\lambda \geq n$.

By direct computation we see that n is an eigenvalue with a 1-dimensional eigenspace. Indeed, from (4.24), $Hu = nu$ iff $A_j u = 0$ for $j = 1, \dots, n$. In each variable separately

$$A_j u(x_j) = 0 \Leftrightarrow u(x_j) = c_1 \exp\left(-\frac{x_j^2}{2}\right).$$

Thus the only tempered solutions of $A_j u = 0$, $i = 1, \dots, n$ are the constant multiples of

$$u_0 = \exp\left(-\frac{|x|^2}{2}\right), \tag{4.25}$$

which is often called the *ground state.*

Now, if λ is an eigenvalue with eigenfunction $u \in \mathcal{S}(\mathbb{R}^n)$ it follows from (4.22) that $\lambda - 2$ is an eigenvalue with eigenfunction $A_j u$. Since all the $A_j u$ cannot vanish unless u is the ground state, it follows that the eigenvalues are contained in the set in (4.23). We can use the same argument to show that if u is an eigenfunction with eigenvalue λ then $C_j u$ is an eigenfunction with eigenvalue $\lambda + 2$. Moreover, $C_j u \equiv 0$ would imply $u \equiv 0$ since $C_j v = 0$ has no non-trivial tempered solutions, the solution in each variable being $\exp(x_j^2/2)$. □

Using the creation operators we can parameterize the eigenspaces quite explicitly.

PROPOSITION 4.5. *For each $k \in \mathbb{N}_0$ there is an isomorphism*

$$\text{(4.26)} \quad \{\textit{Polynomials, homogeneous of degree } k \textit{ on } \mathbb{R}^n\} \ni p \longmapsto p(C) \exp\left(-\frac{|x|^2}{2}\right) \in E_k$$

where E_k is the eigenspace of H with eigenvalue $n + 2k$.

PROOF. Notice that the C_j, $j = 1, \dots, n$ are commuting operators, so $p(C)$ is well-defined. By iteration from (4.22),

$$HC^\alpha u_0 = C^\alpha (H + 2|\alpha|) u_0 = (n + 2|\alpha|) C^\alpha u_0. \tag{4.27}$$

Thus (4.26) is a linear map into the eigenspace as indicated.

To see that (4.26) is an isomorphism consider the action of the annihilation operators. Again from (4.22)

$$|\beta| = |\alpha| \Longrightarrow A^\beta C^\alpha u_0 = \begin{cases} 0 & \beta \neq \alpha \\ 2^{|\alpha|} \alpha! u_0 & \beta = \alpha. \end{cases} \tag{4.28}$$

This allows us to recover the coefficients of p from $p(C)u_0$, so (4.26) is injective. Conversely if $v \in E_k \subset \mathcal{S}(\mathbb{R}^n)$ is orthogonal to all the $C^\alpha u_0$ then

$$\langle A^\alpha v, u_0 \rangle = \langle v, C^\alpha u_0 \rangle = 0 \ \forall \ |\alpha| = k. \tag{4.29}$$

From (4.22), the $A^\alpha v$ are all eigenfunctions of H with eigenvalue n, so (4.29) implies that $A^\alpha v = 0$ for all $|\alpha| = k$. Proceeding inductively in k we see that $A^{\alpha'} A_j v = 0$ for all $|\alpha'| = k - 1$ and $A_j v \in E_{k-1}$ implies $A_j v = 0$, $j = 1, \dots, n$. Since $v \in E_k$, $k > 0$, this implies $v = 0$ so Proposition 4.5 is proved. □

Thus H has eigenspaces as described in (4.26). The same argument shows that for any integer p, positive or negative, the eigenvalues of H^p are precisely $(n + 2k)^p$ with the same eigenspaces E_k. For $p < 0$, H^p is a compact operator on $L^2(\mathbb{R}^n)$; this is obvious for large negative p. For example, if $p \leq -n - 1$ then

$$x_i^\beta D_j^\alpha H \in \Psi^0_{\text{iso}}(\mathbb{R}^n), \ |\alpha| \leq n + 1, |\beta| \leq n + 1 \tag{4.30}$$

are all bounded on L^2. If $S \subset L^2(\mathbb{R}^n)$ is bounded this implies that $H^{-n-1}(S)$ is bounded in $\langle x \rangle^{n+1} C^1_\infty(\mathbb{R}^n)$, so compact in $\langle x \rangle^n C^0_\infty(\mathbb{R}^n)$ and hence in $L^2(\mathbb{R}^n)$. It is a general fact that for compact self-adjoint operators, such as H^{-n-2}, the eigenfunctions span $L^2(\mathbb{R}^n)$. We give a brief proof of this for the sake of 'completeness'.

LEMMA 4.2. *The eigenfunction of H, $u_\alpha = \pi^{-\frac{n}{4}} (2^{|\alpha|} \alpha!)^{-1/2} C^\alpha u_0$ form an orthonormal basis of $L^2(\mathbb{R}^n)$.*

Proof. Let $V \subset L^2(\mathbb{R}^n)$ be the closed subspace consisting of the orthocomplements of all the u_α's. Certainly H^{-n-2} acts on it as a compact self-adjoint operator. Since we have found all the eigenvalues of H, and hence of H^{-n-1}, it has no eigenvalue in V. We wish to conclude that $V = \{0\}$. Set

$$\tau = \|H^{-n-1}\|_V = \sup\{\|H^{-n-1}\varphi\|; \varphi \in V, \|\varphi\| = 1\}.$$

Then there is a weakly convergent sequence $\varphi_j \rightharpoonup \varphi$, $\|\varphi_j\| = 1$, so $\|\varphi\| \le 1$, with $\|H^{-n-1}\varphi_j\| \to \tau$. The compactness of H^{-n-2} allows a subsequence to be chosen such that $H^{-n-1}\varphi_j \to \psi$ in $L^2(\mathbb{R}^n)$. So, by the continuity of H^{-n-1}, $H^{-n-1}\varphi = \psi$ and $\|H^{-n-1}\varphi\| = \tau$, $\|\varphi\| = 1$. If $\varphi' \in V$, $\varphi' \perp \varphi$, $\|\varphi'\| = 1$ then

$$\begin{aligned} \tau^2 \ge \|H^{-n-2}\left(\frac{\varphi + t\varphi'}{\sqrt{1+t^2}}\right)\|^2 &= \tau^2 + 2t\langle H^{-2n-2}\varphi, \varphi'\rangle + 0(t^2) \\ \Longrightarrow \langle H^{-2n-2}\varphi, \varphi'\rangle &= 0 \Longrightarrow H^{-2n-2}\varphi = \tau^2\varphi\,. \end{aligned}$$

This contradicts the fact that H^{-2n-2} has no eigenvalues in V, so $V = \{0\}$ and the eigenbasis is complete. □

Thus, if $u \in L^2(\mathbb{R}^n)$

$$u = \sum_\alpha c_\alpha u_\alpha,\ c_\alpha = \langle u, u_\alpha\rangle \tag{4.31}$$

with convergence in L^2.

Lemma 4.3. *If $u \in \mathcal{S}(\mathbb{R}^n)$ the convergence in (4.31) is rapid, i.e., $|c_\alpha| \le C_N(1+|\alpha|)^{-N}$ for all N and the series converges in $\mathcal{S}(\mathbb{R}^n)$.*

Proof. Since $u \in \mathcal{S}(\mathbb{R}^n)$ implies $H^N u \in L^2(\mathbb{R}^n)$ we see that

$$C_N \ge |\langle H^N u, u_\alpha\rangle| = |\langle u, H^N u_\alpha\rangle| = (n+2|\alpha|)^N |c_\alpha|\ \forall\ \alpha.$$

Furthermore, $2ix_j = C_j - A_j$ and $2D_j = C_j + A_j$ so the polynomial derivatives of the u_α can be estimated (using the Sobolev embedding theorem) by polynomials in α; this implies that the series converges in $\mathcal{S}(\mathbb{R}^n)$. □

Corollary 4.2. *Finite rank elements are dense in $\Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ in the topology of $\mathcal{S}(\mathbb{R}^{2n})$.*

Proof. Consider the approximation (4.31) to the kernel A of an element of $\Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ as an element of $\mathcal{S}(\mathbb{R}^{2n})$. In this case the ground state is

$$U_0 = \exp\left(-\frac{|x|^2}{2} - \frac{|y|^2}{2}\right) = \exp\left(-\frac{|x|^2}{2}\right)\exp\left(-\frac{|y|^2}{2}\right)$$

and so has rank one as an operator. The higher eigenfunctions

$$C^\alpha U_0 = Q_\alpha(x,y)U_0$$

are products of U_0 and a polynomial, so are also of finite rank. □

4.4. L^2 boundedness and compactness

The results above have obvious extension to the case of $N \times N$ matrices of operators, which we denote by $\Psi^0_{\infty-\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$ since they act on $\mathbb{C}^N$ valued functions. Recall that $\Psi^0_{\infty-\mathrm{iso}}(\mathbb{R}^n) \subset \Psi^0_{\infty}(\mathbb{R}^n)$ so, by Proposition 2.6, these operators are bounded on $L^2(\mathbb{R}^n)$. Using the same argument the bound on the L^2 norm can be related to the norm of the principal symbol as an $N \times N$ matrix.

PROPOSITION 4.6. *If* $A \in \Psi^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$ *has homogeneous principal symbol*

$$a = \sigma_L(A)\big|_{\mathbb{S}^{2n-1}} \in \mathcal{C}^\infty(\mathbb{S}^{2n-1}; M(N,\mathbb{C}))$$

then

$$\inf_{E\in\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)} \|A+E\|_{\mathcal{B}(L^2(\mathbb{R}^n;\mathbb{C}^N))} = \sup_{p\in\mathbb{S}^{2n-1}} \|a(p)\|. \tag{4.32}$$

A similar result is true without the assumption that the principal symbol is homogeneous. It is simply necessary to replace the supremum on the right by

$$\lim_{R\to\infty} \sup_{|(x,\xi)|\ge R} \|\sigma_L(A)(x,\xi)\| \tag{4.33}$$

where the norm on the symbol is the Euclidean norm on $N \times N$ matrices.

PROOF. It suffices to prove (4.32) for all single operators $A \in \Psi^0_{\mathrm{iso}}(\mathbb{R}^n)$. Indeed if $j_v(z) = zv$ is the linear map from $\mathbb{C}$ to $\mathbb{C}^N$ defined by $v \in \mathbb{C}^N$ then

$$\|A\|_{\mathcal{B}(L^2(\mathbb{R};\mathbb{C}^N))} = \sup_{\{v,w\in\mathbb{C}^N;\|v\|=\|w\|=1\}} \|j^*_w A j_v\|_{\mathcal{B}(L^2(\mathbb{R}))}. \tag{4.34}$$

Since the symbol of $j^*_w A j_v$ is just $j^*_w \sigma(A) j_v$, (4.32) follows from the corresponding equality for a single operator:

$$\inf_{E\in\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)} \|A+E\|_{\mathcal{B}(L^2(\mathbb{R}^n)} \le \sup_{p\in\mathbb{S}^{2n-1}} |a(p)|,\ a = \sigma_L(A)\big|_{\mathbb{S}^{2n-1}}. \tag{4.35}$$

The construction of the approximate square-root of $C - A^*A$ in Proposition 2.7 only depends on the existence of a positive smooth square-root for $C - |a|^2$, so can be carried out for any

$$C > \sup_{p\in\mathbb{S}^{2n-1}} |a(p)|^2. \tag{4.36}$$

Thus we conclude that with such a value of C

$$\|Au\|^2 \le C\|u\|^2 + \|\langle Gu,u\rangle|\ \forall\ u \in L^2(\mathbb{R}^n),$$

where $G \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$. Since G is an isotropic smoothing operator, for any $\delta > 0$ there is a finite dimensional subspace $W \subset \mathcal{S}(\mathbb{R}^n)$ such that

$$\|\langle Gu,u\rangle\| \le \delta\|u\|^2\ \forall\ u \in W^\perp. \tag{4.37}$$

Thus if we replace A by $A(\mathrm{Id} - \Pi_W) = A + E$ where E is a (finite rank) smoothing operator we see that

$$\|(A+E)u\|^2 \le (C+\delta)\|Gu\|^2\ \forall\ u \in L^2(\mathbb{R}^n) \Longrightarrow \|(A+E)\| \le (C+\delta)^{\frac{1}{2}}.$$

This proves half of the desired estimate (4.34), namely

$$\inf_{E\in\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)} \|A+E\|_{\mathcal{B}(L^2(\mathbb{R}^n)} \le \sup_{p\in\mathbb{S}^{2n-1}} |a(p)|. \tag{4.38}$$

To prove the opposite inequality, leading to (4.32), it is enough to arrive at a contradiction by supposing to the contrary that there is some $A \in \Psi^0_{\mathrm{iso}}(\mathbb{R}^n)$ satisfying the strict inequality

$$\|A\|_{\mathcal{B}(L^2(\mathbb{R}^n))} < \sup_{p\in\mathbb{S}^{2n-1}} |a(p)|.$$

From this it follows that we may choose $c > 0$ such that $c = |a(p)|^2$ for some $p \in \mathbb{S}^{2n-1}$ and yet $A' = A^*A - c$ has a bounded inverse, B. By making an arbitrariy small perturbation of the full symbol of A' we may assume that it vanishes identically near p. By (4.38) we may choose $G \in \Psi^0_{\mathrm{iso}}(\mathbb{R}^n)$ with arbitrariy small L^2 such that $\tilde{A} = A' + B$ has left symbol rapidly vanishing near p. When the norm of the perturbation is small enough, $\tilde{A}$ will still be invertible, with inverse $\tilde{B} \in \mathcal{B}(L^2(\mathbb{R}^n))$. Now choose an element $G \in \Psi^0_{\mathrm{iso}}(\mathbb{R}^n)$ with left symbol supported sufficiently near p, so that $G \circ \tilde{A} \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ but yet the principal symbol of G should not vanish at p. Thus

$$G = G \circ \tilde{A} \circ \tilde{B} : L^2(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n),$$

$$G* = G = \tilde{B}^* \circ \tilde{A}^* \circ G^* : \mathcal{S}'(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n).$$

It follows that $G^*G : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$ is an isotropic smoothing operator. This is the expected contradiction, since G, and hence G^*G, may be chosen to have non-vanishing principal symbol at p. Thus we have proved (4.38) and hence the Proposition. □

It is then easy to characterize the compact operators amongst the polyhomogeneous isotropic operators as those of negative.

