CHAPTER 2

Pseudodifferential operators on Euclidean space

Formula (1.92) for the action of a differential operator (with coefficients in $\mathcal{C}^\infty_\infty(\mathbb{R}^n)$) on $\mathcal{S}(\mathbb{R}^n)$ can be written

$$\begin{aligned} P(x,D)u &= (2\pi)^{-n}\int e^{i(x-y)\cdot\xi}P(x,\xi)u(y)dyd\xi \\ &= (2\pi)^{-n}\int e^{ix\cdot\xi}P(x,\xi)\hat{u}(\xi)d\xi \end{aligned} \tag{2.1}$$

where $\hat{u}(\xi)=\mathcal{F}u(\xi)$ is the Fourier transform of u. We shall generalize this formula by generalizing $P(x,\xi)$ from a polynomial in ξ to a *symbol*, which is to say a smooth function satisfying certain uniformity conditions at infinity. In fact we shall also allow the symbol, or rather the *amplitude*, in the integral (2.1) to depend in addition on the 'incoming' variables, y :

$$A(x,D)u = (2\pi)^{-n}\int e^{i(x-y)\cdot\xi}a(x,y,\xi)u(y)dyd\xi, u\in\mathcal{S}(\mathbb{R}^n). \tag{2.2}$$

Of course it is not immediately clear that this integral is well-defined.

To interpret (2.2) we first look into the definition and properties of symbols. Then we show how this integral can be interpreted as an oscillatory integral and that it thereby defines an operator on $\mathcal{S}(\mathbb{R}^n)$. We then investigate the properties of these *pseudodifferential operators* at some length.

2.1. Symbols

A polynomial, p, in ξ, of degree at most m, satisfies a bound

$$|p(\xi)|\le C(1+|\xi|)^m \quad \forall\ \xi\in\mathbb{R}^n. \tag{2.3}$$

Since successive derivatives, $D^\alpha_\xi p(\xi)$, are polynomials of degree $m-|\alpha|$, for any multiindex α, we get the family of estimates

$$|D^\alpha_\xi p(\xi)|\le C_\alpha(1+|\xi|)^{m-|\alpha|} \quad \forall\ \xi\in\mathbb{R}^n, \alpha\in\mathbb{N}^n_0. \tag{2.4}$$

Of course if $|\alpha|>m$ then $D^\alpha_\xi p\equiv 0$, so we can even take the constants C_α to be independent of α. If we consider the characteristic polynomial $P(x,\xi)$ of a differential operator of order m with coefficients in $\mathcal{C}^\infty_\infty(\mathbb{R}^n)$ (i.e. all derivatives of the coefficients are bounded) (2.4) is replaced by

$$\left|D^\alpha_x D^\beta_\xi P(x,\xi)\right|\le C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|} \quad \forall\ (x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n,\ \alpha,\beta\in\mathbb{N}^n_0. \tag{2.5}$$

There is no particular reason to have the same number of x variables as of ξ variables, so in general we define:

Definition 2.1. *The space $S^m_\infty(\mathbb{R}^p;\mathbb{R}^n)$ of symbols of order m (with coefficients in $\mathcal{C}^\infty_\infty(\mathbb{R}^p)$) consists of those functions $a \in \mathcal{C}^\infty(\mathbb{R}^p \times \mathbb{R}^n)$ satisfying all the estimates*

$$(2.6)\qquad \left|D^\alpha_z D^\beta_\xi a(z,\xi)\right| \le C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|} \text{ on } \mathbb{R}^p\times\mathbb{R}^n \quad \forall\ \alpha\in\mathbb{N}^p_0, \beta\in\mathbb{N}^n_0.$$

For later reference we even define $S^m_\infty(\Omega;\mathbb{R}^n)$ when $\Omega \subset \mathbb{R}^p$ and $\Omega \subset \operatorname{clos}(\operatorname{int}(\Omega))$ as consisting of those $a \in \mathcal{C}^\infty(\operatorname{int}(\Omega)\times\mathbb{R}^n)$ satisfying (2.6) for $(z,\xi) \in \operatorname{int}(\Omega)\times\mathbb{R}^n$.

The estimates (2.6) can be rewritten

$$(2.7)\qquad \|a\|_{N,m} = \sup_{\substack{z\in\operatorname{int}(\Omega)\\ \xi\in\mathbb{R}^n}} \max_{|\alpha|+|\beta|\le N} (1+|\xi|)^{-m+|\beta|}\left|D^\alpha_z D^\beta_\xi a(z,\xi)\right| < \infty.$$

With these *norms* $S^m_\infty(\Omega;\mathbb{R}^n)$ is a Fréchet space, rather similar in structure to $\mathcal{C}^\infty_\infty(\mathbb{R}^n)$. Thus the topology is given by the metric

$$(2.8)\qquad d(a,b) = \sum_{N\ge 0} 2^{-N}\frac{\|a-b\|_{N,m}}{1+\|a-b\|_{N,m}},\ a,b\in S^m_\infty(\Omega;\mathbb{R}^n).$$

The subscript '∞' here is *not* standard notation. It refers to the assumption of uniform boundedness of the derivatives of the 'coefficients'. More standard notation would be just $S^m(\Omega\times\mathbb{R}^n)$, especially for $\Omega=\mathbb{R}^p$, but I think this is too confusing.

A more significant issue is: Why this class precisely? As we shall see below, there are other choices which are not only possible but even profitable to make. However, the present one has several virtues. It is large enough to cover most of the straightforward things we want to do (at least initially) and small enough to 'work' easily. It leads to what I shall refer to as the 'traditional' algebra of pseudodifferential operators.

Now to some basic properties. First notice that

$$(2.9)\qquad (1+|\xi|)^m \le C(1+|\xi|)^{m'}\ \forall\ \xi\in\mathbb{R}^n \Longleftrightarrow m\le m'.$$

Thus we have an inclusion

$$(2.10)\qquad S^m_\infty(\Omega;\mathbb{R}^n) \hookrightarrow S^{m'}_\infty(\Omega;\mathbb{R}^n)\ \forall\ m'\ge m.$$

Moreover this inclusion is continuous, since from (2.7), $\|a\|_{N,m'} \le \|a\|_{N,m}$ if $a\in S^m(\Omega;\mathbb{R}^n)$ and $m'\ge m$. Since these spaces increase with m we think of them as a *filtration* of the big space

$$(2.11)\qquad S^\infty_\infty(\Omega;\mathbb{R}^n) = \bigcup_m S^m_\infty(\Omega;\mathbb{R}^n).$$

Notice that the two '∞s' here are quite different. The subscript refers to the fact that the 'coefficients' are bounded and stands for L^∞ whereas the superscript '∞' stands really for $\mathbb{R}$. The *residual* space of this filtration is

$$(2.12)\qquad S^{-\infty}_\infty(\Omega;\mathbb{R}^n) = \bigcap_m S^m_\infty(\Omega;\mathbb{R}^n).$$

In fact the inclusion (2.10) is *never* dense if $m'>m$. Instead we have the following rather technical, but nevertheless very useful, result.

Lemma 2.1. *For any $m\in\mathbb{R}$ and any $a\in S^m_\infty(\Omega;\mathbb{R}^n)$ there is a sequence in $S^{-\infty}_\infty(\Omega;\mathbb{R}^n)$ which is bounded in $S^m_\infty(\Omega;\mathbb{R}^n)$ and converges to a in the topology of $S^{m'}_\infty(\Omega;\mathbb{R}^n)$ for any $m'>m$; in particular $S^{-\infty}_\infty(\Omega;\mathbb{R}^n)$ is dense in the space $S^m_\infty(\Omega;\mathbb{R}^n)$ in the topology of $S^{m'}_\infty(\Omega;\mathbb{R}^n)$ for $m'>m$.*

The reason one cannot take $m' = m$ here is essentially the same reason that underlies the fact that $\mathcal{S}(\mathbb{R}^n)$ is not dense in $\mathcal{C}^\infty_\infty(\mathbb{R}^n)$. Namely any uniform limit obtained from a converging Schwartz sequence must vanish at infinity. In particular the constant function $1 \in S^0_\infty(\mathbb{R}^p;\mathbb{R}^n)$ cannot be in the closure in this space of $S^{-\infty}_\infty(\mathbb{R}^p;\mathbb{R}^n)$ if $n > 0$.

PROOF. Choose $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ with $0 \le \phi(\xi) \le 1$, $\phi(\xi) = 1$ if $|\xi| < 1, \phi(\xi) = 0$ if $|\xi| > 2$ and consider the sequence

$$a_k(z,\xi) = \phi(\xi/k)a(z,\xi), \quad a \in S^m_\infty(\Omega;\mathbb{R}^n). \tag{2.13}$$

We shall show that $a_k \in S^{-\infty}_\infty(\Omega,\mathbb{R}^n)$ is a bounded sequence in $S^m_\infty(\Omega;\mathbb{R}^n)$ and that $a_k \longrightarrow a$ in $S^{m'}_\infty(\Omega;\mathbb{R}^n)$ for any $m' > m$. Certainly for each N

$$|a_k(z,\xi)| \le C_{N,k}(1+|\xi|)^{-N} \tag{2.14}$$

since ϕ has compact support. Leibniz' formula gives

$$D^\alpha_z D^\beta_\xi a_k(z,\xi) = \sum_{\beta'\le\beta} \binom{\beta'}{\beta} k^{-|\beta'|}(D^{\beta'}\phi)(\xi/k) D^\alpha_z D^{\beta-\beta'}_\xi a(z,\xi). \tag{2.15}$$

On the support of $\phi(\xi/k)$, $|\xi| \le k$ so, using the symbol estimates on a, it follows that a_k is bounded in $S^m_\infty(\Omega;\mathbb{R}^n)$. We easily conclude that

$$\left|D^\alpha_z D^\beta_\xi a_k(z,\xi)\right| \le C_{N,\alpha,\beta,k}(1+|\xi|)^{-N} \quad \forall\ \alpha,\beta,N,k. \tag{2.16}$$

Thus $a_k \in S^{-\infty}_\infty(\Omega;\mathbb{R}^n)$.

So consider the difference

$$(a-a_k)(z,\xi) = (1-\phi)(\xi/k)\ a(z,\xi). \tag{2.17}$$

Now, $|(1-\phi)(\xi/k)| = 0$ in $|\xi| \le k$ so we only need estimate the difference in $|\xi| \ge k$ where this factor is bounded by 1. In this region $1+|\xi| \ge 1+k$ so, since $-m'+m < 0$,

$$\begin{gathered}(1+|\xi|)^{-m'}\left|(a-a_k)(z,\xi)\right| \le \\ (1+k)^{-m'+m}\sup_{z,\xi}|(1+|\xi|)^{-m}|a(z,\xi)| \le (1+k)^{-m'+m}\|a\|_{0,m} \longrightarrow 0.\end{gathered} \tag{2.18}$$

This is convergence with respect to the first symbol norm.

Next consider the ξ derivatives of (2.17). Using Leibniz' formula

$$\begin{aligned}D^\beta_\xi(a-a_k) &= \sum_{\gamma\le\beta}\binom{\beta}{\gamma} D^{\beta-\gamma}_\xi(1-\phi)(\frac{\xi}{k})\cdot D^\gamma_\xi a(z,\xi) \\ &= (1-\phi)(\frac{\xi}{k})\cdot D^\beta_\xi a(z,\xi) - \sum_{\gamma<\beta}\binom{\beta}{\gamma}\left(D^{\beta-\gamma}\phi\right)(\frac{\xi}{k})\cdot k^{-|\beta-\gamma|}D^\gamma_\xi a(z,\xi).\end{aligned}$$

In the first term, $D^\beta_\xi a(z,\xi)$ is a symbol of order $m-|\beta|$, so by the same argument as above

$$\sup_\xi (1+|\xi|)^{-m'+|\beta|}|(1-\phi)(\frac{\xi}{k})D^\beta_\xi a(x,\xi)| \longrightarrow 0 \tag{2.19}$$

as $k \longrightarrow \infty$ if $m' > m$. In all the other terms, $(D^{\beta-\gamma}\phi)(\zeta)$ has compact support, in fact $1 \le |\zeta| \le 2$ on the support. Thus for each term we get a bound

$$\sup_{k\le|\xi|\le 2k}(1+|\xi|)^{-m'+|\beta|}\cdot k^{-|\beta-\gamma|}C\cdot(1+|\xi|)^{m-|\gamma|} \le Ck^{-m'+m}. \tag{2.20}$$

The variables z play the rôle of parameters so we have in fact shown that

$$\sup_{\substack{z\in\Omega\\ \xi\in\mathbb{R}^n}} (1+|\xi|)^{-m'+|\beta|}\big|D_z^\alpha D_\xi^\beta(a-a_k)\big| \longrightarrow 0 \text{ as } k\longrightarrow\infty. \tag{2.21}$$

This means $a_k \longrightarrow a$ in each of the symbol norms, and hence in the topology of $S^{m'}_\infty(\mathbb{R}^p;\mathbb{R}^n)$ as desired. □

In fact this proof suggests a couple of other 'obvious' results. Namely

$$S^m_\infty(\Omega;\mathbb{R}^n)\cdot S^{m'}_\infty(\Omega;\mathbb{R}^n)\subset S^{m+m'}_\infty(\Omega;\mathbb{R}^n). \tag{2.22}$$

This can be proved directly using Leibniz' formula:

$$\begin{aligned}&\sup_\xi(1+|\xi|)^{-m-m'+|\beta|}\big|D_z^\alpha D_\xi^\beta\left(a(z,\xi)\cdot b(z,\xi)\right)\big|\\
&\le\sum_{\substack{\mu\le\alpha\\ \gamma\le\beta}}\binom{\alpha}{\mu}\binom{\beta}{\gamma}\sup_\xi(1+|\xi|)^{-m+|\gamma|}\big|D_z^\mu D_\xi^\gamma a(z,\xi)\big|\\
&\qquad\times\sup_\xi(1+|\xi|)^{-m'+|\beta-\gamma|}\big|D_z^{\alpha-\mu}D_\xi^{\beta-\gamma}b(z,\xi)\big|<\infty.\end{aligned}$$

We also note the action of differentiation:

$$\begin{aligned}D_z^\alpha &: S^m_\infty(\Omega;\mathbb{R}^n)\longrightarrow S^m_\infty(\Omega;\mathbb{R}^n) \text{ and}\\
D_\xi^\beta &: S^m_\infty(\Omega;\mathbb{R}^n)\longrightarrow S^{m-|\beta|}_\infty(\Omega;\mathbb{R}^n).\end{aligned} \tag{2.23}$$

In fact, while we are thinking about these things we might as well show the important consequence of *ellipticity.* A symbol $a\in S^m_\infty(\Omega;\mathbb{R}^n)$ is said to be (globally) elliptic if

$$|a(z,\xi)|\ge\epsilon(1+|\xi|)^m-C(1+|\xi|)^{m-1},\ \epsilon>0 \tag{2.24}$$

or equivalently[1]

$$|a(z,\xi)|\ge\epsilon(1+|\xi|)^m \text{ in } |\xi|\ge C_\epsilon,\ \epsilon>0. \tag{2.25}$$

Lemma 2.2. *If $a\in S^m_\infty(\Omega;\mathbb{R}^n)$ is elliptic there exists $b\in S^{-m}_\infty(\Omega;\mathbb{R}^n)$ such that*

$$a\cdot b-1\in S^{-\infty}_\infty(\Omega;\mathbb{R}^n). \tag{2.26}$$

Proof. Using (2.25) choose ϕ as in the proof of Lemma 2.1 and set

$$b(z,\xi)=\begin{cases}\frac{1-\phi(\xi/2C)}{a(z,\xi)} & |\xi|\ge C\\ 0 & |\xi|\le C.\end{cases} \tag{2.27}$$

Then b is $\mathcal{C}^\infty$ since $b=0$ in $C\le|\xi|\le C+\delta$ for some $\delta>0$. The symbol estimates follow by noting that, in $|\xi|\ge C$,

$$D_z^\alpha D_\xi^\beta b=a^{-1-|\alpha|-|\beta|}\cdot G_{\alpha\beta} \tag{2.28}$$

where $G_{\alpha\beta}$ is a symbol of order $(|\alpha|+|\beta|)m-|\beta|$. This may be proved by induction. Indeed, it is true when $\alpha=\beta=0$. Assuming (2.28) for some α and β, differentiation of (2.28) gives

$$\begin{aligned}D_{z_j}D_z^\alpha D_\xi^\beta b=D_{z_j}a^{-1-|\alpha|-|\beta|}\cdot G_{\alpha\beta}&=a^{-2-|\alpha|-|\beta|}G',\\
G'&=(-1-|\alpha|-|\beta|)(D_{z_j}a)G_{\alpha\beta}+aD_{z_j}G_{\alpha\beta}.\end{aligned}$$

[1]Note it is required that ϵ be chosen to be independent of z here, so this is a notion of uniform ellipticity.

By the inductive hypothesis, G' is a symbol of order $(|\alpha|+1+|\beta|)m-|\beta|$. A similar argument applies to derivatives with respect to the ξ variables. $\square$

2.2. Pseudodifferential operators

Now we proceed to discuss the formula (2.2) where we shall assume that, for some $w, m \in \mathbb{R}$,

$$\begin{gathered} a(x,y,\xi) = (1+|x-y|^2)^{w/2}\tilde{a}(x,y,\xi) \\ \tilde{a} \in S^m_\infty(\mathbb{R}^{2n}_{(x,y)};\mathbb{R}^n_\xi). \end{gathered} \tag{2.29}$$

The extra 'weight' factor (which allows polynomial growth in the direction of $x-y$) turns out, somewhat enigmatically, to both make no difference and be very useful! Notice[2] that if $a \in \mathcal{C}^\infty(\mathbb{R}^{2n}\times\mathbb{R}^n)$ then $a \in (1+|x-y|^2)^{w/2}S^m(\mathbb{R}^{2n};\mathbb{R}^n)$ if and only if

$$|D^\alpha_x D^\beta_y D^\gamma_\xi a(x,y,\xi)| \le C_{\alpha,\beta,\gamma}(1+|x-y|)^w(1+|\xi|)^{m-|\gamma|}\ \forall\ \alpha,\beta,\gamma \in \mathbb{N}^n_0. \tag{2.30}$$

If $m < -n$ then, for each $u \in \mathcal{S}(\mathbb{R}^n)$, the integral in (2.2) is absolutely convergent, locally uniformly in x, since

$$\begin{aligned} |a(x,y,\xi)u(y)| &\le C(1+|x-y|)^w(1+|\xi|)^m(1+|y|)^{-N} \\ &\le C(1+|x|)^w(1+|\xi|)^m(1+|y|)^m,\ m<-n. \end{aligned} \tag{2.31}$$

Here we have used the following simple consequence of the triangle inequality

$$(1+|x-y|) \le (1+|x|)(1+|y|)$$

from which it follows that

$$(1+|x-y|)^w \le \begin{cases} (1+|x|)^w(1+|y|)^w & \text{if } w>0 \\ (1+|x|)^w(1+|y|)^{-w} & \text{if } w \le 0. \end{cases} \tag{2.32}$$

Thus we conclude that, provided $m < -n$,

$$A : \mathcal{S}(\mathbb{R}^n) \longrightarrow (1+|x|^2)^{w/2}\mathcal{C}^0_\infty(\mathbb{R}^n). \tag{2.33}$$

To show that, for general m, A exists as an operator, we prove that its Schwartz kernel exists.

Proposition 2.1. *The map, defined for $m < -n$ as a convergent integral,*

$$\begin{aligned} &(1+|x-y|^2)^{w/2}S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n) \ni a \longmapsto I(a) = \\ &\qquad (2\pi)^{-n}\int e^{i(x-y)\cdot\xi}a(x,y,\xi)d\xi \in (1+|x|^2+|y|^2)^{w/2}\mathcal{C}^0_\infty(\mathbb{R}^{2n}) \end{aligned} \tag{2.34}$$

extends by continuity to

$$I : (1+|x-y|^2)^{w/2}S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^{2n}) \tag{2.35}$$

for each w, $m \in \mathbb{R}$ in the topology of $S^{m'}_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ for any $m' > m$.

---

[2]See Problem 2.5.

PROOF. Since we already have the density of $S^{-\infty}_{\infty}(\mathbb{R}^{2n};\mathbb{R}^n)$ in $S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ in the toplogy of $S^{m'}_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ for any $m' > m$, we only need to show the continuity of the map (2.34) on this residual subspace with respect to the topology of $S^{m'}_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ for any m', which we may as well write as m. What we shall show is that, for each $w, m \in \mathbb{R}$, there are integers $N, k \in \mathbb{N}$ such that, in terms of the norms in (2.7) and (1.6)

$$\big|I(a)(\phi)\big| \le C\|\tilde a\|_{N,m}\|\phi\|_k \quad \forall\ \phi \in \mathcal{S}(\mathbb{R}^{2n}),$$
$$a = (1+|x-y|^2)^{w/2}\tilde a,\ \tilde a \in S^{-\infty}_\infty(\mathbb{R}^{2n};\mathbb{R}^n). \tag{2.36}$$

To see this we just use integration by parts.