Lemma 4.4. *If $A \in \Psi^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$ then, as an operator on $L^2(\mathbb{R}^n;\mathbb{C}^N)$, A is compact if and only if it has negative order.*

Proof. The necessity of the vanishing of the principal symbol for compactness follows from Proposition 4.6 and the sufficiency follows from the density of $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$ in $\Psi^{-1}_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$ in the topology of $\Psi^{-\frac{1}{2}}_{\infty-\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$ and hence in the topology of bounded operators. Thus, such an operator is the norm limit of compact operators so itself is compact. □

Also as a consequence of Proposition 4.6 we can see the necessity of the assumption of ellipticity in Proposition 4.3.

Corollary 4.3. *If $A \in \Psi^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$ then A is Fredholm as an operator on $L^2(\mathbb{R}^n;\mathbb{C}^N)$ if and only if it is elliptic.*

4.5. Sobolev spaces

The space of square-integrable functions plays a basic rôle in the theory of distributions; one reason for this is that it is associated with the embedding of $\mathcal{S}(\mathbb{R}^n)$ in $\mathcal{S}'(\mathbb{R}^n)$. We know that pseudodifferential operators of order 0 are bounded on $L^2(\mathbb{R}^n)$. There is also a natural collection of Sobolev spaces associated to the isotropic calculus. The isotropic Sobolev space of order m may be defined as the collection of distributions mapped into $L^2(\mathbb{R}^n)$ by any one elliptic operator of order $-m$.

Note that a differential operator $P(x, D_x)$ on $\mathbb{R}^n$ is an isotropic pseudodifferential operator if and only if its coefficients are polynomials. The fundamental symmetry between coefficients and differentiation suggest that the isotropic Sobolev spaces of non-negative integral order be defined by

$$H^k_{\mathrm{iso}}(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n); x^\alpha D^\beta_x u \in L^2(\mathbb{R}^n) \text{ if } |\alpha| + |\beta| \le k\},\ k \in \mathbb{N}. \tag{4.39}$$

The norms

$$\|u\|^2_{k,\mathrm{iso}} = \sum_{|\alpha|+|\beta|\le k} \int_{\mathbb{R}^n} |x^\alpha D^\beta_x u|^2\, dx \tag{4.40}$$

turn these into Hilbert spaces. For negative integral orders we identify the isotropic Sobolev spaces with the duals of these spaces

$$H^k_{\mathrm{iso}}(\mathbb{R}^n) = (H^{-k}_{\mathrm{iso}}(\mathbb{R}^n))' \hookrightarrow \mathcal{S}'(\mathbb{R}^n),\ k \in -\mathbb{N}. \tag{4.41}$$

The (continuous) injection into tempered distributions here arises from the density of the image of the inclusion $\mathcal{S}(\mathbb{R}^n) \longrightarrow H^k_{\mathrm{iso}}(\mathbb{R}^n)$.

Lemma 4.5. *For any $k \in \mathbb{Z}$,*

$$\begin{aligned} H^k_{\mathrm{iso}}(\mathbb{R}^n) &= \left\{u \in \mathcal{S}'(\mathbb{R}^n); Au \in L^2(\mathbb{R}^n)\ \forall\ A \in \Psi^{-k}_{\mathrm{iso}}\right\} \\ &= \left\{u \in \mathcal{S}'(\mathbb{R}^n); \exists\ A \in \Psi^{-k}_{\mathrm{iso}} \text{ elliptic and such that } Au \in L^2(\mathbb{R}^n)\right\} \end{aligned} \tag{4.42}$$

and $\mathcal{S}(\mathbb{R}^n) \hookrightarrow H^k_{\mathrm{iso}}(\mathbb{R}^n)$ is dense for each $k \in \mathbb{Z}$.

Proof. [1] For $k \in \mathbb{N}$, the functions $x^\alpha \xi^\beta$ for $|\alpha| + |\beta| = k$ are 'collectively elliptic' in the sense that

$$q_k(x,\xi) = \sum_{|\alpha|+|\beta|=k} (x^\alpha \xi^\beta)^2 \ge c(|x|^2 + |\xi|^2)^k,\ c > 0. \tag{4.43}$$

Thus $Q_k = \sum\limits_{|\alpha|+|\beta|\le k} (D^\beta x^\alpha x^\alpha D^\beta) \in \Psi^{2k}_{\mathrm{iso}}(\mathbb{R}^n)$, which has principal reduced symbol q_k, has a left parameterix $A_k \in \Psi^{-2k}_{\mathrm{iso}}(\mathbb{R}^n)$. This gives the identity

$$\begin{gathered} \sum_{|\alpha|+|\beta|\le k} R_{\alpha,\beta} x^\alpha D^\beta = A_k Q_k = \mathrm{Id} + E, \text{ where} \\ R_{\alpha,\beta} = A_k D^\beta x^\alpha \in \Psi^{-2k+|\alpha|+|\beta|}_{\mathrm{iso}}(\mathbb{R}^n),\ E \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n). \end{gathered} \tag{4.44}$$

Thus if $A \in \Psi^k_{\mathrm{iso}}(\mathbb{R}^n)$

$$Au = -AEu + \sum_{|\alpha|+|\beta|\le k} AR_{\alpha,\beta} x^\alpha D^\beta u.$$

If $u \in H^k_{\mathrm{iso}}(\mathbb{R}^n)$ then by definition $x^\alpha D^\beta u \in L^2(\mathbb{R}^n)$. By the boundedness of operators of order 0 on L^2, all terms on the right are in $L^2(\mathbb{R}^n)$ and we have shown the inclusion of $H^k_{\mathrm{iso}}(\mathbb{R}^n)$ in the first space space on the right in (4.42). The converse is immediate, so this proves the first equality in (4.42) for $k > 0$. Certainly the third space in (4.42) contains in the second. The existence of an elliptic parametrix B for the ellipic operator A proves the converse since any isotropic pseudodifferential operator of order A' of order k can be effectively factorized as

$$A' = A'(BA + E) = B'A + E',\ B' \in \Psi^0_{\infty-\mathrm{iso}}(\mathbb{R}^n),\ E' \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n).$$

Thus, $Au \in L^2(\mathbb{R}^n)$ implies that $A'u \in L^2(\mathbb{R}^n)$.

[1] This is an essentially microlocal proof.

It also follows from second identification that $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^k_{\mathrm{loc}}(\mathbb{R}^n)$. Thus, if $Au \in L^2(\mathbb{R}^n)$ and we choose $f_n \in \mathcal{S}(\mathbb{R}^n)$ with $f_n \to Au$ in $L^2(\mathbb{R}^n)$ then, with B a parametrix for A, $u'_n = Bf_n \to BAu = u + Eu$. Thus $u_n = u'_n - Eu \in \mathcal{S}(\mathbb{R}^n) \to u$ in $L^2(\mathbb{R}^n)$ and $Au_n \to u$ in $L^2(\mathbb{R}^n)$ proving the density.

The Riesz representation theorem shows that $v \in \mathcal{S}'(\mathbb{R}^n)$ is in the dual space, $H^{-k}_{\mathrm{iso}}(\mathbb{R}^n)$, if and only if there exists $v' \in H^k_{\mathrm{iso}}(\mathbb{R}^n)$ such that

$$v(u) = \langle u, v'\rangle_{k,\mathrm{iso}} = \langle u, Q_{2k}v'\rangle_{L^2},\ \forall\ u \in \mathcal{S}(\mathbb{R}^n) \hookrightarrow H^k_{\mathrm{iso}}(\mathbb{R}^n) \\ \text{with } Q_{2k} = \sum_{|\alpha|+|\beta|\le k} D^\beta x^{2\alpha} D^\beta. \tag{4.45}$$

This shows that Q_{2k} is an isomorphism of $H^k_{\mathrm{iso}}(\mathbb{R}^n)$ onto $H^{-k}_{\mathrm{iso}}(\mathbb{R}^n)$ as subspaces of $\mathcal{S}'(\mathbb{R}^n)$. Notice that $Q_{2k} \in \Psi^{2k}_{\mathrm{iso}}(\mathbb{R}^n)$ is elliptic, self-adjoint and invertible, since it is strictly positive. This now gives the same identification (4.42) for $k < 0$.

The case $k = 0$ follows directly from the L^2 boundedness of operators of order 0 so the proof is complete. □

In view of this identification we define the isotropic Sobolev spaces or any real order the same way

$$H^s_{\mathrm{iso}}(\mathbb{R}^n) = \left\{u \in \mathcal{S}'(\mathbb{R}^n); Au \in L^2(\mathbb{R}^n)\ \forall\ A \in \Psi^{-s}_{\mathrm{iso}}\right\},\ s \in \mathbb{R}. \tag{4.46}$$

These are Hilbertable spaces, with the Hilbert norm being given by $\|Au\|_{L^2(\mathbb{R}^n}$ for any $A \in \Psi^s_{\mathrm{iso}}(\mathbb{R}^n)$ which is elliptic and invertible.

Proposition 4.7. *Any element $A \in \Psi^m_{\infty-\mathrm{iso}}(\mathbb{R}^n)$, $m \in \mathbb{R}$, defines a bounded linear operator*

$$A : H^s_{\mathrm{iso}}(\mathbb{R}^n) \longrightarrow H^{s-m}_{\mathrm{iso}}(\mathbb{R}^n),\ \forall\ s \in \mathbb{R}. \tag{4.47}$$

This operator is Fredholm if and only if A is elliptic. For any $s \in \mathbb{R}$, $\mathcal{S}(\mathbb{R}^n) \hookrightarrow H^s_{\mathrm{iso}}(\mathbb{R}^n)$ is dense and $H^{-s}_{\mathrm{iso}}(\mathbb{R}^n)$ may be identified as the dual of $H^s_{\mathrm{iso}}(\mathbb{R}^n)$ with respect to the continuous extension of the L^2 pairing.

Proof. A straightforward application of the calculus, with the exception of the necessity of ellipticity for an isotropic pseudodifferential operator to be Fredholm. This is discussed in the problems beginning at Problem 4.10. □

4.6. Representations

In §1.9 the compactification of Euclidean space to a ball, or half-sphere, is described. We make the following definition, recalling that $\rho \in \mathcal{C}^\infty(\mathbb{S}^{n,+})$ is a boundary defining function.

Definition 4.2. *The space of of 'Laurent functions' on the half-sphere is*

$$\mathcal{L}(\mathbb{S}^{n,+}) = \bigcup_{k\in\mathbb{N}_0} \rho^{-k}\mathcal{C}^\infty(\mathbb{S}^{n,+}), \\ \rho^{-k}\mathcal{C}^\infty(\mathbb{S}^{n,+}) = \{u \in \mathcal{C}^\infty(\mathrm{int}(\mathbb{S}^{n,+})); \rho^k u \in \mathcal{C}^\infty(\mathbb{S}^{n,+}). \tag{4.48}$$

More generally if $m \in \mathbb{R}$ we denote by $\rho^m\mathcal{C}^\infty(\mathbb{S}^{n,+})$ the space of functions which can be written as products $u = \rho^m v$, with $v \in \mathcal{C}^\infty(\mathbb{S}^{n,+})$; again it can be identified with a subspace of the space of $\mathcal{C}^\infty$ functions on the open half-sphere.

PROPOSITION 4.8. *The compactification map* (1.94) *extends from* (1.96) *to give, for each* $m \in \mathbb{R}$, *an identification of* $\rho^{-m}\mathcal{C}^\infty(\mathbb{S}^{n,+})$ *and* $S^m_{\mathrm{cl}}(\mathbb{R}^n)$.

Thus, the fact that the $\Psi^{\mathbb{Z}}_{\mathrm{iso}}(\mathbb{R}^n)$ form an order-filtered $*$-algebra means that $\rho^{\mathbb{Z}}\mathcal{C}^\infty(\mathbb{S}^{2n,+})$ has a non-commutative product defined on it, with $\mathcal{C}^\infty(\mathbb{S}^{2n,+})$ a subalgebra, using the left symbol isomorphism, followed by compactification.

4.7. Symplectic invariance of the isotropic product

The composition law for the isotropic calculus, and in particular for it smoothing part, is derived from its identification as a subalgebra of the (weighted) spaces of pseudodifferential operator on $\mathbb{R}^n$. There is a much more invariant formulation of the product which puts into evidence more of the invariance properties.

Let W be a real symplectic vector space. Thus, W is a vector space equipped with a real, antisymmetic and non-degenerate bilinear form

$$\begin{aligned}(4.49)\quad \omega : W \times W \longrightarrow \mathbb{R},\ \omega(w_1, w_2) + \omega(w_2, w_1) = 0\ \forall\ w_1, w_2 \in W,\\ \omega(w_1, w) = 0\ \forall\ w \in W \Longrightarrow w_1 = 0.\end{aligned}$$

A Lagrangian subspace of W is a vector space $V \subset W$ such that ω vanishes when restricted to V and such that $2\dim V = \dim W$.

LEMMA 4.6. *Every symplectic vector space has a Lagrangian subspace and for any choice of Lagrangian subspace* U_1 *there is a second Lagrangian subspace* U_2 *such that* $W = U_1 \oplus U_2$ *is a* Lagrangian decomposition.

PROOF. First we show that there is a Lagrangian subspace. If $\dim W > 0$ then the antisymmetry of ω shows that any 1-dimensional vector subspace is *isotropic*, that is ω vanishes when restricted to it. Let V be a maximal isotropic subspace, that is an isotropic subspace of maximal dimension amongst isotropic subspaces. Let U be a complement to V in W. Then

$$(4.50)\qquad \omega : V \times U \longrightarrow \mathbb{R}$$

is a non-degenerate pairing. Indeed $u \in U$ and $\omega(v, u) = 0$ for all $v \in V$ then $V + \mathbb{R}\{u\}$ is also isotropic, so $u = 0$ by the assumed maximality. Similarly if $v \in V$ and $\omega(v, u) = 0$ for all $u \in U$ then, recalling that ω vanishes on V, $\omega(v, w) = 0$ for all $w \in W$ so $v = 0$. The pairing (4.50) therefore identifies U with V', the dual of V. In particular $\dim w = 2\dim V$.