Set $\tilde\phi(x,y) = (1+|x-y|^2)^{w/2}\phi(x,y)$. Observe that

$$(1+\xi\cdot D_x)e^{i(x-y)\cdot\xi} = (1+|\xi|^2)e^{i(x-y)\cdot\xi}$$
$$(1-\xi\cdot D_y)e^{i(x-y)\cdot\xi} = (1+|\xi|^2)e^{i(x-y)\cdot\xi}.$$

Thus we can write, for $\tilde a \in S^{-\infty}_\infty$, with $a = (1+|x-y|^2)^{w/2}\tilde a$ and for any $q \in \mathbb{N}$

$$\begin{aligned} I(a)(\phi) = &\iint (2\pi)^{-n}\int e^{i(x-y)\cdot\xi}(1+|\xi|^2)^{-2q} \\ &\qquad (1-\xi\cdot D_x)^q(1+\xi\cdot D_y)^q\big[\tilde a(x,y,\xi)\tilde\phi(x,y)\big]d\xi dxdy \\ &= \sum_{|\gamma|\le 2q}\iint \Big(\int e^{i(x-y)\cdot\xi}a^{(q)}_\gamma(x,y,\xi)d\xi\Big)D^\gamma_{(x,y)}\tilde\phi(x,y)dxdy. \end{aligned} \tag{2.37}$$

Here the $a^{(q)}_\gamma$ arise by expanding the powers of the operator

$$(1-\xi\cdot D_x)^q(1+\xi\cdot D_x)^q = \sum_{|\mu|,|\nu|\le q} C_{\mu,\nu}\xi^{\mu+\nu}D^\mu_x D^\nu_y$$

and applying Leibniz' formula. Thus $a^{(q)}_\gamma$ arises from terms in which $2q-|\gamma|$ derivatives act on $\tilde a$ so it is of the form

$$\begin{aligned} a_\gamma &= (1+|\xi|^2)^{-2q}\sum_{|\mu|\le|\gamma|,|\gamma|\le 2q} C_{\mu,\gamma}\xi^\gamma D^\mu_{(x,y)}\tilde a \\ &\Longrightarrow \|a_\gamma\|_{N,m} \le C_{m,q,N}\|\tilde a\|_{N+2q,m+2q}\ \forall\ m, N, q. \end{aligned}$$

So (for given m) if we take $-2q+m < -n$, e.g. $q > \max(\frac{n+m}{2}, 0)$ and use the integrability of $(1+|x|+|y|)^{-2n-1}$ on $\mathbb{R}^{2n}$, then

$$\big|I(a)(\phi)\big| \le C\|\tilde a\|_{2q,m}\|\tilde\phi\|_{2q+2n+1} \le C\|\tilde a\|_{2q,m}\|\phi\|_{2q+w+2n+1}. \tag{2.38}$$

This is the estimate (2.36), which proves the desired continuity. □

In showing the existence of the Schwartz' kernel in this proof we do *not* really need to integrate by parts in both x *and* y; either separately will do the trick. We can use this observation to show that these pseudodifferential operator act on $\mathcal{S}(\mathbb{R}^n)$.

LEMMA 2.3. *If $a \in (1+|x-y|^2)^{w/2}S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ then the operator A, with Schwartz kernel $I(a)$, is a continuous linear map*

$$A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n). \tag{2.39}$$

We shall denote by $\Psi^m_\infty(\mathbb{R}^n)$ the linear space of operators (2.39), corresponding to $(1+|x-y|^2)^{-w/2}a \in S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ for some w. I call them *pseudodifferential operators 'of traditional type' – or type '1,0'.*[3]

PROOF. Proceeding as in (2.37) but only integrating by parts in y we deduce that, for q large depending on m,

$$Au(\psi) = \sum_{\gamma\le 2q}(2\pi)^{-n}\iint\int e^{i(x-y)\cdot\xi}a_\gamma(x,y,\xi)D^\gamma_y u(y)d\xi\psi(x)dydx,$$
$$a_\gamma \in (1+|x-y|^2)^{w/2}S^{m-q}(\mathbb{R}^{2n};\mathbb{R}^n) \text{ if } a\in(1+|x-y|^2)^{w/2}S^m(\mathbb{R}^{2n};\mathbb{R}^n).$$

The integration by parts is justified by continuity from $S^{-\infty}_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$. Taking $-q+m<-n-|w|$, this shows that Au is given by the convergent integral

$$Au(x) = \sum_{\gamma\le 2q}(2\pi)^{-n}\iint e^{i(x-y)\cdot\xi}a_\gamma(x,y,\xi)D^\gamma_y u(y)d\xi dy, \tag{2.40}$$
$$A:\mathcal{S}(\mathbb{R}^n)\longrightarrow(1+|x|^2)^{\frac{|w|}{2}}\mathcal{C}^0_\infty(\mathbb{R}^n)$$

which is really just (2.33) again. Here $\mathcal{C}^0_\infty(\mathbb{R}^n)$ is the Banach space of bounded continuous functions on $\mathbb{R}^n$, with the supremum norm. The important point is that the weight depends on w but not on m. Notice that

$$D_{x_j}Au(x) = (2\pi)^{-n}\sum_{|\gamma|\le 2q}\iint e^{i(x-y)\cdot\xi}\big(\xi_j + D_{x_j}\big)a_\gamma\cdot D^\gamma_y u(y)dyd\xi$$

and

$$x_jAu(x) = (2\pi)^n\sum_{|\gamma|\le 2q}\iint e^{i(x-y)\cdot\xi}\big(-D_{\xi_j}+y_j\big)a_\gamma\cdot D^\gamma_y u(y)dyd\xi.$$

Proceeding inductively (2.39) follows from (2.33) or (2.40) since we conclude that

$$x^\alpha D^\beta_x Au \in (1+|x|^2)^{\frac{|w|}{2}}\mathcal{C}^0_\infty(\mathbb{R}^n),\ \forall\ \alpha,\beta\in\mathbb{N}^n_0$$

and this implies that $Au\in\mathcal{S}(\mathbb{R}^n)$. □

2.3. Composition

There are two extreme cases of $I(a)$, namely where a is independent of either x or of y. Below we shall prove:

THEOREM 2.1 (Reduction). *Each $A\in\Psi^m_\infty(\mathbb{R}^n)$ can be written uniquely as $I(a')$ where $a'\in S^m_\infty(\mathbb{R}^n_x;\mathbb{R}^n_\xi)$.*

This is the main step in proving the fundamental result of this Chapter, which is that two pseudodifferential operators can be composed to give a pseudodifferential operator and that the orders are additive. Thus our aim is to demonstrate the fundamental

THEOREM 2.2. *[Composition] The space $\Psi^\infty_\infty(\mathbb{R}^n)$ is an order-filtered $*$-algebra on $\mathcal{S}(\mathbb{R}^n)$.*

[3] The meaning of which is explained in Problem 2.16.

We have already shown that each $A \in \Psi^{\infty}_{\infty}(\mathbb{R}^n)$ defines a continuous linear map (2.39). We now want to show that

$$A \in \Psi^m_{\infty}(\mathbb{R}^n) \Longrightarrow A^* \in \Psi^m_{\infty}(\mathbb{R}^n) \tag{2.41}$$

$$A \in \Psi^m_{\infty}(\mathbb{R}^n),\ B \in \Psi^{m'}_{\infty}(\mathbb{R}^n) \Longrightarrow A \circ B \in \Psi^{m+m'}_{\infty}(\mathbb{R}^n), \tag{2.42}$$

since this is what is meant by an order-filtered (the orders add on composition) $*$-algebra (meaning (2.41) holds). In fact we will pick up some more information along the way.

2.4. Reduction

We proceed to prove Theorem 2.1, which we can restate as:

PROPOSITION 2.2. *The range of* (2.34) *(for any w) is the same as the range of I restricted to the image of the inclusion map*

$$S^m_{\infty}(\mathbb{R}^n;\mathbb{R}^n) \ni a \longmapsto a(x,\xi) \in S^m_{\infty}(\mathbb{R}^{2n}_{(x,y)};\mathbb{R}^n).$$

PROOF. Suppose $a \in \big(1+|x-y|^2\big)^{w/2} S^{-\infty}_{\infty}(\mathbb{R}^{2n};\mathbb{R}^n)$ for some w, then

$$I\big((x_j-y_j)a\big) = I\big(-D_{\xi_j}a\big)\ j=1,\dots,n. \tag{2.43}$$

Indeed this is just the result of inserting the identity

$$D_{\xi_j} e^{i(x-y)\cdot\xi} = (x_j-y_j)e^{i(x-y)\cdot\xi}$$

into (2.34) and integrating by parts. Since both sides of (2.43) are continuous on $\big(1+|x-y|^2\big)^{w/2} S^{\infty}_{\infty}(\mathbb{R}^{2n};\mathbb{R}^n)$ the identity holds in general. Notice that if a is of order m then $D_{\xi_j}a$ is of order $m-1$, so (2.43) shows that even though the operator with amplitude $(x_j-y_j)a(x,y,\xi)$ *appears* to have order m, it actually has order $m-1$.

To exploit (2.43) consider the Taylor series (with Legendre's remainder) for $a(x,y,\xi)$ around $x=y$:

$$a(x,y,\xi) = \sum_{|\alpha|\le N-1} \frac{(-i)^{|\alpha|}}{\alpha!}(x-y)^{\alpha}\big(D^{\alpha}_y a\big)(x,x,\xi) + \sum_{|\alpha|=N} \frac{(-i)^{|\alpha|}}{\alpha!}(x-y)^{\alpha}\cdot R_{N,\alpha}(x,y,\xi). \tag{2.44}$$

Here,

$$R_{N,\alpha}(x,y,\xi) = \int_0^1 (1-t)^{N-1}\big(D^{\alpha}_y a\big)(x,(1-t)x+ty,\xi)dt. \tag{2.45}$$

Now,

$$(x-y)^{\alpha}\big(D^{\alpha}_y a\big)(x,y,\xi) \in \big(1+|x-y|^2\big)^{\frac{(w+|\alpha|)}{2}} S^m_{\infty}(\mathbb{R}^{2n};\mathbb{R}^n). \tag{2.46}$$

Applying (2.43) repeatedly we see that if A is the operator with kernel $I(a)$ then

$$A = \sum_{j=0}^{N-1} A_j + R_N,\quad A_j \in \Psi^{m-j}_{\infty}(\mathbb{R}^n),\ R_N \in \Psi^{m-N}_{\infty}(\mathbb{R}^n) \tag{2.47}$$

where the A_j have kernels

$$I\big(\sum_{|\alpha|=j} \frac{i^{|\alpha|}}{\alpha!}\big(D_y^\alpha D_\xi^\alpha a\big)(x,x,\xi)\big). \tag{2.48}$$

To proceed further we need somehow to *sum* this series. Of course we cannot really do this, but we can come close!

2.5. Asymptotic summation

Suppose $a_j \in S_\infty^{m-j}(\mathbb{R}^p;\mathbb{R}^n)$. The fact that the *orders* are decreasing means that these symbols are getting very small, for $|\xi|$ large. The infinite series

$$\sum_j a_j(z,\xi) \tag{2.49}$$

need not converge. However we shall say that it converges asymptotically, or since it is a series we say it is 'asymptotically summable,' if there exists $a \in S_\infty^m(\mathbb{R}^p;\mathbb{R}^n)$ such that,

$$\text{for every } N,\ a - \sum_{j=0}^{N-1} a_j \in S_\infty^{m-N}(\mathbb{R}^p;\mathbb{R}^n). \tag{2.50}$$

We write this relation as

$$a \sim \sum_{j=0}^{\infty} a_j. \tag{2.51}$$

PROPOSITION 2.3. *Any series $a_j \in S_\infty^{m-j}(\mathbb{R}^p;\mathbb{R}^n)$ is asymptotically summable, in the sense of* (2.50)*, and the asymptotic sum is well defined up to an additive term in $S_\infty^{-\infty}(\mathbb{R}^p;\mathbb{R}^n)$.*

PROOF. The uniqueness part is easy. Suppose a and a' both satisfy (2.50). Taking the difference

$$a - a' = \big(a - \sum_{j=0}^{N-1} a_j\big) - \big(a' - \sum_{j=0}^{N-1} a_j\big) \in S_\infty^{m-N}(\mathbb{R}^p;\mathbb{R}^n). \tag{2.52}$$

Since $S_\infty^{-\infty}(\mathbb{R}^p;\mathbb{R}^n)$ is just the intersection of the $S_\infty^{-N}(\mathbb{R}^p;\mathbb{R}^n)$ over N it follows that $a - a' \in S_\infty^{-\infty}(\mathbb{R}^p;\mathbb{R}^n)$, proving the uniqueness.