Now, choose any Lagrangian subspace U_1. We proceed to show that there is a complementary Lagrangian subspace. Certainly there is a 1-dimensional subspace which does not meet U_1. Let V be an isotropic subspace which does not meet U_1 and is of maximal dimension amongst such subspaces. Suppose that $\dim V < \dim U_1$. Choose $w \in W$ with $w \notin V \oplus U_1$. Then $V \ni v \longrightarrow \omega(w, v)$ is a linear functional on U_1. Since U_1 can be completed to a complement, any such linear functional can be written $\omega(u_1, v)$ for some $u_1 \in U_1$. It follows that $\omega(w - u_1, v) = 0$ for all $v \in V$. Thus $V \oplus \mathbb{R}\{w - u_1\}$ a non-trivial isotropic extension of V, contradicting the assumed maximality. Thus $V = U_2$ is a complement of U_1. □

Given such a Lagrangian decomposition of the symplectic vector space W, let $X_1, \dots X_n$ be a basis for the dual of U_1, and let $\Xi_1, \dots, \Xi_n$ be the dual basis, of U_1 itself. The pairing (4.50) with $U = U_1$ and $V = U_2$ identifies $U_2 = U_1'$ so the Ξ_i

can also be regarded as a basis of the dual of U_2. Thus $X_1 \dots X_n, \Xi_1, \dots, \Xi_n$ gives a basis of $W' = U_1' \oplus U_2'$. The symplectic form can then be written

$$\omega(w_1, w_2) = \sum_{i=1}^{n} (\Xi_i(w_1)X_i(w_2) - \Xi_i(w_2)X_i(w_1)). \tag{4.51}$$

This is the *Darboux* form of ω. If the X_i, Ξ_i are thought of as linear functions x_i, ξ_i on W now considered as a manifold then these are *Darboux coordinates* in which(4.51) becomes

$$\omega = \sum_{i=1}^{n} d\xi_i \wedge dx_i. \tag{4.52}$$

The symplectic form ω defines a volume form on W, namely the n-fold wedge product ω^n. In Darboux coordinates this is just, up to sign, the Lebesgue form $d\xi dx$.

Proposition 4.9. *On any symplectic vector space, W, the bilinear map on $\mathcal{S}(W)$,*
(4.53)

$$a\#b(w) = (2\pi)^{-2n} \int_{W^2} e^{i\omega(w_1,w_2)} a(w + w_1)b(w + w_2)\omega^n(w_1)\omega^2(w_2),\ \dim W = 2n$$

defines an associative product isomorphic to the composition of $\Psi^{-\infty}_{\text{iso}}(U_1)$ for any Lagrangian decomposition $W = U_1 \oplus U_2$.

Corollary 4.4. *Extended by continuity in the symbol space* (4.53) *defines a filtered product on $S^\infty(W)$ which is isomorphic to the isotropic algebra on $\mathbb{R}^{2n}$ and is invariant under symplectic linear transformation of W.*

Proof. Written in the form (4.53) the symplectic invariance is immediate. That is, if F is a linear transformation of W which preserves the symplectic form, $\omega(Fw_1, Fw_2) = \omega(w_1, w_2)$ then

$$F^*(a\#b) = (F^*a)\#(F^*b)\ \forall\ a, b \in \mathcal{S}(W). \tag{4.54}$$

The same result holds for general symbols once the continuity is established.

Let us start from the Weyl quatization of the isotropic algebra. As usual for computations we may assume that the amplitudes are of order $-\infty$. Thus, $A \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ may be written

$$Au(x) = \int A(x, y)u(y) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a(\frac{1}{2}(x + y), \xi)u(y)dyd\xi. \tag{4.55}$$

Both the kernel $A(x, y)$ and the amplitude $a(x, \xi)$ are elements of $\mathcal{S}(\mathbb{R}^{2n})$. The relationship (4.55) and its inverse may be written

$$\begin{aligned} A(s + \frac{t}{2}, s - \frac{t}{2}) &= (2\pi)^{-n} \int e^{it\cdot\xi} a(s, \xi)d\xi, \\ a(x, \xi) &= \int e^{-it\cdot\xi} A(x + \frac{t}{2}, x - \frac{t}{2})dt. \end{aligned} \tag{4.56}$$

If A has Weyl symbol a and B has Weyl symbol b let c be the Weyl symbol of the composite $A \circ B$. Using (4.56) and (4.55)

$$c(s,\zeta) = \int e^{-it\cdot\zeta} A(s+\frac{t}{2},z)B(z,s-\frac{t}{2})dt$$

$$= (2\pi)^{-2n} \int\int\int dt\,dz\,d\xi\,d\eta e^{i\Phi} a(\frac{s}{2}+\frac{t}{4}+\frac{z}{2},\xi)a(\frac{z}{2}+\frac{s}{2}-\frac{t}{4},\eta)$$

$$\text{where } \Phi = -t\cdot\zeta + (s+\frac{t}{2}-z)\cdot\xi + (z-s+\frac{t}{2})\cdot\eta.$$

Changing variables of integration to $X = \frac{z}{2}+\frac{t}{4}-\frac{s}{2}$, $Y = \frac{z}{2}-\frac{t}{4}-\frac{s}{2}$, $\Xi = \xi - \zeta$ and $H = \eta - \zeta$ this becomes

$$c(s,\zeta) = (2\pi)^{-2n}4^n \int\int\int dY\,dX\,d\Xi\,dH \\ e^{2i(X\cdot H - Y\cdot\Xi)} a(X+s,\Xi+\zeta)a(Y+s,H+\zeta).$$

This reduces to (4.53), written out in Darboux coordinates, after the change of variable $H' = 2H$, $\Xi' = 2\Xi$ and $\zeta' = 2\zeta$. Thus the precise isomorphism with the product in Weyl form is given by

$$A(x,y) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a_\omega(\frac{1}{2}(x+y),2\xi)u(y)dyd\xi \tag{4.57}$$

so that composition of kernels reduces to (4.53). □

4.8. Metaplectic group

The discussion of the metaplectic group in this section might, or might not, be relevant for later material. For the moment you can freely ignore it, but it is amusing enough. The operators constructed here are 'Fourier integral operators' in the isotropic sense – but by no means all such Fourier integral operator. In particular they correspond to linear symplectic transformations of the underlying space, rather than more general homogeneous symplectic diffeomorphisms.

As we shall see below the discussion of the metaplectic group reduces to the computation of some constants, these are bound up with the standard formula for the Fourier transform of 'Gaussians'. Namely, if $z \in \mathbb{C}$ has positive real part then

$$\mathcal{F}(\exp(-zx^2)) = \frac{\sqrt{\pi}}{\sqrt{z}} \exp(-\frac{1}{4z}\xi^2) \tag{4.58}$$

where the square-root is the standard branch, having positive real part for z in this half-plane. One can carry out the integrals directly. In fact both sides are holomorphic in $\Re z > 0$ so it suffices to check the formula on the positive real axis in z where it is easy.

Now, recall that the symplectic group on $\mathbb{R}^{2n}$, denoted $\mathrm{Sp}(2n)$, is the group of linear transformations preserving a given non-degenerate antisymmetric bilinear form. We will take the standard (well, standard up to sign and maybe constants) Darboux form

$$\omega_D((x,\xi),(x',\xi')) = \xi'\cdot x - \xi\cdot x'. \tag{4.59}$$

Recall that this is not a restriction in the sense that

LEMMA 4.7 (Linear Darboux Theorem). *If ω is a non-degenerate antisymetric real bilinear form on a (necessarily even-dimensional) real vector space then there is a linear isomorphism to $\mathbb{R}^{2n}$ reducing ω to the Darboux form* (4.59).

BRIEF PROOF. Construct a basis by induction. First choose a non-zero element e_1 and then a second element e_2 such that $\omega(e_1, e_2) = 1$, which is possible by the assumed non-degeneracy. Then look at the subspace spanned by those vector satisfying $\omega(e_1, f) = \omega(e_2, f) = 0$. This is complementary to the span of e_1, e_2 and ω is the direct sum of ω_D for $n = 1$ on the span of e_1, e_2 and ω on this complement. After a finite number of steps one arrives at (4.59) with the x's corresponding to the odd basis elements and the ξ's to the even ones. □

We will need properties of the symplectic group below, but I will just work them out as the need arises.

Let me define a group of operators on $L^2(\mathbb{R}^n)$ which also map $\mathcal{S}(\mathbb{R}^n)$ to itself, by the crude method of taking products of some obvious invertible operators. The basic list is:-

(F.1) Multiplication by constants.

(F.2) Multiplication by functions $e^{iq(x)}$ where q is a real quadratic form,

(F.3) The Fourier transforms in each variable

(4.60)
$$\mathcal{F}_j u(x', \tau, x'') = \int e^{-t\tau} u(x', t, x'')dt, \ x' = (x_1, \dots, x_j), \ x'' = (x_{j+1}, \dots, x_n).$$

(F.4) Pull-back under any linear isomorphism

$$T^* u(x) = u(Tx), \ T \in \mathrm{GL}(n, \mathbb{R}). \tag{4.61}$$

Obviously the multiples of the identity in (F.1) commute with the other operators. Moreover

$$e^{iq(x)} T^* = T^* e^{iq'(x)}, \ q'(x) = q((T^t)^{-1}x) \tag{4.62}$$

so (F.2) and (F.4) may be interchanged.

In fact it is convenient to reorganize the products of these elements. Observe that conjugation by the Fourier transform (in all variables) of an operator (F.2)

$$\mathcal{F}^{-1} e^{iq(\cdot)} \mathcal{F} u(x) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} e^{iq(\xi)} u(u) dy d\xi. \tag{4.63}$$

gives a convolution operator which we can, and will, denote $e^{iq(D)}$. Then the operators in this list which are 'close to the identity' are

(S.1) Multiplication by constants near 1

(S.2) Multiplication by functions $e^{iq(x)}$ where q is a small real quadratic form,

(S.3) Application of $e^{iq(D)}$ where q is a small real quadratic form and

(S.4) Pull-back under any linear isomorphism close to the identity.

For definiteness sake:-

DEFINITION 4.3. *Let $\mathcal{M}(2n)$ denote the space of operators on $\mathcal{S}(\mathbb{R}^n)$ which are finite products of elements of the form*

$$M = ce^{iq_2} L_2^* \mathcal{F}_I e^{iq_1} \mathcal{F}_I L_1^* \tag{4.64}$$

where $\mathcal{F}_I$ denotes the product of the Fourier transforms in the variables corresponding to $i \in I$ for some subset $I \subset \{1, \dots, n\}$.

As we shall see, these operators form a Lie group; it contains but is not equal to the metaplectic group. The products of elements in (S.1) – (S.4) give a neighbourhood of the identity in this group.

First we need to see how these operators are related to the symplectic group.

LEMMA 4.8. *If M is of the form* (4.64) *then*

$$M(x_j u) = (\sum_i A_{kj} x_k + \sum_k B_{kj} D_k) Mu,$$
$$M(D_j u) = (\sum_i iC_{kj} x_k + \sum_k D_{kj} D_k) Mu \tag{4.65}$$

where A, B, C and D are real $n \times n$ matrices and

$$S(M) = \begin{pmatrix} A & C \\ B & D \end{pmatrix} \in \mathrm{Sp}(2n). \tag{4.66}$$

Furthermore all symplectic matrices arise this way and all symplectic matrices close to the identity arise from products operators in (S.1) – (S.4) (one of each of type).

PROOF. To prove (4.66) we will check that it holds for each of the factors in (4.64). Then from (4.65)

$$S(M_1 M_2) = S(M_1) S(M_2), \tag{4.67}$$

i.e. this will be a group homomorphism.

For (F.1), (4.65) and (4.66) are obvious, with the matrix being the identity. For M as in (F.2), $A = \mathrm{Id}$, $B = 0$, $D = \mathrm{Id}$ and $Cx = -q'(x)$ is given by the derivative of q and

$$S(M) = \begin{pmatrix} \mathrm{Id} & C \\ 0 & \mathrm{Id} \end{pmatrix} \in \mathrm{Sp}(2n) \text{ for any symmetric } C. \tag{4.68}$$

The matrix for $\mathcal{F}_l$ is the identity outside the 2×2 block corresponding to x_l and D_l where it is just

$$\begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix} \tag{4.69}$$

which is certainly symplectic. Finally the matrix for L^* is just

$$\begin{pmatrix} L & 0 \\ 0 & (L^{-1})^t \end{pmatrix}. \tag{4.70}$$

So this gives a group homomorphism, we proceed to check the surjectivity of this map to $\mathrm{Sp}(2n)$. By the conjugation result (4.62) an operator of the form (4.64) remains so under conguation by some L^*. This in particular conjugates the upper left block A in S to $L^{-1}AL$. The rank of the matrix is a complete invariant under conjugation by $\mathrm{GL}(n, \mathbb{R})$ so we may arrange that

$$A = \pi_k \text{ projection onto the first } k \text{ components} \tag{4.71}$$

without affecting the overall problem. The symplectic condition then implies that

$$S = \begin{pmatrix} \pi_k & C' \\ B' & D' \end{pmatrix}, \ \pi_k B'(\mathrm{Id} - \pi_k) = 0 \tag{4.72}$$

and where $(\mathrm{Id}-\pi_k)B'(\mathrm{Id}-\pi_k)$ must be an isomorphism on the range of $\mathrm{Id}-\pi_k$ to preserve invertibility. Thus a further choice of L_1, not affecting the special form of A allows us to arrange that

$$S = \begin{pmatrix} \pi_k & C' \\ -(\mathrm{Id}-\pi_k) + B'' & D' \end{pmatrix},\ B''\pi_k = B''. \tag{4.73}$$

Again from the symplectic condition it follows that B'' is symmetric. Now choosing I to be the first k elements arranges that $A = \mathrm{Id}$. For the new matrix, B must be symmetric. □

To proceed further consider the operators of the form (4.64) for which $S(M) = \mathrm{Id}$.