So to the existence of an asymptotic sum. To construct this (by Borel's method) we cut off each term 'near infinity in ξ'. Thus fix $\phi \in \mathcal{C}^\infty(\mathbb{R}^n)$ with $\phi(\xi) = 0$ in $|\xi| \le 1$, $\phi(\xi) = 1$ in $|\xi| \ge 2$, $0 \le \phi(\xi) \le 1$. Consider a decreasing sequence

$$\epsilon_0 > \epsilon_1 > \cdots > \epsilon_j \downarrow 0. \tag{2.53}$$

We shall set

$$a(z,\xi) = \sum_{j=0}^{\infty} \phi(\epsilon_j\xi) a_j(z,\xi). \tag{2.54}$$

Since $\phi(\epsilon_j\xi) = 0$ in $|\xi| < 1/\epsilon_j \to \infty$ as $j \to \infty$, only finitely many of these terms are non-zero in any ball $|\xi| \le R$. Thus $a(z,\xi)$ is a well-defined $\mathcal{C}^\infty$ function. Of course we need to consider the seminorms, in $S_\infty^m(\mathbb{R}^p;\mathbb{R}^n)$, of each term.

The first of these is

$$\sup_z \sup_\xi (1+|\xi|)^{-m} \, \big|\phi(\epsilon_j\xi)\big| \, \big|a_j(z,\xi)\big|. \tag{2.55}$$

Now $|\xi| \geq \frac{1}{\epsilon_j}$ on the support of $\phi(\epsilon_j\xi)a_j(z,\xi)$ and since a_j is a symbol of order $m-j$ this allows us to estimate (2.55) by

$$\begin{aligned} &\sup_z \sup_{|\xi|\geq\frac{1}{\epsilon_j}} (1+|\xi|)^{-j} \cdot \big[\big(1+|\xi|\big)^{-m+j}\big|a_j(z,\xi)\big|\big] \\ &\qquad \leq \big(1+\frac{1}{\epsilon_j}\big)^{-j} \cdot C_j \leq \epsilon_j^j \cdot C_j \end{aligned}$$

where the C_j's are *fixed* constants, independent of ϵ_j.

Let us look at the higher symbol estimates. As usual we can apply Leibniz' formula:

$$\begin{aligned} &\sup_z \sup_\xi (1+|\xi|)^{-m+|\beta|}\big|D_z^\alpha D_\xi^\beta \phi(\epsilon_j\xi)a_j(z,\xi)\big| \\ &\leq \sum_{\mu\leq\beta} \sup_z \sup_\xi (1+|\xi|)^{|\beta|-|\mu|-j}\epsilon_j^{|\beta|-|\mu|}\big|\big(D^{\beta-\mu}\phi\big)(\epsilon_j\xi)\big| \\ &\qquad \times (1+|\xi|)^{-m+j+|\mu|}\big|D_z^\alpha D_\xi^\mu a_j(z,\xi)\big|. \end{aligned}$$

The term with $\mu=\beta$ we estimate as before and the others, with $\mu\neq\beta$ are supported in $\frac{1}{\epsilon_j} \leq |\xi| \leq \frac{2}{\epsilon_j}$. Then we find that for all j

$$\|\phi(\epsilon_j\xi)a_j(z,\xi)\|_{N,m} \leq C_{N,j}\epsilon_j^j \tag{2.56}$$

where $C_{N,j}$ is independent of ϵ_j.

So we see that for each given N we can arrange that, for instance,

$$\|\phi(\epsilon_j\xi)a_j(z,\xi)\|_{N,m} \leq C_N\frac{1}{j^2}$$

by choosing the ϵ_j to satify

$$C_{N,j}\epsilon_j^j \leq \frac{1}{j^2} \;\forall\; j \geq j(N).$$

Notice the crucial point here, we can arrange that for *each* N the sequence of norms in (2.56) is dominated by $C_N j^{-2}$ by fixing $\epsilon_j < \epsilon_{j,N}$ for large j. Thus we can arrange convergence of *all* the sums

$$\sum_j \|\phi(\epsilon_j\xi)a_j(z,\xi)\|_{N,m}$$

by diagonalization, for example setting $\epsilon_j = \frac{1}{2}\epsilon_{j,j}$. Thus by choosing $\epsilon_j \downarrow 0$ rapidly enough we ensure that the series (2.54) converges. In fact the same argument allows us to ensure that for every N

$$\sum_{j\geq N} \phi(\epsilon_j\xi)a_j(z,\xi) \text{ converges in } S_\infty^{m-N}(\mathbb{R}^p;\mathbb{R}^n). \tag{2.57}$$

This certainly gives (2.50) with a defined by (2.54). □

2.6. Residual terms

Now we can apply Proposition 2.3 to the series in (2.48), that is we can find $b \in S^m_\infty(\mathbb{R}^n_x;\mathbb{R}^n_\xi)$ satisfying

$$b(x,\xi) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!}\big(D^\alpha_y a\big)(x,x,\xi). \tag{2.58}$$

Let $B = I(b)$ be the operator defined by this amplitude (which is independent of y). Now (2.47) says that

$$A - B = \sum_{j=0}^{N-1} A_j + R_N - B$$

and from (2.50) applied to (2.58)

$$B = \sum_{j=0}^{N-1} A_j + R'_N,\ \ R'_N \in \Psi^{m-N}_\infty(\mathbb{R}^n)$$

Thus

$$A - B \in \Psi^{-\infty}_\infty(\mathbb{R}^n) = \bigcap_N \Psi^N_\infty(\mathbb{R}^n). \tag{2.59}$$

Notice that, at this stage, we do *not* know that $A - B$ has kernel $I(c)$ with $c \in S^{-\infty}_\infty(\mathbb{R}^{2n},\mathbb{R}^n)$, just that it has kernel $I(c_N)$ with $c_N \in S^N_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ for each N.

However:

Proposition 2.4. *An operator $A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ is an element of the space $\Psi^{-\infty}_\infty(\mathbb{R}^n)$ if and only if its Schwartz kernel is $\mathcal{C}^\infty$ and satisfies the estimates*

$$\big|D^\alpha_x D^\beta_y K(x,y)\big| \le C_{N,\alpha,\beta}(1+|x-y|)^{-N}\ \forall\ \alpha,\beta,N. \tag{2.60}$$

Proof. Suppose first that $A \in \Psi^{-\infty}_\infty(\mathbb{R}^n)$, which means that $A \in \Psi^N_\infty(\mathbb{R}^n)$ for every N. The Schwartz kernel, K_A, of A is therefore given by (2.34) with the amplitude $a_N \in S^N_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$. For $N << -n-1-p$ the integral converges absolutely and we can integrate by parts to show that

$$\begin{aligned}&(x-y)^\alpha D^\beta_x D^\gamma_y K_A(x,y)\\ &= (2\pi)^{-N}\int e^{i(x-y)\cdot\xi}(-D_\xi)^\alpha(D_x+i\xi)^\beta(D_y-i\xi)^\gamma a_N(x,y,\xi)d\xi\end{aligned}$$

which converges absolutely, and uniformly in x, y, provided $|\beta|+|\gamma|+N-|\alpha| < -n$. Thus

$$\sup\big|(x-y)^\alpha D^\beta_x D^\gamma_y K\big| < \infty\ \forall\ \alpha,\beta,\gamma$$

which is another way of writing (2.60) i.e.

$$\sup\big(1+|x-y|^2\big)^N\big|D^\beta_x D^\gamma_y K\big| < \infty\ \forall\ \beta,\gamma,N.$$

Conversely suppose that (2.60) holds. Define

$$g(x,z) = K(x,x-z). \tag{2.61}$$

The estimates (2.60) become

$$\sup\big|D^\alpha_x z^\gamma D^\beta_z g(x,z)\big| < \infty\ \forall\ \alpha,\beta,\gamma. \tag{2.62}$$

That is, g is rapidly decreasing with all its derivatives in z. Taking the Fourier transform,

$$b(x,\xi) = \int e^{-iz\cdot\xi} g(x,z)dz \tag{2.63}$$

the estimate (2.62) translates to

$$\begin{gathered}\sup_{x,\xi} \left| D_x^\alpha \xi^\beta D_\xi^\gamma b(x,\xi) \right| < \infty \ \forall \ \alpha,\beta,\gamma \\ \Longleftrightarrow b \in S_\infty^{-\infty}(\mathbb{R}_x^n;\mathbb{R}_\xi^n).\end{gathered} \tag{2.64}$$

Now the inverse Fourier transform in (2.63), combined with (2.61) gives

$$K(x,y) = g(x,x-y) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} b(x,\xi)d\xi \tag{2.65}$$

i.e. $K = I(b)$. This certainly proves the proposition and actually gives the stronger result.

$$A \in \Psi_\infty^{-\infty}(\mathbb{R}^n) \Longleftrightarrow A = I(c),\ c \in S_\infty^{-\infty}(\mathbb{R}_x^n;\mathbb{R}_\xi^n). \tag{2.66}$$

□

This also finishes the proof of Proposition 2.2 since in (2.58), (2.59) we have shown that

$$A = B + R,\ B = I(b), \quad R \in \Psi_\infty^{-\infty}(\mathbb{R}^n) \tag{2.67}$$

so in fact

$$A = I(e),\ e \in S_\infty^m(\mathbb{R}_x^n;\mathbb{R}_\xi^n),\ e \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} \big(D_y^\alpha D_\xi^\alpha a\big)(x,x,\xi). \tag{2.68}$$

□

2.7. Proof of Composition Theorem

First consider the adjoint formula. If

$$A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$$

the adjoint is the operator

$$A^* : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$$

defined by duality:

$$A^*u(\bar\phi) = u(\overline{A\phi})\ \forall\ \phi \in \mathcal{S}(\mathbb{R}^n). \tag{2.69}$$

Certainly $A^*u \in \mathcal{S}'(\mathbb{R}^n)$ if $u \in \mathcal{S}'(\mathbb{R}^n)$ since

$$A^*u(\psi) = u(\overline{A\overline{\psi}}) \text{ and } \mathcal{S}(\mathbb{R}^n) \ni \psi \longmapsto \overline{A\overline{\psi}} \in \mathcal{S}(\mathbb{R}^n) \tag{2.70}$$

is clearly continuous. In terms of Schwartz kernels,

$$\begin{gathered}A\phi(x) = \int K_A(x,y)\phi(y)\,dy,\ \phi \in \mathcal{S}(\mathbb{R}^n) \\ A^*u(x) = \int K_{A^*}(x,y)u(y)\,dy,\ u \in \mathcal{S}(\mathbb{R}^n).\end{gathered} \tag{2.71}$$

We then see that

$$\begin{aligned}\int K_{A^*}(x,y)u(y)\overline{\phi(x)}dydx &= \int \overline{K_A(x,y)\phi(y)}dyu(x)dx \\ \Longrightarrow K_{A^*}(x,y) &= \overline{K_A(y,x)}\end{aligned} \tag{2.72}$$

where we are using the uniqueness of Schwartz' kernels.