Proposition 4.10. *The space of operators $\mathcal{M}$ defined by* (4.64) *is a group with a multiplicative short exact sequence*

$$\mathbb{C}^* \longrightarrow \mathcal{M} \longrightarrow \mathrm{Sp}(2n). \tag{4.74}$$

Proof. Consider the elements of $\mathcal{M}$ such that $S(M) = \mathrm{Id}$. By definition of $S(M)$ these have the property that they commute with x_k and D_j for all j. Recalling the proof of the invertibility of the Fourier transform, this shows that the kernel of M satisfies the differential equations

$$(x_j - y_j)M(x,y) = 0,\ (D_{x_j} + D_{y_j})M(x,y) = 0 \Longrightarrow M(x,y) = c\delta(x,y) \tag{4.75}$$

for some constant c. Thus

$$\ker(S : \mathcal{M} \longrightarrow \mathrm{Sp}(2n)) = \mathbb{C}^*\,\mathrm{Id}. \tag{4.76}$$

Now Lemma 4.8 combined with this argument shows that $\mathcal{M}$ actually consists of the operators of the form (4.64), without having to take further products. Indeed, given a finite product $M_1M_2\dots M_p$, of elements of $\mathcal{M}$ we can use Lemma 4.8 to find a single element $M \in \mathcal{M}$ such that $S(M) = S(M_1)\dots S(M_p)$. Composing on the right with M gives a product $M^{-1}M_1\dots M_p$ which commutes with x_j and D_j as above, so is a multiple of the identity, which proves that $M_1\dots M_p$ is of the form (4.64). The inverse of an element of $\mathcal{M}(2n)$, as an operator is not quite of the same form directly, but the same argument applies. □

Thus $\mathcal{M}$ is two real dimensions larger than $\mathrm{Sp}(2n)$. Notice that all the elements in the products (4.64) are unitary up to positive constant multiples – and all multiples occur. So we can kill one dimension by looking at the unitary elements

$$\mathbb{S} \longrightarrow (\mathcal{M}(2n) \cap \mathrm{U}(L^2(\mathbb{R}^n)) \longrightarrow \mathrm{Sp}(2n) \text{ is exact.} \tag{4.77}$$

In fact we can do more than this, namely we can define in a reasonably natural way a lift of a neighbourhood of the identity in $\mathrm{Sp}(2n)$ into $\mathcal{M} \cap \mathrm{U}(L^2(\mathbb{R}^n)$. If $S \in \mathrm{Sp}(2n)$ is close to the identity then it has a 'generating function'. Namely if we write

$$S(x,\xi) = (x',\xi') \text{ then } \frac{\partial \xi'(x,\xi)}{\partial \xi} \text{ is invertible} \tag{4.78}$$

since it is close to the identity. So, the corresponding linear map is invertible, and x and ξ' may be introduced as linear coordinates on the graph

$$\xi = \Xi(x,\xi'),\ x' = X'(x,\xi') \text{ on the graph of } S. \tag{4.79}$$

Now the symplectic condition can be rewritten as

$$-d\xi'\wedge dx' + d\xi\wedge dx = d(\Xi\cdot dx + X'\cdot d\xi') = 0 \Longrightarrow \Xi\cdot dx + X'\cdot d\xi = d\Phi(x,\xi') \tag{4.80}$$

where Φ is a quadratic form (since there is no 1-dimensional cohomolgy in $\mathbb{R}^{2n}$ such a smooth function exists but by homogeneity we may replace it by its quadratic part at the origin) such that

$$\Xi(x,\xi') = \frac{\partial\Phi(x,\xi'}{\partial x},\ X'(x,\xi') = \frac{\partial\Phi(x,\xi'}{\partial\xi'} \text{ defines } S'. \tag{4.81}$$

So now we 'lift' S to the element

$$M(S)u(x) = c(S)\int e^{i\Phi(x,\eta)}\hat{u}(\eta)d\eta \tag{4.82}$$

defined by the construction of the generating function above.

Proposition 4.11. *For S in a small neighbourhood of the identity in* $\mathrm{Sp}(2n)$ *here is a unique choice of $c(S) > 0$ in* (4.82) *such that $M \in \mathcal{M}\cap \mathrm{U}(L^2(\mathbb{R}^n)$ and this choice is smooth in S, the subgroup of* $\mathrm{Mp}(2n) \subset \mathcal{M}(2n)$ *generated by the finite products of these elements is a Lie group giving a 2-fold cover*

$$\mathbb{Z}_2 \longrightarrow \mathrm{Mp}(2n) \longrightarrow \mathrm{Sp}(2n). \tag{4.83}$$

This is either the metaplectic group or else is a faithful representation of it, depending on your attitude; I will call it the metaplectic group!

Proof. For S close to the identity the discussion above shows that Φ is close to $x\cdot\eta$ as a quadratic form, meaning that

$$\Phi_S(x,\eta) = q_2(x) + Lx\cdot\eta + q_1(\eta),\ L\in \mathrm{GL}(n,\mathbb{R}). \tag{4.84}$$

In fact L is close to the identity. The definition of $M(S)$ in (4.82) can therefore be rewritten

$$M(S) = c(S)e^{iq_2}L^*\mathcal{F}^{-1}e^{iq_1}\mathcal{F}. \tag{4.85}$$

The desired unitarity then fixes $c(S) > 0$ and in fact

$$c(S) = \sqrt{|\det T|} \tag{4.86}$$

and it follows that $M(S)$ depends smoothly on $S\in \mathrm{Sp}(2n)$, near Id.

The next important thing to check is that this lift is multiplicative near the identity, i.e. gives a local Lie group. From the discussion above we know that

$$M(S_1)M(S_2) = e^{i\theta}M(S_1S_2),\ S_1,\ S_2\in \mathrm{Sp}(2n) \text{ near } \mathrm{Id} \tag{4.87}$$

up to the possibility of factor of absolute value 1 – we proceed to show that there is no such factor locally, although as we shall see there is one globally.

Lemma 4.9. *If q_i, $i = 1,2$, are real quadratic forms which are sufficiently small and $L\in \mathrm{GL}(n,\mathbb{R})$ is sufficiently close to the identity then there exist unique small quadratic forms q_i', $i = 1,2$ and $L'\in \mathrm{GL}(n,\mathbb{R})$ close to the identity such that*

$$e^{iq_1(x)}Le^{iq_2(D)} = \delta' e^{iq_2'(D)}e^{iq_1'(x)}L',\ \delta' > 0. \tag{4.88}$$

PROOF. We can move the linear transformations to the left, so it suffices to show the existence of q_i' and L' such that

$$M = e^{iq_1(x)}e^{iq_2(D)} = \delta e^{iq_2'(D)}e^{iq_1'(x)}L',\ \delta > 0 \tag{4.89}$$

under the same hypotheses. By making an overal orthogonal transformation, we may suppose that q_2 is a non-degenerate quadratic form in the duals of the first variables in a splitting $x = (x', x'')$ and is trivial in the second variables. Since e^{iq_2} is a product of terms in each of the variables, it suffices (by renumbering the coordinates) to consider the case that $q_2 = \xi_1^2$. Then, after another orthogonal transformation close to the identity, we may suppose that $q_1 = ax^2 + bxy$ where a, b and c are all small and we are reduced to two variables which we denote x and y with $q = c\xi^2$. Now we will show directly that

$$e^{ibxy+cb^2y^2}e^{icD_x^2} = e^{icD_x^2}e^{ibxy}T^*,\ T^*x = x - 2cby,\ T^*y = y \tag{4.90}$$

where, up to a constant of absolute value one, this comes from the computation of the corresponding symplectic transformations. To see (4.90) insert the Fourier transform on the right and change variables

$$\begin{aligned}
&(e^{icD_x^2}e^{ibxy}T^*)u(x,y)\\
&= (2\pi)^{-1}\int e^{ic\xi^2+i(x-x')\xi}e^{ibx'y}u(x'-2cby,y)dx'd\xi\\
&= e^{ibxy}(2\pi)^{-1}\int e^{ic\xi^2+i(x-x''-2cby)\xi+ib(x''+2cby)y)}u(x'',y)dx''d\xi\\
&= e^{ibxy+cb^2y^2}(2\pi)^{-1}\int e^{ic(\xi-by)^2+i(x-x'')(\xi-by)}u(x'',y)dx''d\xi\\
&= e^{ibxy+cb^2y^2}(2\pi)^{-1}\int e^{ic(\xi')^2+i(x-x'')\xi'}u(x'',y)dx''d\xi'\\
&= e^{ibxy+cb^2y}e^{icD_x^2}u.
\end{aligned} \tag{4.91}$$

where $x'' = x' - 2cby$ and $\xi' = \xi - by$. Whilst these are really oscillatory integrals, the formal manipulation is easily justified by regularization, as usual.

Since (4.90) can be rewritten

$$e^{ibxy+cb^2y^2}e^{icD_x^2} = e^{ibxy}e^{icD_x^2}e^{cb^2y^2} = e^{icD_x^2}e^{ibxy}T^* \Longrightarrow e^{ibxy}e^{icD_x^2} = e^{icD_x^2}e^{ibxy-icb^2y^2}T^* \tag{4.92}$$

we are reduced to the case $q_1 = ax^2$, $q_2 = c\xi^2$ which is purely one-dimensional. By a similar computation it can be checked that

$$\begin{gathered}
e^{iax^2}e^{icD_x^2} = DT^*e^{ic'D^2}e^{ia'x^2}\\
\text{if } a' = \frac{a}{1-4ac},\ c' = c(1-4ac),\ T^*x = (1-4ac)x,\ D = 1-4ac
\end{gathered} \tag{4.93}$$

where again the basic formula comes from comparing the symplectic transformations, namely under the operator on the left

$$x \longmapsto (1-4ac)x + 2xD_x,\ D_x \longmapsto D_x - 2ax \tag{4.94}$$

and on the right, before the application of T^*,

$$x \longmapsto x + 2cD_x,\ D_x \longmapsto (1-4a'c')D_x - 2a'x. \tag{4.95}$$

Comparing these leads to (4.93). Thus we know that the sides are equal up to a multiplicative constant and this can be computed by applying the operators to one non-trivial function. For example applying the operators to e^{-x^2}, and using (4.58), the right side gives

$$\begin{aligned}(4.96)\qquad &\frac{D}{\sqrt{A}\sqrt{1-ia'}}\exp(-(\frac{1}{4A}-ia)(1-4ac)^2x^2)\\ &\qquad=\frac{2D}{\sqrt{1-4ac}\sqrt{1-4ic}}\exp(-Bx^2),\\ &\qquad\qquad\text{since } A=\frac{1}{4(1-ia')}-ic'=(1-4ac)\frac{1-4ic}{4(1-ia')}\end{aligned}$$

and on the left

$$(4.97)\qquad \frac{2}{\sqrt{1-4ic}}\exp(-Bx^2),\ B=\frac{1-4ac-ia}{1-4ic}$$

which gives the formula for D and shows most significantly that it is positive. □

Returning to the proof of the Propostion, we have now checked that the lift is well-defined near the identity and defines a local group. In fact it follows from this discussion that all the operators of the form

$$(4.98)\qquad M=e^{iq_1(x)}Le^{iq_2(D)},\ \det L>0,$$

where we no longer assume that the quadratic forms are small, are products of elements from a neighbourhood of the identity, and hence are in $\mathrm{Mp}(2n)$ and have a unique representation (4.98). Indeed, we can certainly connect such an element to the identity by connecting L to the identity by a curve $L_t\in\mathrm{GL}(n,\mathbb{R})$ and replacing q_1 and q_2 by tq_1 and tq_2. The corresponding element $M_t\in\mathrm{Mp}(2n)$ for small t and by continuity it follows that it is in $\mathrm{Mp}(2n)$ for all $t\in[0,1]$. Indeed, let T be the supremum of those t for which it remains in the group, and is therefore a finite product of elements in a fixed small neighbourhood of the identity for each $t<T$. Consider the image curve $S(M_t)$ in $\mathrm{Sp}(2n)$. For $0\le s<\epsilon$ for some $\epsilon>0$, $S(M_t)=R_sS(M_{t-\epsilon+s})$ where $[0,\epsilon]\ni s\longmapsto R_s$ is a curve starting at the identity in $\mathrm{Sp}(2n)$. Thus, from the discussion above, R_s has a unique lift N_s as in Lemma 4.9. The uniqueness of the representation shows that $M_{t-\epsilon+s}=N_sM_{t-\epsilon}$ for $s<\epsilon$ and since $M_{t-\epsilon}$ has a finite product representation, so does M_t for $t\le\epsilon$ and so this is true of M_T. Thus, $M_t\in\mathrm{Mp}(2n)$ and the unqueness follows from the earlier discussion.

In fact we can now check that the metaplectic group, defined by iterated composition of the elements near the identity just discussed, consists precisely of the unitary operators of the form

$$(4.99)\qquad \pm D\exp\left(i\frac{\pi}{2}(1-\operatorname{sgn}\det(T)-|I|)\right)e^{iq_1(x)}e^{iq_2(D)}T^*\mathcal{F}_I,\ D>0.$$

Notice that if $I=0$, so no explicit partial Fourier transforms are present, then the complex factor is $\pm i$ if $\det T<0$ and ± 1 if $\det T>0$ which shows that $\mathcal{M}$ is a double cover of $\mathrm{Sp}(2n)$. □

Theorem 4.3. *The metaplectic group of operators on $\mathcal{S}(\mathbb{R}^n)$ acts by conjugation on $\Psi^k_{\mathrm{iso}}(\mathbb{R}^n)$ and gives an action of $\mathrm{Sp}(2n)$ as a group of outer automorphisms of the algebra.*

4.9. Complex order

The identification of polyhomogeneous symbols of order zero on $\mathbb{R}^{2n}$ with the smooth functions on the radial compactification allows us to define the isotropic operators of a given complex order $z \in \mathbb{C}$. Namely, we use the left quantization map to identify

$$\Psi^z_{\mathrm{iso}}(\mathbb{R}^n) = \rho^{-z}\mathcal{C}^\infty(\mathbb{S}^{2n,1}) \subset \Psi^{\Re z}_{\infty-\mathrm{iso}}(\mathbb{R}^n). \tag{4.100}$$

Here, $\rho \in \mathcal{C}^\infty(\mathbb{S}^{2n,1})$ is a boundary defining function. Any other boundary defining function is of the form $a\rho$ with $0 < a \in \mathcal{C}^\infty(\mathbb{S}^{2n,1})$. It follows that the definition is independent of the choice of ρ since $a^z \in \mathcal{C}^\infty(\mathbb{S}^{2n,1})$ for any $z \in \mathbb{Z}$.

In fact it is even more useful to consider holomorphic families. Thus if $\Omega \subset \mathbb{C}$ is an open set and $h : \Omega \longrightarrow \mathbb{C}$ is holomorphic then we may consider holomorphic families of order h as elements of

$$\Psi^{h(z)}_{\mathrm{iso}}(\mathbb{R}^{2n}) = \big\{A : \Omega \longrightarrow \Psi^\infty_{\infty-\mathrm{iso}}(\mathbb{R}^{2n}); \\ \Omega \ni z \longmapsto \rho^{h(z)}A(z) \in \mathcal{C}^\infty(\mathbb{S}^{2n,1}) \text{ is holomorphic.}\big\} \tag{4.101}$$

Note that a map from $\Omega \subset \mathbb{C}$ into $\mathcal{C}^\infty(\mathbb{S}^{2n1,})$ is said to be holomorphic it is defines an element of $\mathcal{C}^\infty(\Omega \times \mathbb{S}^{2n,1})$ which satisies the Cauchy-Riemann equation in the first variable.

Proposition 4.12. *If h and g are holomorphic functions on an open set $\Omega \subset \mathbb{C}$ and $A(z)$, $B(z)$ are holomorphic familes of isotropic operators of orders $h(z)$ and $g(z)$ then the composite family $A(z) \circ B(z)$ is holomorphic of order $h(z) + g(z)$.*

Proof. It suffices to consider an arbitrary open subset $\Omega' \subset \Omega$ with compact closure inside Ω. Then h and g have bounded real parts, so $A(z)$, $B(z) \in \Psi^M_{\infty-\mathrm{iso}}(\mathbb{R}^{2n})$ for $z \in \Omega'$ for some fixed M. It follows that the composite $A(z)\circ B(z) \in \Psi^{2M}_{\infty-\mathrm{iso}}(\mathbb{R}^{2n})$. The symbol is given by the usual formula. Furthermore □

4.10. Resolvent and spectrum

One direct application of analytic Fredholm theory is to the resolvent of an elliptic operator of positive order. For simplicity we assume that $A \in \Psi^m_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$ with $m \in \mathbb{N}$, although the case of non-integral positive order is only slightly more complicated.

Proposition 4.13. *If $A \in \Psi^m_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$, $m \in \mathbb{N}$, and there exists one point $\lambda' \in \mathbb{C}$ such that $A - \lambda'$ and $A^* - \overline{\lambda'}$ both have trivial null space, then*

$$(A - \lambda)^{-1} \in \Psi^{-m}_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N) \tag{4.102}$$

is a meromorphic family with all residues finite rank smoothing operators; the span of the ranges of the residues at any $\tilde\lambda$ is the linear space of generalized eigenvalues, the solutions of

$$(A - \tilde\lambda)^p u = 0 \text{ for some } p \in \mathbb{N}. \tag{4.103}$$

Proof. Since A is elliptic and of positive integral order, m, $A - \lambda \in \Psi^m_{\mathrm{iso}}(\mathbb{R}^n)$ is and entire elliptic family. By hypothesis, its inverse exists for some $\lambda' \in \mathbb{C}$. Thus, by Proposition **??** $(A - \lambda)^{-1} \in \Psi^{-m}_{\mathrm{iso}}(\mathbb{R}^n)$ is a meromorphic family in the complex plane, with all residues finite rank smoothing operators.

Let $\tilde{\lambda}$ be a pole of $A-\lambda$. Since we can replace A by $A-\tilde{\lambda}$ we may suppose without loss of generality that $\tilde{\lambda}=0$. Thus, for some k the product $\lambda^k(A-\lambda)^{-1}$ is holomorphic near $\lambda=0$. Differentiating the identities

$$(A-\lambda)[\lambda^k(A-\lambda)^{-1}]=\lambda^k\operatorname{Id}=[\lambda^k(A-\lambda)^{-1}](A-\lambda)$$

up to k times gives the relations

$$\begin{gathered}A\circ R_{k-j}=R_{k-j}\circ A=R_{k-j+1},\ j=0,\cdots,k-1,\\ A\circ R_0=R_0\circ A=\operatorname{Id}+R_1,\ \text{where}\\ (A-\lambda)^{-1}=R_k\lambda^{-k}+R_{k-1}\lambda^{-k+1}+\cdots+R_0+\cdots,\ R_{k+1}=0.\end{gathered}\tag{4.104}$$

Thus $A^p\circ R_{k-p+1}=0=R_{k-p+1}\circ A^p$ for $0<p\le k$, which shows that all the residues, R_j, $1\le j\le k$, have ranges in the generalized eigenfunctions. □

Notice also from (4.104) that the range of R_{k-j+1} is contained in the range of R_{k-j} for each $j=0,\ldots,k-1$, and conversely for the null spaces

$$\begin{gathered}\operatorname{Ran}(R_k)\subset\operatorname{Ran}(R_{k-1})\subset\cdots\subset\operatorname{Ran}(R_1)\\ \operatorname{Nul}(R_k)\supset\operatorname{Nul}(R_{k-1})\supset\cdots\supset\operatorname{Nul}(R_1).\end{gathered}$$

Thus,

$$u\in\operatorname{Ran}(R_p),\ p\ge1\Longleftrightarrow\exists\ u_1\in\operatorname{Ran}(R_1)\ \text{s.t.}\ A^{p-1}u_1=u.\tag{4.105}$$

4.11. Residue trace

We have shown, in Proposition 3.4, the existence of a unique trace functional on the residual algebra $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$. We now follow ideas originating with Seeley, [**12**], and developed by Guillemin [**6**], [**7**] and Wodzicki [**15**], [**14**] to investigate the traces on the full algebra $\Psi^{\mathbb{Z}}_{\mathrm{iso}}(\mathbb{R}^n)$ of polyhomogeneous operators of integral order. We will prove the existence of a trace but defer until later the proof of its uniqueness.

Observe that for $A\in\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ the kernel can be written

$$A(x,y)=(2\pi)^{-n}\int e^{i(x-y)\xi}a_L(x,\xi)\,d\xi$$

and hence the trace, from (3.24), becomes

$$\operatorname{Tr}(A)=(2\pi)^{-n}\int_{\mathbb{R}^{2n}}a_L(x,\xi)\,dx\,d\xi\,,\tag{4.106}$$

just the integral of the left-reduced symbol. In fact this is true for *any* amplitude (of order $-\infty$) representing A :

(4.107)

$$A=(2\pi)^{-n}\int e^{i(x-y)}a(x,y,\xi)\,d\xi\Longrightarrow\operatorname{Tr}(A)=(2\pi)^{-n}\int_{\mathbb{R}^{2n}}a(x,x,\xi)\,dx\,d\xi\,.$$

The integral in (4.106) extends by continuity to $a_L\in S^m_\infty(\mathbb{R}^{2n})$ provided $m<-2n$. Thus, as a functional,

$$\operatorname{Tr}:\Psi^{-2n-\epsilon}_{\infty,\mathrm{iso}}(\mathbb{R}^n)\to\mathbb{C},\ \text{for any}\ \epsilon>0.\tag{4.108}$$

To extend it further we need somehow to *regularize* the resultant divergent integral in (4.106) (and to pay the price in terms of properties). One elegant way to do this is to use a holomorphic family as discussed in Section 4.9. Notice that we are passing from the algebra-with-bounds in (4.108) to polyhomogeneous operators.

LEMMA 4.10. *If* $A(z) \in \Psi^z_{\mathrm{iso}}(\mathbb{R}^n)$ *is a holomorphic family then* $f(z) = \mathrm{Tr}(A(z))$, *defined by* (4.107) *when* $\Re(z) < -2n$, *extends to a meromorphic function of* z *with at most simple poles on the divisor*

$$\{-2n, -2n+1, \dots, -1, 0, 1, \dots\} \subset \mathbb{C}\,.$$

PROOF. We know that $A(z) \in \Psi^z_{\mathrm{iso}}(\mathbb{R}^n)$ is a holomorphic family if and only if its left-reduced symbol is of the form

$$\sigma_L(A(z)) = (1 + |x|^2 + |z|^2)^{z/2} a(z; x, \xi)$$

where $a(z; x, y)$ is an entire function with values in $S^0_{\mathrm{phg}}(\mathbb{R}^n)$. For $\Re z < -2n$ the trace of $A(z)$ is

$$f(z) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} (1 + |x|^2 + |\xi|^2)^{z/2} a(z; x, \xi)\, dx\, d\xi\,.$$

Consider the part of this integral on the ball

$$f_1(z) = (2\pi)^{-n} \int_{|x|^2 + |\xi|^2 \le 1} (1 + |x|^2 + |\xi|^2)^{z/2} a(z, x, y)\, dx\, d\xi\,.$$

This is clearly an entire function of z, since the integrand is entire and the domain compact.

To analyze the remaining part $f_2(z) = f(z) - f_1(z)$ let us introduce polar coordinates

$$r = (|x|^2 + |\xi|^2)^{1/2},\ \theta = \frac{(x, \xi)}{r} \in \mathbb{S}^{2n-1}\,.$$

The integral, convergent in $\Re z < -2n$, becomes

$$f_2(z) = (2\pi)^{-n} \int_1^\infty \int_{\mathbb{S}^{2n-1}} (1 + r^2)^{z/2} \tilde{a}(z; r, \theta)\, d\theta r^{2n-1}\, dr.$$

Let us now pass to the radical compactification of $\mathbb{R}^{2n}$ or more prosaically, introduce $t = 1/r \in [0, 1]$ as variable of integration, so

$$f_2(z) = (2\pi)^{-n} \int_0^1 \int_{\mathbb{S}^{2n-1}} t^{-z} (1 + t^2)^{z/2} \tilde{a}(z; \frac{1}{t}, \theta)\, d\theta t^{-2n} \frac{dt}{t}\,.$$

Now the definition of $S^0_{\mathrm{phg}}(\mathbb{R}^{2n})$ reduces to the statement that

$$b(z; t, \theta) = (1 + t^2)^{z/2} \tilde{a}(z; \frac{1}{t}, \theta) \in \mathcal{C}^\infty(\mathbb{C} \times [0, 1] \times \mathbb{S}^{2n-1}) \tag{4.109}$$

is holomorphic in z.

If we replace b by its Taylor series at $t = 0$ to high order,

$$b(z; t, \theta) = \sum_{j=0}^k \frac{t^j}{j!} b_j(z; \theta) + t^{k+1} b_{(k)}(z; t, \theta)\,, \tag{4.110}$$

where $b_{(k)}(z; t, \theta)$ has the same regularity (4.109), then $f_2(z)$ is decomposed as

$$f_2(z) = (2\pi)^{-n} \sum_{j=0}^k \int_0^1 \int_{\mathbb{S}^{2n-1}} \frac{t^{-z+j}}{j!} b_j(z; \theta) t^{-2n} \frac{dt}{t} + f_2^{(k)}(z). \tag{4.111}$$

The presence of this factor t^k in the remainder in (4.110) shows that $f_2^{(k)}(z)$ is holomorphic in $\Re z < -2n + k$. On the other hand the individual terms in the sum in (4.111) can be computed (for $\Re z < -2n$) as

$$(2\pi)^{-n}\left[\frac{t^{-z+j-2n}}{(-z+j-2n)}\right]_0^1 \int_{\mathbb{S}^{2n-1}} b_j(z,\theta)\frac{d\theta}{j!} = (2\pi)^{-n}\frac{1}{(z-j+2n)}\int_{\mathbb{S}^{2n-1}} b_j(z,\theta)\frac{d\theta}{j!}.$$

Each of these terms extends to be meromorphic in the entire complex plane, with a simple pole (at most) at $z = -2n + j$. This shows that $f(z)$ has a meromorphic continuation as claimed. □

By this argument we have actually computed the residues of the analytic continuation of $\mathrm{Tr}(A(z))$ as

$$\lim_{z\to -2n+j}(z-j+2n)\,\mathrm{Tr}(A(z)) = (2\pi)^{-n}\int_{\mathbb{S}^{2n-1}} a_j(\theta)\,d\theta \tag{4.112}$$

when $a_j(\theta) \in \mathcal{C}^\infty(S^{2n-1})$ is the function occurring in the asymptotic expansion of the left symbol of $A(z)$:

$$\sigma_L(A(z)) \sim \sum_{j=0}^{\infty}(|x|^2+|\xi|^2)^{z/2-j}\tilde{a}_j(z,\theta) \tag{4.113}$$
$$|x|^2+|\xi|^2\to\infty,\ \theta = \frac{(x,\xi)}{(|x|^2+|\xi|^2)^{1/2}},\ a_j(\theta) = \tilde{a}_j(-2n+j,\theta).$$

More generally, if $m \in \mathbb{Z}$ and $A(z) \in \Psi^{m+z}_{\mathrm{iso}}(\mathbb{R}^n)$ is a holomorphic family then

$$\begin{gathered}\mathrm{Tr}(A(z)) \text{ is meromorphic with at most}\\ \text{simple poles at } -2n-m+\mathbb{N}_0\,.\end{gathered}$$

Indeed this just follows by considering the family $A(z-m)$.

We are especially interested in the behavior at $z = 0$. Since the residue there is an integral of the term of order $-2n$, we know that

$$\begin{gathered}A(z) \in \Psi^{m+z}_{\mathrm{iso}}(\mathbb{R}^n) \text{ holomorphic with } A(0) = 0\\ \Longrightarrow \mathrm{Tr}(A(z)) \text{ is regular at } z = 0\,.\end{gathered} \tag{4.114}$$

This allows us to make the following definition:

$$\begin{gathered}\mathrm{Tr}_{\mathrm{Res}}(A) = \lim_{z\to 0} z\,\mathrm{Tr}(A(z)) \text{ if}\\ A(z) \in \Psi^{m+z}_{\mathrm{iso}}(\mathbb{R}^n) \text{ is holomorphic with } A(0) = A\,.\end{gathered}$$

We know that such a holomorphic family exists, since we showed in Section 4.9 the existence of a holomorphic family $F(z) \in \Psi^{z}_{\mathrm{iso}}(\mathbb{R}^n)$ with $F(0) = \mathrm{Id}$; $A(z) = AF(z)$ is therefore an example. Similarly we know that $\mathrm{Tr}_{\mathrm{Res}}(A)$ is independent of the choice of holomorphic family $A(z)$ because of (4.114) applied to the difference, which vanishes at zero.