This proves (2.41) since

$$\begin{aligned}\overline{K_A(y,x)} &= \overline{[\frac{1}{(2\pi)^n}\int e^{i(y-x)\cdot\xi}a(y,x,\xi)d\xi]} \\ &= \frac{1}{(2\pi)^n}\int e^{i(x-y)\cdot\xi}\bar{a}(y,x,\xi)d\xi\end{aligned} \tag{2.73}$$

i.e. $A^* = I(\bar{a}(y,x,\xi))$. Thus one advantage of allowing general operators (2.34) is that closure under the passage to adjoint is immediate.

For the composition formula we need to apply Proposition 2.2 twice. First to $A \in \Psi^m_\infty(\mathbb{R}^n)$, to write it with symbol $a(x,\xi)$

$$\begin{aligned}A\phi(x) &= (2\pi)^{-n}\int e^{i(x-y)\cdot\xi}a(x,\xi)\phi(y)dyd\xi \\ &= (2\pi)^{-n}\int e^{ix\cdot\xi}a(x,\xi)\hat{\phi}(\xi)d\xi.\end{aligned}$$

Then we also apply Proposition 2.2 to B^*,

$$B^*u(x) = (2\pi)^{-n}\int e^{ix\cdot\xi}\bar{b}(x,\xi)\hat{u}(\xi)d\xi.$$

Integrating this against a test function $\phi \in \mathcal{S}(\mathbb{R}^n)$ gives

$$\begin{aligned}\langle B\phi, u\rangle = \langle \phi, B^*u\rangle &= (2\pi)^{-n}\int\int e^{-ix\cdot\xi}\phi(x)b(x,\xi)\overline{\hat{u}(\xi)}d\xi dx \\ \Longrightarrow \widehat{B\phi}(\xi) &= \int e^{-iy\cdot\xi}b(y,\xi)\phi(y)dy.\end{aligned} \tag{2.74}$$

Inserting this into the formula for $A\phi$ shows that

$$\Longrightarrow AB(u) = (2\pi)^{-n}\int e^{i(x-y)\cdot\xi}a(x,\xi)b(y,\xi)u(y)dyd\xi.$$

Since $a(x,\xi)b(y,\xi) \in S^{m+m'}_\infty(\mathbb{R}^{2n}_{(x,y)};\mathbb{R}^n_\xi)$ this shows that $AB \in \Psi^{m+m'}_\infty(\mathbb{R}^n)$ as claimed.

2.8. Quantization and symbols

So, we have now shown that there is an 'oscillatory integral' interpretation of

$$K(x,y) = (2\pi)^{-n}\int e^{i(x-y)\cdot\xi}a(x,y,\xi)d\xi = I(a) \tag{2.75}$$

which defines, for any $w \in \mathbb{R}$, a continuous linear map

$$I : (1+|x-y|^2)^{\frac{w}{2}}S^\infty_\infty(\mathbb{R}^{2n};\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^{2n})$$

the range of which is the space of pseudodifferential operators on $\mathbb{R}^n$;

$$\begin{aligned}&A \in \Psi^m_\infty(\mathbb{R}^n) \Longleftrightarrow A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n) \text{ and} \\ &\exists\, w \text{ s.t. } K_A(x,y) = I(a),\ a \in \left(1+|x-y|^2\right)^{\frac{w}{2}}S^m_\infty\left(\mathbb{R}^{2n};\mathbb{R}^n\right).\end{aligned} \tag{2.76}$$

Furthermore, we have shown in Proposition 2.2 that the special case, $w = 0$ and $\partial_y a \equiv 0$, gives an isomorphism

$$\Psi^m_\infty(\mathbb{R}^n) \underset{q_L}{\overset{\sigma_L}{\rightleftarrows}} S^m_\infty(\mathbb{R}^n;\mathbb{R}^n). \tag{2.77}$$

The map here, $q_L = I$ on symbols independent of y, is the *left quantization map* and its inverse σ_L is the *left full symbol map.* Next we consider some more consequences of this reduction theorem.

As well as the left quantization map leading to the isomorphism (2.77) there is a right quantization map, similarly derived from (2.75):

$$q_R(a) = (2\pi)^{-n}\int e^{i(x-y)\cdot\xi}a(y,\xi)d\xi, \quad a \in S^m_\infty\left(\mathbb{R}^n;\mathbb{R}^n\right). \tag{2.78}$$

In fact using the adjoint operator, $*$, on operators and writing as well $*$ for complex conjugation of symbols shows that

$$q_R = * \cdot q_L \cdot * \tag{2.79}$$

is also an isomorphism, with inverse σ_R[4]

$$\Psi^m_\infty(\mathbb{R}^n) \underset{q_R}{\overset{\sigma_R}{\rightleftarrows}} S^m_\infty\left(\mathbb{R}^n;\mathbb{R}^n\right). \tag{2.80}$$

Using the proof of the reduction theorem we find:

LEMMA 2.4. *For any* $a \in S^m_\infty\left(\mathbb{R}^n;\mathbb{R}^n\right)$,

$$\sigma_L\left(q_R(a)\right)(x,\xi) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!}D^\alpha_x D^\alpha_\xi a(x,\xi) \sim e^{i<D_x,D_\xi>}a. \tag{2.81}$$

For the moment the last asymptotic equality is just to help in remembering the formula, which is the same as given by the formal Taylor series expansion at the origin of the exponential.

PROOF. This follows from the general formula (2.68). □

2.9. Principal symbol

One important thing to note from (2.81) is that

$$D^\alpha_x D^\alpha_\xi a(x,\xi) \in S^{m-|\alpha|}_\infty\left(\mathbb{R}^n;\mathbb{R}^n\right) \tag{2.82}$$

so that for *any* pseudodifferential operator

$$A \in \Psi^m_\infty(\mathbb{R}^n) \Longrightarrow \sigma_L(A) - \sigma_R(A) \in S^{m-1}_\infty\left(\mathbb{R}^n;\mathbb{R}^n\right). \tag{2.83}$$

For this reason we consider the general quotient spaces

$$S^{m-[1]}_\infty\left(\mathbb{R}^p;\mathbb{R}^n\right) = S^m_\infty\left(\mathbb{R}^p;\mathbb{R}^n\right)/S^{m-1}_\infty\left(\mathbb{R}^p;\mathbb{R}^n\right) \tag{2.84}$$

and, for $a \in S^m_\infty(\mathbb{R}^p;\mathbb{R}^n)$, write $[a]$ for its image, i.e. equivalence class, in the quotient space $S^{m-[1]}_\infty\left(\mathbb{R}^p;\mathbb{R}^n\right)$. The 'principal symbol map'

$$\begin{gathered}\sigma_m : \Psi^m_\infty(\mathbb{R}^n) \longrightarrow S^{m-[1]}_\infty\left(\mathbb{R}^n;\mathbb{R}^n\right)\\ \text{is defined by } \sigma_m(A) = [\sigma_L(A)] = [\sigma_R(A)].\end{gathered} \tag{2.85}$$

[4]This involves the left and right symbols, see Problem 5.1 for another the more centrist 'Weyl' quantization.

As distinct from σ_L or σ_R, σ_m depends on m, i.e. one needs to know that the order is at most m before it is defined.

The isomorphism (2.77) is replaced by a weaker (but very useful) exact sequence.

LEMMA 2.5. *For every $m \in \mathbb{R}$*

$$0 \hookrightarrow \Psi^{m-1}_\infty(\mathbb{R}^n) \hookrightarrow \Psi^m_\infty(\mathbb{R}^n) \xrightarrow{\sigma_m} S^{m-[1]}_\infty(\mathbb{R}^n;\mathbb{R}^n) \longrightarrow 0$$

is a short exact sequence (the 'principal symbol sequence' or simply the 'symbol sequence').

PROOF. This is just the statement that the range of each map is the null space of the next i.e. that σ_m is surjective, which follows from (2.77), and that the null space of σ_m is just $\Psi^{m-1}_\infty(\mathbb{R}^n)$ and this is again (2.77) and the definition of σ_m. □

The fundamental result proved above is that

$$\Psi^m_\infty(\mathbb{R}^n) \cdot \Psi^{m'}_\infty(\mathbb{R}^n) \subset \Psi^{m+m'}_\infty(\mathbb{R}^n). \tag{2.86}$$

In fact we showed that if $A = q_L(a)$, $a \in S^m_\infty(\mathbb{R}^n;\mathbb{R}^n)$ and $B = q_R(b)$, $b \in S^{m'}_\infty(\mathbb{R}^n;\mathbb{R}^n)$ then the composite operator has Schwartz kernel

$$K_{A\cdot B}(x,y) = I\left(a(x,\xi)b(y,\xi)\right)$$

Using the formula (2.68) again we see that

$$\sigma_L(A\cdot B) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} D^\alpha_\xi\big[a(x,\xi)D^\alpha_x b(x,\xi)\big]. \tag{2.87}$$

Of course $b = \sigma_R(B)$ so we really want to rewrite (2.87) in terms of $\sigma_L(B)$.

LEMMA 2.6. *If $A \in \Psi^m_\infty(\mathbb{R}^n)$ and $B \in \Psi^{m'}_\infty(\mathbb{R}^n)$ then $A \circ B \in \Psi^{m+m'}_\infty(\mathbb{R}^n)$ and*

$$\sigma_{m+m'}(A\circ B) = \sigma_m(A)\cdot\sigma_{m'}(B), \tag{2.88}$$

$$\sigma_L(A\circ B) \sim \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} D^\alpha_\xi \sigma_L(A)\cdot D^\alpha_x\sigma_L(B). \tag{2.89}$$

PROOF. The simple formula (2.88) is already immediate from (2.87) since all terms with $|\alpha| \geq 1$ are of order $m+m'-|\alpha| \leq m+m'-1$. To get the 'full' formula (2.89) we can insert into (2.87) the inverse of (2.81), namely

$$\sigma_R(x,\xi) \sim \sum_\alpha \frac{(-i)^{|\alpha|}}{\alpha!} D^\alpha_\xi D^\alpha_x \sigma_L(x,\xi) \sim e^{-i<D_x,D_\xi>}\sigma_L(x,\xi).$$

This gives the double sum (still asymptotically convergent)

$$\sigma_L(A\circ B) \sim \sum_\beta\sum_\alpha \frac{i^{|\alpha|}}{\alpha!} D^\alpha_\xi\big[\sigma_L(A) D^\alpha_x \frac{i^{|\beta|}}{\beta!} D^\beta_x D^\beta_\xi \sigma_L(B)\big].$$

Setting $\gamma = \alpha+\beta$ this becomes

$$\sigma_L(A\circ B) \sim \sum_\gamma \frac{i^{|\gamma|}}{\gamma!} \sum_{0\leq\alpha\leq\gamma} \frac{\gamma!(-1)^{|\gamma-\alpha|}}{\alpha!(\gamma-\alpha)!} D^\alpha_\xi\big[\sigma_L(A)\times D^{\gamma-\alpha}_\xi D^\gamma_x \sigma_L(B)\big].$$

Then Leibniz' formula shows that this sum over α can be rewritten as

$$\sigma_L(A\circ B)\sim\sum_{\gamma}\frac{i^{|\gamma|}}{\gamma!}D_\xi^\gamma\sigma_L(A)\cdot D_x^\gamma\sigma_L(B)$$
$$\sim e^{i<D_y,D_\xi>}\sigma_L(A)(x,\xi)\sigma_L(B)(y,\eta)\big|_{y=x,\eta=\xi}.$$

This is just (2.89). □

The simplicity of (2.88) over (2.89) is achieved at the expense of enormous loss of information. Still, many problems can be solved using (2.88) which we can think of as saying that the principal symbol maps give a homomorphism, for instance from the filtered algebra $\Psi^0_\infty(\mathbb{R}^n)$ to the *commutative* algebra $S^{0-[1]}_\infty(\mathbb{R}^n;\mathbb{R}^n)$.