Lemma 4.11. *The residue functional* $\mathrm{Tr}_{\mathrm{Res}}(A)$, $A \in \Psi^{\mathbb{Z}}_{\mathrm{iso}}(\mathbb{R}^n)$, *is a trace:*

$$\mathrm{Tr}_{\mathrm{Res}}([A,B]) = 0\ \forall\ A,B \in \Psi^{\mathbb{Z}}_{\mathrm{iso}}(\mathbb{R}^n) \tag{4.115}$$

which vanishes on $\Psi^{-2n-1}_{\text{iso}}(\mathbb{R}^n)$ *and is given explicitly by*

$$\operatorname{Tr}_{\text{Res}}(A) = (2\pi)^{-n} \int_{\mathcal{S}^{2n-1}} a_{-2n}(\theta)\, d\theta \tag{4.116}$$

where $a_{-2n}(\theta)$ *is the term of order* $-2n$ *in the expansion of the left (or right) symbol of* a.

PROOF. We have already shown that $\operatorname{Tr}_{\text{Res}}(A)$ is well-defined and (4.116) follows from (4.112) with $a_{-2n}(\theta)$ the term of order $-2n$ in the left-reduced symbol of $A = A(0)$. On the other hand, the same argument applies for the right-reduced symbol.

To see (4.115) just note that if $A(z)$ and $B(z)$ are holomorphic families with $A(0) = A$, and $B(0) = B$ then $C(z) = [A(z), B(z)]$ is a holomorphic family with $C(0) = [A, B]$. On the other hand, $\operatorname{Tr}(C(z)) = 0$ when $\Re z \gg 0$, so the analytic continuation of $\operatorname{Tr}(C(z))$ vanishes identically and (4.115) follows. □

As we shall see below, $\operatorname{Tr}_{\text{Res}}$ is the unique trace (up to a multiple of course) on $\Psi^{\mathbb{Z}}_{\text{iso}}(\mathbb{R}^n)$.

4.12. Exterior derivation

Let $A(z) \in \Psi^z_{\text{iso}}(\mathbb{R}^n)$ be a holomorphic family with $A(0) = \operatorname{Id}$. Then

$$G(z) = A(z) \cdot A(-z) \in \Psi^0_{\text{iso}}(\mathbb{R}^n)$$

is a holomorphic family of fixed order with $G(0) = \operatorname{Id}$. By analytic Fredholm theory

$$G^{-1}(z) \in \Psi^0_{\text{iso}}(\mathbb{R}^n) \text{ is a meromorphic family with finite rank poles.} \tag{4.117}$$

It follows that $A^{-1}(z) = A(-z)G^{-1}(z)$ is a meromorphic family of order $-z$ with at most finite rank poles and regular near 0. Set

$$\Psi^m_{\text{iso}}(\mathbb{R}^n) \ni B \mapsto A(z)BA^{-1}(z) = B(z)\,. \tag{4.118}$$

Thus $B(z)$ is a meromorphic family of order m with $B(0) = B$. The derivative gives a linear map.

$$\Psi^m_{\text{iso}}(\mathbb{R}^n) \ni B \mapsto D_A B = \frac{d}{dz} A(z)BA^{-1}(z)|_{z=0} \in \Psi^m_{\text{iso}}(\mathbb{R}^n)\,. \tag{4.119}$$

PROPOSITION 4.14. *For any holomorphic family of order* z, *with* $A(0) = \operatorname{Id}$, *the map* (4.119), *defined through* (4.118), *is a derivation and for two choices of* $A(z)$ *the derivations differ by an inner derivation.*

PROOF. Since

$$A(z)B_1B_2A^{-1}(z) = A(z)B_1A^{-1}(z)A(z)B_2A^{-1}(z)$$

it follows that

$$\frac{d}{dz} A(z)B_1B_2A^{-1}(z)|_{z=0} = (D_AB_1) \circ B_2 + B_1 \circ (D_AB_2)\,.$$

If $A_1(z)$ and $A_2(z)$ are two holomorphic families of order z with $A_1(0) = A_2(0) = \operatorname{Id}$ then

$$A_2(z) = A_1(z)G(z)$$

when $G(z) \in \Psi^\infty_{\text{iso}}(\mathbb{R}^n)$ is a meromorphic family, with finite rank poles. Thus

$$\begin{aligned} A_2(z)BA_2^{-1}(z) &= A_1(z)G(z)BG^{-1}(z)A_1^{-1}(z) \\ &= A_1(z)BA^{-1}(z) + zA_1(z)H(z)A_1^{-1}(z)\,. \end{aligned}$$

Here $H(z) = (G(z)BG^{-1}(z) - B)/z$ is a holomorphic family of degree m with $H(0) = G'(0)B - BG'(0)$. Thus

$$\frac{d}{dz}A_2(z)BA_2^{-1}(z)|_{z=0} = \frac{d}{dz}A_1(z)BA^{-1}(z)|_{z=0} + [G'(0), B]$$

which shows that the two derivations differ by an inner derivation, which is to say commutation with an element of $\Psi^0_{\text{iso}}(\mathbb{R}^n)$. □

Note that in fact

$$D_A : \Psi^m_{\text{iso}}(\mathbb{R}^n) \to \Psi^{m-1}_{\text{iso}}(\mathbb{R}^n)\ \forall\ m$$

since the symbol of $A(z)BA^{-1}(z)$ is equal to the principal symbol of B for all z.

For the specific choice of $A(z) = H(z)$ given by

$$\sigma_L(H(z)) = (1 + |x|^2 + |\xi|^2)^{z/2}$$

we shall set

$$D_A B = D_H B\,.$$

Observe that $\frac{1}{2}\log(1 + |x|^2 + |\xi|^2) \in S^{\epsilon}_{\infty}(\mathbb{R}^{2n})\ \forall\ \epsilon > 0$. Thus $\log(1 + |x|^2 + |\xi|^2)$, defined by Weyl quantization, is an element of $\Psi^{-\epsilon}_{\infty-\text{iso}}(\mathbb{R}^n)$ for all $\epsilon > 0$. By differentiation the symbols satisfy

$$D_H B = [\frac{1}{2}\log(1 + |x|^2 + |D|^2), B] + [G, B]$$

where $G \in \Psi^{-1}_{\text{iso}}(\mathbb{R}^n)$. Thus D_H is *not* itself an interior derivation. It is therefore an *exterior* derivation.

4.13. Regularized trace

In Section 4.11 we defined the residue trace of B as the residue at $z = 0$ of the analytic continuation of $\text{Tr}(BA(z))$, where $A(z)$ is a holomorphic family of order z with $A(0) = \text{Id}$. Next we consider the functional

$$\overline{\text{Tr}}_{\text{A}}(B) = \lim_{z=0}(\text{Tr}(BA(z)) - \frac{1}{z}\text{Tr}_{\text{Res}}(B))\,. \tag{4.120}$$

In contrast to the residue trace, $\overline{\text{Tr}}_{\text{A}}(z)$ *does* depend on the choice of analytic family $A(z)$.

Lemma 4.12. *If $A_i(z)$, $i = 1, 2$, are two holomorphic families of order z with $A_i(0) = \text{Id}$ and $G'(0) = \frac{d}{dz}A_2(z)A_1^{-2}(z)|_{z=0}$ then*

$$\overline{\text{Tr}}_{\text{A}_2}(B) - \overline{\text{Tr}}_{\text{A}_1}(B) = \text{Tr}_{\text{Res}}(BG'(0))\,. \tag{4.121}$$

Proof. Writing $G(z) = A_2(z)A_1^{-1}(z)$, which is a meromorphic family of order 0 with $G(0) = \text{Id}$,

$$\begin{aligned}\text{Tr}(BA_2(z)) &= \text{Tr}(BG(z)A_1(z)) \\ &= \text{Tr}(BA_1(z)) + z\,\text{Tr}(BG'(0)A_1(z)) + z^2\,\text{Tr}(H(z)A_1(z))\end{aligned}$$

where $H(z) = \frac{B}{z^2}(G(z) - \text{Id} - zG'(0))$ is then meromorphic with only finite rank poles and is regular near $z = 0$. Thus the analytic continuation of $z^2\,\text{Tr}(H(z)A(z))$ vanishes at zero from which (4.121) follows. □

This regularized trace $\overline{\text{Tr}}_{\text{A}}(B)$ therefore only depends on the first order, in z, term in $A(z)$ at $z = 0$. It is important to note that it is *not* itself a trace.

LEMMA 4.13. *If* $B_1,\ B_2 \in \Psi^{\mathbb{Z}}_{\mathrm{iso}}(\mathbb{R}^n)$ *then*

$$\overline{\operatorname{Tr}}_{\mathrm{A}}([B_1, B_2]) = \operatorname{Tr}_{\mathrm{Res}}(B_2 D_A B_1)\,. \tag{4.122}$$

PROOF. Since $\overline{\operatorname{Tr}}_{\mathrm{A}}([B_1, B_2])$ is the regularized value at 0 of the analytic continuation of the trace of

$$\begin{aligned} B_1 B_2 A(z) - B_2 B_1 A(z) &= B_2[A(z), B_1] + [B_1, B_2 A(z)] \\ &= B_2([A(z), B]A^{-1}(z))A_1(z) + [B_1 B_2 A(z)]\,. \end{aligned} \tag{4.123}$$

The second term on the right in (4.123) has zero trace before analytic continuation. Thus $\overline{\operatorname{Tr}}_A([B_1, B_2])$ is the regularized value of the analytic continuation of the trace of $Q(z)A(z)$ where

$$Q(z) = B_2[A(z), B_1]A^{-1}(z) = zD_A B_1 + z^2 L(z)$$

with $L(z)$ meromorphic of fixed order and regular at $z = 0$. Thus (4.122) follows. □

Note that

$$\operatorname{Tr}_{\mathrm{Res}}(D_A B) = 0\ \forall\ B \in \Psi^{\mathbb{Z}}_{\mathrm{iso}}(\mathbb{R}^n) \tag{4.124}$$

and any family A. Indeed the residue trace is the residue of $z = 0$ of the analytic continuation of $\operatorname{Tr}(H(z)A(z))$ when $A(z)$ is any meromorphic family of fixed order with $H(0) = D_A B$. In particular we can take

$$H(z) = \tfrac{1}{z}(A(z)BA^{-1}(z) - B)\,.$$

Then $H(z)A(z) = \frac{1}{z}[A(z), B]$ so the trace vanishes before analytic continuation.

4.14. Projections

4.15. Complex powers

4.16. Index and invertibility

We have already seen that the elliptic elements

$$E^0_{\mathrm{iso}}(\mathbb{R}^n; \mathbb{C}^N) \subset \Psi^0_{\mathrm{iso}}(\mathbb{R}^n; \mathbb{C}^N) \hookrightarrow \mathcal{B}(L^2(\mathbb{R}^n; \mathbb{C}^N)) \tag{4.125}$$

define Fredholm operators. The *index* of such an operator

$$\operatorname{Ind}(A) = \dim \operatorname{Nul}(A) - \dim \operatorname{Nul}(A^*) \tag{4.126}$$

is a measure of its non-invertibility. Set

$$E^0_{\mathrm{iso},k}(\mathbb{R}^n; \mathbb{C}^N) = \left\{A \in E^0_{\mathrm{iso}}(\mathbb{R}^n; \mathbb{C}^N); \operatorname{Ind}(A) = k\right\},\ k \in \mathbb{Z}. \tag{4.127}$$

PROPOSITION 4.15. *If* $A \in E^0_{\mathrm{iso}}(\mathbb{R}^n; \mathbb{C}^N)$ *and* $\operatorname{Ind}(A) = 0$ *then there exists* $E \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n; \mathbb{C}^N)$ *such that* $A + E$ *is invertible in* $\mathcal{B}(L^2(\mathbb{R}^n; \mathbb{C}^N))$ *and the inverse then lies in* $\Psi^0_{\mathrm{iso}}(\mathbb{R}^n; \mathbb{C}^N)$.

PROOF. Let B be the generalized inverse of A, assumed to be elliptic. The assumption that $\operatorname{Ind}(A) = 0$ means that $\operatorname{Nul}(A)$ and $\operatorname{Nul}(A^*)$ have the same dimension. Let $e_1, \cdots, e_p \in \mathcal{S}(\mathbb{R}^n; \mathbb{C}^N)$ and $f_1, \cdots, f_p \in \mathcal{S}(\mathbb{R}^n; \mathbb{C}^N)$ be bases of $\operatorname{Nul}(A)$ and $\operatorname{Nul}(A^*)$. Then consider

$$E = \sum_{j=1}^{p} f_j(x)\overline{e_j(y)} \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n; \mathbb{C}^N). \tag{4.128}$$

By construction E is an isomorphism (in fact an arbitrary one) between $\operatorname{Nul}(A)$ and $\operatorname{Nul}(A^*)$. Thus $A+E$ is continuous, injective and surjective, hence has an inverse in $\mathcal{B}(L^2(\mathbb{R}^n;\mathbb{C}^N))$. Indeed this inverse is $B+E^{-1}$ where E^{-1} is the inverse of E as a map from $\operatorname{Nul}(A)$ to $\operatorname{Nul}(A^*)$. This shows that A can be perturbed by a smoothing operator to be invertible. □

Let

$$G^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N) \subset E^0_{\mathrm{iso},0}(\mathbb{R}^n;\mathbb{C}^N) \subset E^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N) \subset \Psi 0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N) \tag{4.129}$$

denote the group of the invertible elements (invertibility being either in $\mathcal{B}(L^2(\mathbb{R};\mathbb{C}^N)$ or in $\Psi^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$) in the ring of elliptic elements of index 0.

Corollary 4.5. *The first inclusion in* (4.129) *is dense in the topology of* $\Psi^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$.

Proof. This follows from the proof of Proposition 4.15, since $A+sE$ is invertible for all $s \neq 0$. □

We next derive some simple formulæ for the index of an element of $E^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$. First observe that the trace of a finite dimensional projection is its rank, the dimension of its range. Thus

$$\operatorname{Ind}(A) = \operatorname{Tr}(\Pi_{\operatorname{Nul}(A)}) - \operatorname{Tr}(\Pi_{\operatorname{Nul}(A^*)}) \tag{4.130}$$

where the trace may be reinterpreted as the trace on smoothing operators. The identities, (4.15), satisfied by the generalized inverse of A shows that this can be rewritten

$$\operatorname{Ind}(A) = -\operatorname{Tr}(BA - \mathrm{Id}) + \operatorname{Tr}(AB - \mathrm{Id}) = \operatorname{Tr}([A,B]). \tag{4.131}$$

Here $[A,B] = \Pi_{\operatorname{Nul}(A)} - \Pi_{\operatorname{Nul}(A^*)}$ is a smoothing operator, even though both A and B are elliptic of order 0.

Lemma 4.14. *If* $A \in E^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$ *the identity* (4.131)*, which may be rewritten*

$$\operatorname{Ind}(A) = \operatorname{Tr}([A,B]), \tag{4.132}$$

holds for any parametrix B.

Proof. If B' is a parametrix and B is the generalized inverse then $B' - B = E \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$. Thus

$$[A,B'] = [A,B] + [A,E].$$

Since $\operatorname{Tr}([A,E] = 0$, one of the arguments being a smoothing operator, (4.132) follows in general from the particular case (4.131). □

Note that it follows from (4.132) that $\operatorname{Ind}(A) = \operatorname{Ind}(A+E)$ if E is smoothing. In fact the index is even more stable than this as we shall see, since it is locally constant on $E^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$. In any case this shows that

$$\begin{gathered}\operatorname{Ind} : \mathcal{E}^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N) \longrightarrow \mathbb{Z},\ \operatorname{Ind}(a) = \operatorname{Ind}(A) \text{ if } a = [A],\\ \mathcal{E}^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N) = E^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)/\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)\\ \subset \mathcal{A}^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N) = \Psi^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)/\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)\end{gathered} \tag{4.133}$$

is well-defined.

The argument of the trace functional in (4.132) is a smoothing operator, but we may still rewrite the formula in terms of the regularized trace, with respect to the standard regularizer $H(z)$ with left symbol $(1+|x|^2+|\xi|^2)^{\frac{z}{2}}$. The advantage of doing so is that we can then use the trace defect formula (4.122). Thus for any elliptic isotropic operator of order 0

$$\operatorname{Ind}(A) = \operatorname{Tr}_{\mathrm{Res}}(BD_H A). \tag{4.134}$$

Here B is a parametrix for A. The residue trace is actually a functional

$$\operatorname{Tr}_{\mathrm{Res}} : \mathcal{A}^{\mathbb{Z}}_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N) \longrightarrow \mathbb{C},$$

so if we write a^{-1} for the inverse of a in the ring $\mathcal{E}^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$ then

$$\operatorname{Ind}(a) = \operatorname{Tr}_{\mathrm{Res}}(a^{-1}D_H a),\ D_H : \mathcal{A}^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N) \longrightarrow \mathcal{A}^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N) \tag{4.135}$$

being the induced derivation (since D_H clearly preserves the ideal $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$.

From this simple formula we can easily deduce two elementary properties of elliptic operators. These actually hold in general for Fredholm operators, although the proofs here are not valid in that generality. Namely

$$\operatorname{Ind} : \mathcal{E}^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N) \longrightarrow \mathbb{Z} \text{ is locally constant and} \tag{4.136}$$

$$\operatorname{Ind}(a_1a_2) = \operatorname{Ind}(a_1) + \operatorname{Ind}(a_2)\ \forall\ a_1,\ a_2 \in \mathcal{E}^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N). \tag{4.137}$$

The first of these follows the continuity of the formula (4.135) since under deformation of a in $\mathcal{E}^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$ the inverse a^{-1} varies continuously, so Ind is continuous and integer-valued, hence locally constant. Similarly the second, logarithmic additivity, property follows from the fact that D_H is a derivation, so

$$D_H(a_1a_2) = (D_Ha_1)a_2 + a_1D_Ha_2$$

and the the trace property of $\operatorname{Tr}_{\mathrm{Res}}$ which shows that

$$\begin{aligned}\operatorname{Ind}(a_1a_2) &= \operatorname{Tr}_{\mathrm{Res}}((a_1a_2)^{-1}D_H(a_1a_2) = \operatorname{Tr}(a_2^{-1}a_1^{-1}\left((D_Ha_1)a_2 + a_1D_Ha_2\right) \\ &= \operatorname{Tr}(a_2^{-1}a_1^{-1}(D_Ha_1)a_2) + \operatorname{Tr}(a_2^{-1}D_Ha_2) = \operatorname{Ind}(a_1) + \operatorname{Ind}(a_2).\end{aligned} \tag{4.138}$$

4.17. Variation 1-form

In the previous section we have seen that the index

$$\operatorname{Ind} : E^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N) \longrightarrow \mathbb{Z} \tag{4.139}$$

is a multiplicative map which is the obstruction to perturbative invertibility. In the next two sections we will derive a closely related obstruction to the perturbative invertibility of a family of elliptic operators. Thus, suppose

$$Y \ni y \longmapsto A_y \in E^0_{\mathrm{iso},0}(\mathbb{R}^n;\mathbb{C}^N) \tag{4.140}$$

is a family of elliptic operators depending smoothly on a parameter in the compact manifold Y. We are interested in the *families perturbative invertibility question.* That is, does there exist a smooth family

$$Y \ni y \longmapsto E_y \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N) \text{ such that } (A_y + E_y) \in G^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)\ \forall\ y. \tag{4.141}$$

We have assumed that the operators have index zero since this is necessary (and sufficient) for E_y to exist for any one $y \in Y$. Thus the issue is the smoothness (really just the continuity) of the perturbation E_y.

We shall start by essentially writing down such a putative obstruction directly and then subsequently we shall investigate its topological origins.

PROPOSITION 4.16. *If a smooth family* (4.140), *parameterized by a compact manifold* Y, *is perturbatively invariant in the sense that there is a smooth family as in* (4.141), *then the closed* 2*-form on* Y

$$\beta = \frac{1}{2}\operatorname{Tr}_{\mathrm{Res}}(a_y^{-1}d_y a_y \wedge a_y^{-1}d_y a_y \cdot a_y^{-1}D_H a_y) \in \mathcal{C}^\infty(Y;\Lambda^2), \quad a_y = [A_y] \in \mathcal{E}^0_{\mathrm{iso},0}(\mathbb{R}^n;\mathbb{C}^N), \tag{4.142}$$

is exact.

PROOF. Note first that β is indeed a smooth form, since the full symbolic inverse depends smoothly on parameters. Next we show that β is always closed. The 1-forms $a_y^{-1}d_y a_y a_y^{-1}$ and da_y are exact so differentiating directly gives

$$\begin{aligned} d\beta &= \frac{1}{2}\operatorname{Tr}_{\mathrm{Res}}(a_y^{-1}d_y a_y \wedge a_y^{-1}d_y a_y \wedge d(a_y^{-1}D_H a_y)) \\ &= -\frac{1}{2}\operatorname{Tr}_{\mathrm{Res}}(a_y^{-1}d_y a_y \wedge a_y^{-1}d_y a_y \wedge a_y^{-1}d_y a_y^{-1}D_H a_y)) \\ &\qquad +\frac{1}{2}\operatorname{Tr}_{\mathrm{Res}}(a_y^{-1}d_y a_y \wedge a_y^{-1}d_y a_y \wedge a_y^{-1}D_H(da_y)) \\ &= \frac{1}{2}\operatorname{Tr}_{\mathrm{Res}}(a_y^{-1}d_y a_y \wedge a_y^{-1}d_y a_y \wedge D_H(a_y^{-1}da_y)). \end{aligned} \tag{4.143}$$

Using the trace property and the commutativity of a 2-form with other forms the last expression can be written

$$\frac{1}{6}\operatorname{Tr}_{\mathrm{Res}}(D_H(a_y^{-1}d_y a_y \wedge a_y^{-1}d_y a_y \wedge a_y^{-1}da_y)) = 0 \tag{4.144}$$

by property (4.124) of the residue trace.

Now, suppose that a smooth perturbation as in (4.141) does exist. We can replace A_y by $A_y + E_y$ without affecting β, since the residue trace vanishes on the ideal of smoothing operators. Thus we can assume that A_y itself is invertible. Then consider the 1-form defined using the regularized trace

$$\overline{\alpha} = \overline{\operatorname{Tr}}_{\mathrm{H}}(A_y^{-1}d_y A_y). \tag{4.145}$$

This is an extension of the 1-form $d\log\det_F$ on $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$. The extension is not in general closed, because the regularized trace does not satisfy the trace condition. Using the stanadard formula for the variation of the inverse, $dA_y^{-1} = -A_y^{-1}dA_y A_y^{-1}$, the exterior derivative is the 2-form

$$d\overline{\alpha} = -\overline{\operatorname{Tr}}_{\mathrm{H}}(A_y^{-1}(d_y A)A_y^{-1}d_y A_y). \tag{4.146}$$

The 2-form argument is a commutator. Indeed, in terms of local coordinates we can write

$$\begin{aligned} A_y^{-1}(d_y A)A_y^{-1}d_y A_y &= \sum_{j,k=1}^{p} A_y^{-1}(\frac{\partial A}{\partial y_j})A_y^{-1}(\frac{\partial A}{\partial y_k})dy_j \wedge dy_k \\ &= \frac{1}{2}\sum_{j,k=1}^{p}\left(A_y^{-1}(\frac{\partial A}{\partial y_j})A_y^{-1}(\frac{\partial A}{\partial y_k}) - A_y^{-1}(\frac{\partial A}{\partial y_k})A_y^{-1}(\frac{\partial A}{\partial y_j})\right)dy_j \wedge dy_k \\ &= \frac{1}{2}\sum_{j,k=1}^{p}[A_y^{-1}(\frac{\partial A}{\partial y_j}), A_y^{-1}(\frac{\partial A}{\partial y_k})]dy_j \wedge dy_k \end{aligned}$$

Applying the trace defect formula (4.122) shows that

$$d\overline{\alpha} = -\frac{1}{2}\operatorname{Tr}_{\mathrm{Res}}\left(A_y^{-1}d_yA_y \wedge D_H(A_y^{-1}d_yA_y)\right), \tag{4.147}$$

locally and hence globally.

Expanding the action of the derivation D_H gives

$$d\overline{\alpha} = \beta - \frac{1}{2}\operatorname{Tr}_{\mathrm{Res}}\left(A_y^{-1}d_yA_y \wedge A_y^{-1}d_y(D_HA_y)\right) = \beta - d\gamma, \text{ where}$$
$$\gamma = \frac{1}{2}\operatorname{Tr}_{\mathrm{Res}}\left(A_y^{-1}d_yA_y \wedge .A_y^{-1}D_HA_y\right). \tag{4.148}$$

We conclude that if A_y has an invertible lift then β is exact. □

Note that the form γ in (4.148) is well-defined as a form on $\mathcal{E}^0_{\mathrm{iso},0}(\mathbb{R}^n;\mathbb{C}^N)$, and is independent of the perturbation. Thus the cohomology class which we have constructed as the obstruction to perturbative invertibility can be written

$$[\beta] = [\beta - d\gamma] \in H^2(\mathcal{E}^0_{\mathrm{iso},0}(\mathbb{R}^n;\mathbb{C}^N)). \tag{4.149}$$

4.18. Determinant bundle

To better explain the topological origin of the cohomology class (4.149) we construct the determinant bundle. This was originally introduced for families of Dirac operators by Quillen [**11**]. Recall that the Fredholm determinant is a character

$$\det_{\mathrm{Fr}} : \mathrm{Id} + \Psi^{-2n-1}_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N) \longrightarrow \mathbb{C},$$
$$\det_{\mathrm{Fr}}(AB) = \det_{\mathrm{Fr}}(A)\det_{\mathrm{Fr}}(B) \forall\ A,\ B \in \mathrm{Id} + \Psi^{-2n-1}_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N). \tag{4.150}$$

As we shall see, it is not possible to extend the Fredholm determinant as a multiplicative function to $G^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$, essentially because of the non-extendibility of the trace.

However in trying to extend the determinant we can consider the possible values it would take on a point $A \in G^0_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$ as the set of pairs (A,z), $z \in \mathbb{C}$. Thus we simple consider the product

$$D^0 = G^0 \times \mathbb{C}, \tag{4.151}$$

where from now on we simplify the notation and write $G^0 = G_{\mathrm{iso}}(\mathbb{R}^n;\mathbb{C}^N)$ etc. Although it is not reasonable to expect full multiplicative of the determinant, it is more reasonable to expect the determinant of $A(\mathrm{Id}+B)$, $B \in \Psi^{-2n-1}$ to be related to the product of determinants. Thus it is natural to identify pairs in D^0,

$$(A,z) \sim_p (A',z') \text{ if}$$
$$A, A' \in G^0,\ A' = A(\mathrm{Id}+B),\ z' = \det_{\mathrm{Fr}}(\mathrm{Id}+B)z,\ B \in \Psi^p,\ p < -2n. \tag{4.152}$$

The equivalence relations here are slightly different, depending on p. In all cases the action of the determinant is linear, so the quotient is a line bundle.

Lemma 4.15. *For any integer $p < -2n$, and also $p = -\infty$, the quotient*

$$\mathcal{D}^0_p = D^0/\sim_p \tag{4.153}$$

is a smooth line bundle over $\mathcal{G}^0_p = G^0/G^p$.