2.10. Ellipticity

We say that an element of $\Psi^m_\infty(\mathbb{R}^n)$ is *elliptic* if it is invertible modulo an error in $\Psi^{-\infty}_\infty(\mathbb{R}^n)$ with the approximate inverse of order $-m$ i.e.

$$\begin{aligned}&A\in\Psi^m_\infty(\mathbb{R}^n)\text{ is elliptic}\\&\Longleftrightarrow\exists\quad B\in\Psi^{-m}_\infty(\mathbb{R}^n)\text{ s.t. }\ A\circ B-\mathrm{Id}\in\Psi^{-\infty}_\infty(\mathbb{R}^n).\end{aligned}\tag{2.90}$$

Thus ellipticity, here by definition, is invertibility in $\Psi^m_\infty(\mathbb{R}^n)/\Psi^{-\infty}_\infty(\mathbb{R}^n)$, so the inverse lies in $\Psi^{-m}_\infty(\mathbb{R}^n)/\Psi^{-\infty}_\infty(\mathbb{R}^n)$. The point about ellipticity is that it is a phenomenon of the *principal symbol.*

THEOREM 2.3. *The following conditions on $A\in\Psi^m_\infty(\mathbb{R}^n)$ are equivalent*

$$A\text{ is elliptic}\tag{2.91}$$

$$\exists\ [b]\in S^{-m-[1]}_\infty(\mathbb{R}^n;\mathbb{R}^n)\ \text{ s.t. }\ \sigma_m(A)\cdot[b]\equiv1\ \text{ in }\ S^{0-[1]}_\infty(\mathbb{R}^n;\mathbb{R}^n)\tag{2.92}$$

$$\exists\ b\in S^{-m}_\infty(\mathbb{R}^n;\mathbb{R}^n)\ \text{ s.t. }\ \sigma_L(A)\cdot b-1\in S^{-\infty}_\infty(\mathbb{R}^n;\mathbb{R}^n)\tag{2.93}$$

$$\exists\ \epsilon>0\ \text{ s.t. }\ \big|\sigma_L(A)(x,\xi)\big|\ge\epsilon(1+|\xi|)^m\ \text{ in }\ |\xi|>\frac{1}{\epsilon}.\tag{2.94}$$

PROOF. We shall show

$$(2.91)\Longrightarrow(2.92)\Longrightarrow(2.93)\Longleftrightarrow(2.94)\Longrightarrow(2.91).\tag{2.95}$$

In fact Lemma 2.2 shows the equivalence of (2.93) and (2.94). Since we know that $\sigma_0(\mathrm{Id})=1$ applying the identity (2.88) to the definition of ellipticity in (2.90) gives

$$\sigma_m(A)\cdot\sigma_{-m}(B)\equiv1\text{ in }S^{0-[1]}_\infty(\mathbb{R}^n,\mathbb{R}^n),\tag{2.96}$$

i.e. that (2.91) $\Longrightarrow$ (2.92).

Now assuming (2.96) (i.e. (2.92)), and recalling that $\sigma_m(A)=[\sigma_L(A)]$ we find that a representative b_1 of the class $[b]$ must satisfy

$$\sigma_L(A)\cdot b_1=1+e_1,\quad e_1\in S^{-1}_\infty(\mathbb{R}^n;\mathbb{R}^n),\tag{2.97}$$

this being the meaning of the equality of residue classes. Now for the remainder, $e_1\in S^{-1}_\infty(\mathbb{R}^n;\mathbb{R}^n)$, the Neumann series

$$f\sim\sum_{j\ge1}(-1)^je_1^j\tag{2.98}$$

is asymptotically convergent, so $f\in S^{-1}_\infty(\mathbb{R}^n;\mathbb{R}^n)$ exists, and

$$(1+f)\cdot(1+e_1)=1+e_\infty,\quad e_\infty\in S^{-\infty}_\infty(\mathbb{R}^n;\mathbb{R}^n).\tag{2.99}$$

Then multiplying (2.97) by $(1+f)$ gives

$$\sigma_L(A)\cdot\{b_1(1+f)\} = 1 + e_\infty \tag{2.100}$$

which proves (2.93), since $b = b_1(1+f) \in S^{-m}_\infty(\mathbb{R}^n;\mathbb{R}^n)$. Of course

$$\sup(1+|\xi|)^N|e_\infty| < \infty \quad \forall\ N \tag{2.101}$$

so

$$\exists\ C \text{ s.t. } \ |e_\infty(x,\xi)| < \frac{1}{2} \text{ in } |\xi| > C. \tag{2.102}$$

From (2.100) this means

$$\big|\sigma_L(A)(x,\xi)\big|\cdot\big|b(x,\xi)\big| \ge \frac{1}{2},\ |\xi| > C. \tag{2.103}$$

Since $|b(x,\xi)| \le C(1+|\xi|)^{-m}$ (being a symbol of order $-m$), (2.103) implies

$$\inf_{|\xi|\ge C}\big|\sigma_L(A)(x,\xi)\big|(1+|\xi|)^{-m} \ge C > 0. \tag{2.104}$$

which shows that (2.93) implies (2.94).

Conversely, as already remarked, (2.94) implies (2.93).

Now suppose (2.93) holds. Set $B_1 = q_L(b)$ then from (2.88) again

$$\sigma_0(A\circ B_1) = [q_m(A)]\cdot[b] \equiv 1. \tag{2.105}$$

That is,

$$A\circ B_1 - \mathrm{Id} = E_1 \in \Psi^{-1}_\infty(\mathbb{R}^n). \tag{2.106}$$

Consider the Neumann series of operators

$$\sum_{j\ge1}(-1)^j E_1^j. \tag{2.107}$$

The corresponding series of (left-reduced) symbols is asymptotically summable so we can choose $F \in \Psi^{-1}_\infty(\mathbb{R}^n)$ with

$$\sigma_L(F) \sim \sum_{j\ge1}(-1)^j\sigma_L(E_1^j). \tag{2.108}$$

Then

$$(\mathrm{Id}+E_1)(\mathrm{Id}+F) = \mathrm{Id}+E_\infty,\ E_\infty \in \Psi^{-\infty}_\infty(\mathbb{R}^n). \tag{2.109}$$

Thus $B = B_1(\mathrm{Id}+F) \in \Psi^{-m}_\infty(\mathbb{R}^n)$ satisfies (2.90) and it follows that A is elliptic. □

In the definition of ellipticity in (2.90) we have taken B to be a 'right parametrix', i.e. a right inverse modulo $\Psi^{-\infty}_\infty(\mathbb{R}^n)$. We can just as well take it to be a *left* parametrix.

Lemma 2.7. *$A \in \Psi^m_\infty(\mathbb{R}^n)$ is elliptic if and only if there exists $B' \in \Psi^{-m}_\infty(\mathbb{R}^n)$ such that*

$$B'\circ A = \mathrm{Id}+E',\ E' \in \Psi^{-\infty}_\infty(\mathbb{R}^n) \tag{2.110}$$

and then if B satisfies (2.90), $B - B' \in \Psi^{-\infty}_\infty(\mathbb{R}^n)$.

PROOF. Certainly (2.110) implies $\sigma_{-m}(B')\cdot\sigma_m(A)\equiv 1$, and the multiplication here is commutative so (2.92) holds and A is elliptic. Conversely if A is elliptic we get in place of (2.106)

$$B_1 \circ A - \mathrm{Id} = E_1' \in \Psi_\infty^{-1}(\mathbb{R}^n).$$

Then defining F' as in (2.108) with E_1' in place of E_1 we get $(\mathrm{Id}+F')(\mathrm{Id}+E_1') = \mathrm{Id}+E_\infty'$ and then $B' = (\mathrm{Id}+F')\circ B_1$ satisfies (2.110). Thus 'left' ellipticity as in (2.110) is equivalent to right ellipticity. Applying B to (2.110) gives

$$B'\circ(\mathrm{Id}+E) = B'\circ(A\circ B) = (\mathrm{Id}+E')\circ B \tag{2.111}$$

which shows that $B - B' \in \Psi_\infty^{-\infty}(\mathbb{R}^n)$. $\square$

Thus a left parametrix of an elliptic element of $\Psi_\infty^m(\mathbb{R}^n)$ is always a right, hence two-sided, parametrix and such a parametrix is unique up to an additive term in $\Psi_\infty^{-\infty}(\mathbb{R}^n)$.

2.11. Elliptic regularity and the Laplacian

One of the main reasons that the 'residual' terms *are* residual is that they are smoothing operators.

LEMMA 2.8. *If $E \in \Psi_\infty^{-\infty}(\mathbb{R}^n)$ then*

$$E : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)\cap\mathcal{C}^\infty(\mathbb{R}^n). \tag{2.112}$$

PROOF. This follows from Proposition 2.4 since we can regard the kernel as a $\mathcal{C}^\infty$ function of x taking values in $\mathcal{S}(\mathbb{R}^n_y)$. $\square$

Directly from the existence of parametrices for elliptic operators we can deduce the regularity of solutions to elliptic (pseudodifferential) equations.