PROOF. The projection is just the quotient in the first factor and this clearly defines a commutative square

$$\begin{array}{ccc} D^0 & \xrightarrow{[\sim_p]} & \mathcal{D}^0_p \\ \downarrow \pi & & \downarrow \pi \\ G^0 & \xrightarrow{/G^p} & \mathcal{G}^0_p. \end{array} \tag{4.154}$$

□

4.19. Index bundle

4.20. Index formulæ

4.21. Isotropic essential support

4.22. Isotropic wavefront set

4.23. Isotropic FBI transform

4.24. Problems

PROBLEM 4.1. Define the *isotropic* Sobolev spaces of integral order by

$$H^k_{\text{iso}}(\mathbb{R}^n) = \begin{cases} \left\{u \in L^2(\mathbb{R}^n); x^\alpha D^\beta_x u \in L^2(\mathbb{R}^n) \ \forall\ |\alpha| + |\beta| \le k\right\} & k \in \mathbb{N} \\ \left\{ u \in \mathcal{S}'(\mathbb{R}^n); u = \sum\limits_{|\alpha|+|\beta| \le -k} x^\alpha D^\beta_x u_{\alpha,\beta},\ u_{\alpha,\beta} \in L^2(\mathbb{R}^n) \right\} & k \in -\mathbb{N}. \end{cases} \tag{4.155}$$

Show that if $A \in \Psi^p_{\text{iso}}(\mathbb{R}^n)$ with p an integer, then $A : H^k_{\text{iso}}(\mathbb{R}^n) \longrightarrow H^{k-p}_{\text{iso}}(\mathbb{R}^n)$ for any integral k. Deduce (using the properties of elliptic isotropic operators) that the general definition

$$H^m_{\text{iso}}(\mathbb{R}^n) = \left\{u \in \mathcal{S}'(\mathbb{R}^n); Au \in L^2(\mathbb{R}^n),\ \forall\ A \in \Psi^{-m}_{\text{iso}}(\mathbb{R}^n)\right\},\ m \in \mathbb{R} \tag{4.156}$$

is consistent with (4.155) and has the properties

$$A \in \Psi^M_{\text{iso}}(\mathbb{R}^n) \Longrightarrow A : H^m_{\text{iso}}(\mathbb{R}^n) \longrightarrow H^{m-M}_{\text{iso}}(\mathbb{R}^n), \tag{4.157}$$

$$\bigcap_m H^m_{\text{iso}}(\mathbb{R}^n) = \mathcal{S}(\mathbb{R}^n),\ \bigcup_m H^m_{\text{iso}}(\mathbb{R}^n) = \mathcal{S}'(\mathbb{R}^n) \tag{4.158}$$

$$A \in \Psi^m_{\text{iso}}(\mathbb{R}^n),\ u \in \mathcal{S}'(\mathbb{R}^n),\ Au \in H^{m'}(\mathbb{R}^n) \Longrightarrow u \in H^{m'-m}(\mathbb{R}^n), \tag{4.159}$$

PROBLEM 4.2. Show that if $\epsilon > 0$ then

$$H^\epsilon_{\text{iso}}(\mathbb{R}^n) \subsetneq (1+|x|)^{-\epsilon} L^2(\mathbb{R}^n) \cap H^\epsilon(\mathbb{R}^n)$$

Deduce that $H^\epsilon_{\text{iso}}(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n)$ is a compact inclusion.

PROBLEM 4.3. Using Problem 4.2, or otherwise, show that each element of $\Psi^{-\epsilon}_{\text{iso}}(\mathbb{R}^n)$, $\epsilon > 0$, defines a compact operator on $L^2(\mathbb{R}^n)$.

PROBLEM 4.4. Show that if $E \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ then there exists $F \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ such that

$$(\text{Id} + E)(\text{Id} + F) = \text{Id}_G \text{ with } G \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n) \text{ of finite rank,}$$

that is, $G \cdot \mathcal{S}(\mathbb{R}^n)$ is finite dimensional.

Problem 4.5. Using Problem 4.4 show that an elliptic element $A \in \Psi^m_{\text{iso}}(\mathbb{R}^n)$ has a parametrix $B \in \Psi^{-m}_{\text{iso}}(\mathbb{R}^n)$ *up to finite rank error*; that is, such that $A \circ B - \operatorname{Id}$ and $B \circ A - \operatorname{Id}$ are finite rank elements of $\Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$. Deduce that such an elliptic A defines a Fredholm operator

$$A : H^M_{\text{iso}}(\mathbb{R}^n) \longrightarrow H^{M-m}_{\text{iso}}(\mathbb{R}^n)$$

for any M. [The requirements for an operator A between Hilbert spaces to be Fredholm are that it be bounded, have finite-dimensional null space and closed range with a finite-dimensional complement.]

Problem 4.6. [The harmonic oscillator] Show that the 'harmonic oscillator'

$$H = |D|^2 + |x|^2,\ Hu = \sum_{j=1}^n D_j^2 u + |x|^2 u,$$

is an elliptic element of $\Psi^2_{\text{iso}}(\mathbb{R}^n)$. Consider the 'creation' and 'annihilation' operators

$$C_j = D_j + ix_j,\ A_j = D_j - ix_j = C_j^*, \tag{4.160}$$

and show that

$$H = \sum_{j=1}^n C_j A_j + n = \sum_{j=1}^n A_j C_j - n, \tag{4.161}$$

$$[A_j, H] = 2A_j,\ [C_j, H] = -2C_j,\ [C_l, C_j] = 0,\ [A_l, A_j] = 0,\ [A_l, C_j] = 2\delta_{lk} \operatorname{Id},$$

where $[A, B] = A \circ B - B \circ A$ is the commutator bracket and δ_{lk} is the Kronecker symbol. Knowing that $(H - \lambda)u = 0$, for $\lambda \in \mathbb{C}$ and $u \in \mathcal{S}'(\mathbb{R}^n)$ implies $u \in \mathcal{S}(\mathbb{R}^n)$ (why?) show that

$$E_\lambda = \{u \in \mathcal{S}'(\mathbb{R}^n); (H - \lambda)u = 0\} \neq \{0\} \Longleftrightarrow \lambda \in n + 2\mathbb{N}_0 \tag{4.162}$$

$$\text{and } E_{-n+2k} = \left\{ \sum_{|\alpha|=k} c_\alpha C^\alpha \exp(-|x|^2/2),\ c_\alpha \in \mathbb{C} \right\},\ k \in \mathbb{N}_0. \tag{4.163}$$

Problem 4.7. [Definition of determinant of matrices.]

Problem 4.8. [Proof that $d\alpha = 0$ in (3.32).] To prove that the 1-form is closed it suffices to show that it is closed when restricted to any 2-dimensional submanifold. Thus we may suppose that $A = A(s, t)$ depends on 2 parameters. In terms of these parameters

$$\alpha = \operatorname{Tr}(A(s,t)^{-1} \frac{dA(s,t)}{ds}) ds + \operatorname{Tr}(A(s,t)^{-1} \frac{dA(s,t)}{dt}) dt. \tag{4.164}$$

Show that the exterior derivative can be written

$$d\alpha = \operatorname{Tr}([A(s,t)^{-1} \frac{dA(s,t)}{dt}, A(s,t)^{-1} \frac{dA(s,t)}{ds}]) ds \wedge dt \tag{4.165}$$

and hence that it vanishes.

Problem 4.9. If E and F are vector spaces, show that the space of operators $\Psi^m_{\text{iso}}(\mathbb{R}^n; E, F)$ from $\mathcal{S}'(\mathbb{R}^n; E)$ to $\mathcal{S}'(\mathbb{R}^n; F)$ is well-defined as the matrices with entries in $\Psi^m_{\text{iso}}(\mathbb{R}^n)$ for any choice of bases of E and F.

Problem 4.10. Necessity of ellipticity for a psuedodifferential operator to be Fredholm on the isotropic Sobolev spaces.

(1) Reduce to the case of operators of order 0.
(2) Construct a sequence in L^2 such that $\|u_n\| = 1$, $u_n \to 0$ weakly and $Au_n \to 0$ strongly in L^2.

PROBLEM 4.11. [Koszul complex] Consider the form bundles over $\mathbb{R}^n$. That is $\Lambda^k\mathbb{R}^n$ is the vector space of dimension $\binom{n}{k}$ consisting of the totally antisymmetric k-linear forms on $\mathbb{R}^n$. If $e_1, e_2, \dots, e_n$ is the standard basis for $\mathbb{R}^n$ then for a k-tuple α e^α defined on basis elements by

$$e^\alpha(e_{i_1}, \dots, e_{i_k}) = \prod_{j=1}^{k} \delta_{1_j \alpha_j}$$

extends uniquely to a k-linear map. Elements $dx^\alpha \in \Lambda^k\mathbb{R}^n$ are defined by the total antisymmetrization of the e^α. Explicitly,

$$dx^\alpha(v_1, \dots, v_k) = \sum_\pi \operatorname{sgn} \pi e^\alpha(v_{\pi_1}, \dots, v_{\pi_n})$$

where the sum is over permutations π of $\{1, \dots, n\}$ and $\operatorname{sgn}\pi$ is the parity of π. The dx^α for strictly increasing k-tuples α of elements of $\{1, \dots, n\}$ give a basis for $\Lambda^k\mathbb{R}^n$. The wedge product is defined by $dx^\alpha \wedge dx^\beta = dx^{\alpha,\beta}$.

Now let $\mathcal{S}'(\mathbb{R}^n; \Lambda^k)$ be the tensor product, that is $u \in \mathcal{S}'(\mathbb{R}^n; \Lambda^k)$ is a finite sum

$$u = \sum_\alpha u_\alpha dx^\alpha. \tag{4.166}$$

The annihilation operators in (4.160) define an operator, for each k,

$$D : \mathcal{S}'(\mathbb{R}^n; \Lambda^k) \longrightarrow \mathcal{S}'(\mathbb{R}^n; \Lambda^{k+1}),\ Du = \sum_{j=1}^{n} A_j u_\alpha dx^j \wedge dx^\alpha.$$

Show that $D^2 = 0$. Define inner products on the $\Lambda^k\mathbb{R}^n$ by declaring the basis introduced above to be orthonormal. Show that the adjoint of D, defined with respect to these inner products and the L^2 pairing is

$$D^* : \mathcal{S}'(\mathbb{R}^n; \Lambda^k) \longrightarrow \mathcal{S}'(\mathbb{R}^n; \Lambda^{k-1}),\ D^*u = \sum_{j=1}^{n} C_j u_\alpha \iota_j dx^\alpha.$$

Here, ι_j is 'contraction with e_j;' it is the adjoint of $dx^j \wedge$. Show that $D + D^*$ is an elliptic element of $\Psi^1_{\text{iso}}(\mathbb{R}^n; \Lambda^*)$. Maybe using Problem 4.6 show that the null space of $D + D^*$ on $\mathcal{S}'(\mathbb{R}^n; \Lambda^*\mathbb{R}^n)$ is 1-dimensional. Deduce that

$$\begin{aligned} &\{u \in \mathcal{S}'(\mathbb{R}^n); Du = 0\} = \mathbb{C}\exp(-|x|^2/2), \\ &\qquad\qquad \{u \in \mathcal{S}'(\mathbb{R}^n; \Lambda^k); Du = 0\} = (\mathcal{S}'(\mathbb{R}^n; \Lambda^{k-1}),\ k \geq 1. \end{aligned} \tag{4.167}$$

Observe that, as an operator from $\mathcal{S}'(\mathbb{R}^n; \Lambda^{\text{odd}})$ to $\mathcal{S}'(\mathbb{R}^n; \Lambda^{\text{even}})$, $D + D^*$ is an elliptic element of $\Psi^1_{\text{iso}}(\mathbb{R}^n; \Lambda^{\text{odd}}, \Lambda^{\text{even}})$ and has index 1.

PROBLEM 4.12. [Isotropic essential support] For an element of $S^m(\mathbb{R}^n)$ define (isotropic) essential support, or operator wavefront set, of $A \in \Psi^m_{\text{iso}}(\mathbb{R}^n)$ by

$$\text{WF}_{\text{iso}}(A) = \text{cone}\,\text{supp}(\sigma_L(A)) \subset \mathbb{R}^{2n} \setminus \{0\}. \tag{4.168}$$

Show that $\mathrm{WF}_{\mathrm{iso}}(A) = \operatorname{cone}\operatorname{supp}(\sigma_L(A))$ and check the following

$$\mathrm{WF}'_{\mathrm{iso}}(A+B) \cup \mathrm{WF}'_{\mathrm{iso}}(A \circ B) \subset \mathrm{WF}'_{\mathrm{iso}}(A) \cap \mathrm{WF}'_{\mathrm{iso}}(B), \tag{4.169}$$

$$\mathrm{WF}'_{\mathrm{iso}}(A) = \emptyset \Longleftrightarrow A \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n). \tag{4.170}$$

PROBLEM 4.13. [Isotropic partition of unity] Show that if $U_i \subset \mathbb{S}^{n-1}$ is an open cover of the unit sphere and $\tilde{U}_i = \{Z \in \mathbb{R}^{2n} \setminus \{0\}; \frac{Z}{|Z|} \in U_i\}$ is the corresponding *conic* open cover of $\mathbb{R}^{2n} \setminus \{0\}$ then there exist (finitely many) operators $A_i \in \Psi^0_{\mathrm{iso}}(\mathbb{R}^n)$ with $\mathrm{WF}'_{\mathrm{iso}}(A_i) \subset \tilde{U}_i$, such that

$$\mathrm{Id} - \sum_i A_i \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n). \tag{4.171}$$

PROBLEM 4.14. Suppose $A \in \Psi^m_{\mathrm{iso}}(\mathbb{R}^n)$, is elliptic and has index zero as an operator on $\mathcal{S}'(\mathbb{R}^n)$. Show that there exists $E \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ such that $A + E$ is an isomorphism of $\mathcal{S}'(\mathbb{R}^n)$.

PROBLEM 4.15. [Isotropic wave front set] For $u \in \mathcal{S}'(\mathbb{R}^n)$ define

$$\mathrm{WF}_{\mathrm{iso}}(u) = \bigcap \left\{ \mathrm{WF}'_{\mathrm{iso}}(A); A \in \Psi^0_{\mathrm{iso}}(\mathbb{R}^n),\ Au \in \mathcal{S}(\mathbb{R}^n) \right\}. \tag{4.172}$$