PROPOSITION 2.5. *If $A \in \Psi_\infty^m(\mathbb{R}^n)$ is elliptic and $u \in \mathcal{S}'(\mathbb{R}^n)$ satifies $Au = 0$ then $u \in \mathcal{C}^\infty(\mathbb{R}^n)$.*

PROOF. Let $B \in \Psi_\infty^{-m}(\mathbb{R}^n)$ be a parametrix for A. Then $B\circ A = \mathrm{Id}+E$, $E \in \Psi_\infty^{-\infty}(\mathbb{R}^n)$. Thus,

$$u = (BA - E)u = -Eu \tag{2.113}$$

and the conclusion follows from Lemma 2.8. $\square$

Suppose that $g_{ij}(x)$ are the components of an '∞-metric' on $\mathbb{R}^n$, i.e.

$$g_{ij}(x) \in \mathcal{C}_\infty^\infty(\mathbb{R}^n), i,j = 1,\dots,n$$

$$\Big|\sum_{i,j=1}^n g_{ij}(x)\xi_i\xi_j\Big| \geq \epsilon|\xi|^2 \quad \forall\ x\in\mathbb{R}^n, \xi\in\mathbb{R}^n,\ \epsilon>0. \tag{2.114}$$

The *Laplacian* of the metric is the second order differential operator

$$\Delta_g = \sum_{i,j=1}^n \frac{1}{\sqrt{g}} D_{x_i} g^{ij}\sqrt{g} D_{x_j} \tag{2.115}$$

where

$$g(x) = \det g^{ij}(x),\ g^{ij}(x) = (g_{ij}(x))^{-1}.$$

The Laplacian is determined by the integration by parts formula

$$\int_{\mathbb{R}^n} \sum_{i,j} g^{ij}(x) D_{x_i}\phi \cdot \overline{D_{x_j}\psi} dg = \int \Delta_g \phi \cdot \overline{\psi} dg \ \forall\ \phi, \psi \in \mathcal{S}(\mathbb{R}^n) \tag{2.116}$$

where

$$dg = \sqrt{g} dx. \tag{2.117}$$

Our assumption in (2.114) shows that $\Delta = \varDelta_g \in \mathrm{Diff}^2_\infty(\mathbb{R}^n) \subset \Psi^2_\infty(\mathbb{R}^n)$ is in fact elliptic, since

$$\sigma_2(\Delta) = \sum_{i,j=1} g^{ij} \xi_i \xi_j. \tag{2.118}$$

Thus Δ has a two-sided parametrix $B \in \Psi^{-2}_\infty(\mathbb{R}^n)$

$$\Delta \circ B \equiv B \circ \Delta \equiv \mathrm{Id} \quad \mod \quad \Psi^{-\infty}_\infty(\mathbb{R}^n). \tag{2.119}$$

In particular we see from Proposition 2.5 that $\Delta u = 0$, $u \in \mathcal{S}'(\mathbb{R}^n)$ implies $u \in \mathcal{C}^\infty(\mathbb{R}^n)$.

2.12. L^2 boundedness

So far we have thought of pseudodifferential operators, the elements of $\Psi^m_\infty(\mathbb{R}^n)$ for some m, as defining continuous linear operators on $\mathcal{S}(\mathbb{R}^n)$ and, by duality, on $\mathcal{S}'(\mathbb{R}^n)$. Now that we have proved the composition formula we can use it to prove other 'finite order' regularity results. The basic one of these is L^2 boundedness:

Proposition 2.6. *[Boundedness] If $A \in \Psi^0_\infty(\mathbb{R}^n)$ then, by continuity from $\mathcal{S}(\mathbb{R}^n)$, A defines a bounded linear operator*

$$A : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n). \tag{2.120}$$

Our proof will be in two stages, the first part is by direct estimation. Namely, *Schur's lemma* gives a useful criterion for an integral operator to be bounded on L^2.

Lemma 2.9 (Schur). *If $K(x, y)$ is locally integrable on $\mathbb{R}^{2n}$ and is such that*

$$\sup_{x\in\mathbb{R}^n} \int_{\mathbb{R}^n} |K(x, y)| dy, \ \sup_{y\in\mathbb{R}^n} \int_{\mathbb{R}^n} |K(x, y)| dx < \infty \tag{2.121}$$

then the operator $K : \phi \longmapsto \int_{\mathbb{R}^n} K(x, y)\phi(y)dy$ is bounded on $L^2(\mathbb{R}^n)$.

Proof. Since $\mathcal{S}(\mathbb{R}^n)$ is dense[5] in $L^2(\mathbb{R}^n)$ we only need to show the existence of a constant, C, such that

$$\int \left|K\phi(x)\right|^2 dx \le C \int |\phi|^2 \quad \forall\ \phi \in \mathcal{S}(\mathbb{R}^n). \tag{2.122}$$

Writing out the integral on the left

$$\begin{aligned} &\int \Big| \int K(x, y)\phi(y)dy\Big|^2 dx \\ &= \iiint K(x, y)\overline{K(x, z)}\phi(y)\overline{\phi(z)}\ dydzdx \end{aligned} \tag{2.123}$$

[5]Ssee Problem 2.18

is certainly absolutely convergent and

$$\begin{aligned}&\int \left|K\phi(x)\right|^2 dx\\ &\le \left(\iiint \left|K(x,y)K(x,z)\right|\left|\phi(y)\right|^2 dydxdz\right)^{\frac12}\\ &\times \left(\iiint \left|K(x,y)K(x,z)\right|\left|\phi(z)\right|^2 dzdxdy\right)^{\frac12}.\end{aligned}$$

These two factors are the same. Since

$$\int \left|K(x,y)\right|\left|K(x,z)\right| dx\,dz \le \sup_{x\in\mathbb{R}^n} \int \left|K(x,z)\right| dz \cdot \sup_{y\in\mathbb{R}^n} \int \left|K(x,y\right| dx$$

(2.122) follows. Thus (2.121) gives (2.122). □

This standard lemma immediately implies the L^2 boundedness of the 'residual terms.' Thus, if $K \in \Psi^{-\infty}_{\infty}(\mathbb{R}^n)$ then its kernel satisfies (2.60). This in particular implies

$$\left|K(x,y)\right| \le C\,(1+|x-y|)^{-n-1}$$

and hence that K satisfies (2.121). Thus

$$\text{each } K \in \Psi^{-\infty}_{\infty}(\mathbb{R}^n) \text{ is bounded on } L^2(\mathbb{R}^n). \tag{2.124}$$

2.13. Square root and boundedness

To prove the general result, (2.120), we shall use the clever idea, due to Hörmander, of using the (approximate) square root of an operator. We shall say that an element $[a] \in S^{m-[1]}_{\infty}(\mathbb{R}^n;\mathbb{R}^n)$ is positive if there is some $0 < a \in S^m(\mathbb{R}^n;\mathbb{R}^n)$ in the equivalence class.

Proposition 2.7. *Suppose $A \in \Psi^m_{\infty}(\mathbb{R}^n)$, $m > 0$, is self-adjoint, $A = A^*$, and elliptic with a positive principal symbol, then there exists $B \in \Psi^{m/2}_{\infty}(\mathbb{R}^n)$, $B = B^*$, such that*

$$A = B^2 + G, \quad G \in \Psi^{-\infty}_{\infty}(\mathbb{R}^n). \tag{2.125}$$

Proof. This is a good exercise in the use of the symbol calculus. Let $a \in S^m_{\infty}(\mathbb{R}^n;\mathbb{R}^n)$, $a > 0$, be a positive representative of the principal symbol of A. Now[6]

$$b_0 = a^{\frac12} \in S^{m/2}_{\infty}(\mathbb{R}^n;\mathbb{R}^n). \tag{2.126}$$

Let $B_0 \in \Psi^{m/2}_{\infty}(\mathbb{R}^n)$ have principal symbol b_0. We can assume that $B_0 = B_0^*$, since if not we just replace B_0 by $\frac12(B_0 + B_0^*)$ which has the same principal symbol.

The symbol calculus shows that $B_0^2 \in \Psi^m_{\infty}(\mathbb{R}^n)$ and

$$\sigma_m(B_0^2) = \left(\sigma_{m/2}(B_0)\right)^2 = b_0^2 = a_0 \mod S^{m-1}_{\infty}.$$

Thus

$$A - B_0^2 = E_1 \in \Psi^{m-1}_{\infty}(\mathbb{R}^n). \tag{2.127}$$

[6]See Problem 2.19 for an outline of the proof

Then we proceed inductively. Suppose we have chosen $B_j \in \Psi_\infty^{m/2-j}(\mathbb{R}^n)$, with $B_j^* = B_j$, for $j \le N$ such that

$$A - \left(\sum_{j=0}^{N} B_j\right)^2 = E_{N+1} \in \Psi_\infty^{m-N-1}(\mathbb{R}^n). \tag{2.128}$$

Of course we *have* done this for $N = 0$. Then see the effect of adding $B_{N+1} \in \Psi_\infty^{m/2-N-1}(\mathbb{R}^n)$:

$$A - \left(\sum_{j=0}^{N+1} B_j\right)^2 = E_{N+1} - \left(\sum_{j=0}^{N} B_j\right) B_{N+1} - B_{N+1}\left(\sum_{j=0}^{N} B_j\right) - B_{N+1}^2. \tag{2.129}$$

On the right side all terms are of order $m - N - 2$, except for

$$E_{N+1} - B_0 B_{N+1} - B_{N+1} B_0 \in \Psi_\infty^{m-N-1}(\mathbb{R}^n). \tag{2.130}$$

The principal symbol, of order $m - N - 1$, of this is just

$$\sigma_{m-N-1}(E_{N+1}) - 2\ b_0 \cdot \sigma_{\frac{m}{2}-N-1}(B_{N+1}). \tag{2.131}$$

Thus if we choose $B_{N+1} \in \Psi_\infty^{\frac{m}{2}-N-1}(\mathbb{R}^n)$ with

$$\sigma_{m/2-N-1}(B_{N+1}) = \frac{1}{2}\frac{1}{b_0} \cdot \sigma_{m-N-1}(E_{N+1})$$

and replace B_{N+1} by $\frac{1}{2}(B_{N+1} + B_{N+1}^*)$, we get the inductive hypothesis for $N+1$. Thus we have arranged (2.128) for every N. Now define $B = \frac{1}{2}(B' + (B')^*)$ where

$$\sigma_L(B') \sim \sum_{j=0}^{\infty} \sigma_L(B_j). \tag{2.132}$$

Since all the B_j are self-adjoint B also satisfies (2.132) and from (2.128)

$$A - B^2 = A - \left(\sum_{j=0}^{N} B_j + B_{(N+1)}\right)^2 \in \Psi_\infty^{m-N-1}(\mathbb{R}^n) \tag{2.133}$$

for every N, since $B_{(N+1)} = B - \sum_{j=0}^{N} B_j \in \Psi_\infty^{m/2-N-1}(\mathbb{R}^n)$. Thus $A - B^2 \in \Psi_\infty^{-\infty}(\mathbb{R}^n)$ and we have proved (2.125), and so Proposition 2.7. $\square$

Here is Hörmander's argument to prove Proposition 2.6. We want to show that

$$\|A\phi\| \le C\|\phi\| \quad \forall\ \phi \in \mathcal{S}(\mathbb{R}^n) \tag{2.134}$$

where $A \in \Psi_\infty^0(\mathbb{R}^n)$. The square of the left side can be written

$$\int A\phi \cdot \overline{A\phi} dx = \int \phi \cdot \overline{(A^* A\phi)} dx.$$

So it suffices to show that

$$\langle \phi, A^* A\phi \rangle \le C\|\phi\|^2. \tag{2.135}$$

Now $A^*A \in \Psi^0_\infty(\mathbb{R}^n)$ with $\sigma_0(A^*A) = \overline{\sigma_0(A)}\sigma_0(A) \in \mathbb{R}$. If $C > 0$ is a large constant,

$$C > \sup_{x,\xi} \big|\sigma_L(A^*A)(x,\xi)\big|$$

then $C - A^*A$ has a positive representative of its principal symbol. We can therefore apply Proposition 2.7 to it:

$$C - A^*A = B^*B + G, \quad G \in \Psi^{-\infty}_\infty(\mathbb{R}^n). \tag{2.136}$$

This gives

$$\begin{aligned} \langle \phi, A^*A\phi\rangle =& C\langle\phi,\phi\rangle - \langle\phi, B^*B\phi\rangle - \langle \phi, G\phi\rangle \\ =& C\|\phi\|^2 - \|B\phi\|^2 - \langle\phi, G\phi\rangle. \end{aligned} \tag{2.137}$$

The second term on the right is negative and, since $G \in \Psi^{-\infty}_\infty(\mathbb{R}^n)$, we can use the residual case in (2.124) to conclude that

$$\big|\langle\phi, G\phi\rangle\big| \le C'\|\phi\|^2 \Longrightarrow \|A\phi\|^2 \le C\|\phi\|^2 + C'\|\phi\|^2,$$

so (2.120) holds and Proposition 2.6 is proved.

2.14. Sobolev boundedness

Using the basic boundedness result, Proposition 2.6, and the calculus of pseudodifferential operators we can prove more general results on the action of pseudodifferential operators on Sobolev spaces.

Recall that for any positive integer, k,

$$H^k(\mathbb{R}^n) = \big\{u \in L^2(\mathbb{R}^n);\ D^\alpha u \in L^2(\mathbb{R}^n)\ \forall\ |\alpha| \le k\big\}. \tag{2.138}$$

Using the Fourier transform we find

$$u \in H^k(\mathbb{R}^n) \Longrightarrow \xi^\alpha \hat{u}(\xi) \in L^2(\mathbb{R}^n) \quad \forall\ |\alpha| \le k. \tag{2.139}$$

Now these finitely many conditions can be written as just the one condition

$$\big(1 + |\xi|^2\big)^{k/2}\,\hat{u}(\xi) \in L^2(\mathbb{R}^n). \tag{2.140}$$

Notice that $a(\xi) = (1+|\xi|^2)^{k/2} = \langle\xi\rangle^k \in S^k_\infty(\mathbb{R}^n)$. Here we use the notation

$$\langle\xi\rangle = \big(1+|\xi|^2\big)^{\frac{1}{2}} \tag{2.141}$$

for a smooth (symbol) of the size of $1+|\xi|$, thus (2.140) just says

$$u \in H^k(\mathbb{R}^n) \Longleftrightarrow u \in \mathcal{S}'(\mathbb{R}^n) \text{ and } \langle D\rangle^k u \in L^2(\mathbb{R}^n). \tag{2.142}$$

For *negative* integers

$$H^k(\mathbb{R}^n) = \big\{u \in \mathcal{S}'(\mathbb{R}^n); u = \sum_{|\beta| \le -k} D^\beta u_\beta,\ u_\beta \in L^2(\mathbb{R}^n)\big\},\ -k \in \mathbb{N}. \tag{2.143}$$

The same sort of discussion applies, showing that

$$u \in H^k(\mathbb{R}^n) \Longleftrightarrow u \in \mathcal{S}'(\mathbb{R}^n) \text{ and } \langle D\rangle^k u \in L^2(\mathbb{R}^n),\ k \in \mathbb{Z}. \tag{2.144}$$

In view of this we define the Sobolev space $H^m(\mathbb{R}^n)$, for any real order, by

$$u \in H^m(\mathbb{R}^n) \Longleftrightarrow u \in \mathcal{S}'(\mathbb{R}^n) \text{ and } \langle D\rangle^m u \in L^2(\mathbb{R}^n). \tag{2.145}$$

It is a Hilbert space with

$$\|u\|^2_m = \|\langle D\rangle^m u\|^2_{L^2} = \int (1+|\xi|^2)^m |\hat{u}(\xi)|^2 d\xi. \tag{2.146}$$

Clearly we have

$$H^m(\mathbb{R}^n) \supseteq H^{m'}(\mathbb{R}^n) \text{ if } m' \geq m. \tag{2.147}$$

Notice that it is rather unfortunate that these spaces get *smaller* as m gets bigger, as opposed to the spaces $\Psi^m_\infty(\mathbb{R}^n)$ which get *bigger* with m. Anyway that's life and we have to think of

$$\begin{cases} H^\infty(\mathbb{R}^n) = \bigcap\limits_m H^m(\mathbb{R}^n) & \text{as the residual space} \\ H^{-\infty}(\mathbb{R}^n) = \bigcup\limits_m H^m(\mathbb{R}^n) & \text{as the big space.} \end{cases} \tag{2.148}$$

It is important to note that

$$\mathcal{S}(\mathbb{R}^n) \subsetneq H^\infty(\mathbb{R}^n) \subsetneq H^{-\infty}(\mathbb{R}^n) \subsetneq \mathcal{S}'(\mathbb{R}^n). \tag{2.149}$$

In particular we do *not* capture all the tempered distributions in $H^{-\infty}(\mathbb{R}^n)$. We therefore consider *weighted* versions of these Sobolev spaces:

$$\langle x\rangle^q H^m(\mathbb{R}^n) = \left\{u \in \mathcal{S}'(\mathbb{R}^n);\ \langle x\rangle^{-q} u \in H^m(\mathbb{R}^n)\right\}. \tag{2.150}$$

Theorem 2.4. *For each $q, m, M \in \mathbb{R}$ each $A \in \Psi^M_\infty(\mathbb{R}^n)$ defines a continuous linear map*

$$A : \langle x\rangle^q H^m(\mathbb{R}^n) \longrightarrow \langle x\rangle^q H^{m-M}(\mathbb{R}^n). \tag{2.151}$$

Proof. Let us start off with $q = 0$, so we want to show that

$$A : H^m(\mathbb{R}^n) \longrightarrow H^{m-M}(\mathbb{R}^n),\ A \in \Psi^M_\infty(\mathbb{R}^n). \tag{2.152}$$

Now from (2.145) we see that

$$\begin{aligned} u \in H^m(\mathbb{R}^n) &\iff \langle D\rangle^m u \in L^2(\mathbb{R}^n) \\ &\iff \langle D\rangle^{m-M}\langle D\rangle^M u \in L^2(\mathbb{R}^n) \iff \langle D\rangle^M u \in H^{m-M}(\mathbb{R}^n) \quad \forall\ m, M. \end{aligned} \tag{2.153}$$

That is,

$$\langle D\rangle^M : H^m(\mathbb{R}^n) \longleftrightarrow H^{m-M}(\mathbb{R}^n) \quad \forall\ m, M. \tag{2.154}$$

To prove (2.152) it suffices to show that

$$B = \langle D\rangle^{-M+m} \cdot A \cdot \langle D\rangle^{-m} : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n) \tag{2.155}$$

since then $A = \langle D\rangle^{-m+M} \cdot B \cdot \langle D\rangle^m$ maps $H^m(\mathbb{R}^n)$ to $H^{m-M}(\mathbb{R}^n)$:

$$\begin{tikzcd} H^m(\mathbb{R}^n) \arrow[r, "A"] \arrow[d, "\langle D\rangle^m"'] & H^{m-M}(\mathbb{R}^n) \arrow[d, "\langle D\rangle^{m-M}"] \\ L^2(\mathbb{R}^n) \arrow[r, "B"'] & L^2(\mathbb{R}^n). \end{tikzcd} \tag{2.156}$$

Since $B \in \Psi^0_\infty(\mathbb{R}^n)$, by the composition theorem, we already know (2.155).

Thus we have proved (2.152). To prove the general case, (2.151), we proceed in the same spirit. Thus $\langle x\rangle^q$ is an isomorphism from $H^m(\mathbb{R}^n)$ to $\langle x\rangle^q H^m(\mathbb{R}^n)$, by definition. So to get (2.151) we need to show that

$$Q = \langle x\rangle^{-q} \cdot A \cdot \langle x\rangle^q :\ H^m(\mathbb{R}^n) \longrightarrow H^{m-M}(\mathbb{R}^n), \tag{2.157}$$

i.e. satisfies (2.152). Consider the Schwartz kernel of Q. Writing A in left-reduced form, with symbol a,

$$K_Q(x,y) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} \langle x\rangle^{-q} a(x,\xi)d\xi \cdot \langle y\rangle^{q}. \tag{2.158}$$

Now if we check that

$$\langle x\rangle^{-q}\langle y\rangle^{q} a(x,\xi) \in \left(1+|x-y|^2\right)^{\frac{|q|}{2}} S^M_\infty\left(\mathbb{R}^{2n};\mathbb{R}^n\right) \tag{2.159}$$

then we know that $Q \in \Psi^M_\infty(\mathbb{R}^n)$ and we get (2.157) from (2.152). Thus we want to show that

$$\langle x-y\rangle^{-|q|}\frac{\langle y\rangle^q}{\langle x\rangle^q} a(x,\xi) \in S^M_\infty(\mathbb{R}^{2n};\mathbb{R}^n) \tag{2.160}$$

assuming of course that $a(x,\xi) \in S^M_\infty(\mathbb{R}^n;\mathbb{R}^n)$. By interchanging the variables x and y if necessary we can assume that $q < 0$. Consider separately the two regions

$$\begin{aligned} &\left\{(x,y); |x-y| < \frac{1}{4}(|x|+|y|)\right\} = \Omega_1 \\ &\left\{(x,y); |x-y| > \frac{1}{8}(|x|+|y|)\right\} = \Omega_2. \end{aligned} \tag{2.161}$$

In Ω_1, x is "close" to y, in the sense that

$$|x| \le |x-y| + |y| \le \frac{1}{4}(|x|+|y|) + |y| \Longrightarrow |x| \le \frac{4}{3}\cdot\frac{5}{4}|y| \le 2|y|. \tag{2.162}$$

Thus

$$\langle x-y\rangle^{-q}\cdot\frac{\langle x\rangle^{-q}}{\langle y\rangle^{-q}} \le C \quad \text{in } \Omega_1. \tag{2.163}$$

On the other hand in Ω_2,

$$|x|+|y| < 8|x-y| \Longrightarrow |x| < 8|x-y| \tag{2.164}$$

so again

$$\langle x-y\rangle^{-q}\frac{\langle x\rangle^{-q}}{\langle y\rangle^{-q}} \le C. \tag{2.165}$$

In fact we easily conclude that

$$\langle x-y\rangle^{-q}\frac{\langle y\rangle^{q}}{\langle x\rangle^{q}} \in \mathcal{C}^\infty_\infty(\mathbb{R}^n) \qquad \forall\ q, \tag{2.166}$$

since differentiation by x or y makes all terms "smaller". This proves (2.160), hence (2.159) and (2.157) and therefore (2.151), i.e. the theorem is proved. □

We can capture any tempered distribution in a weighted Sobolev space; this is really Schwartz' representation theorem which says that any $u \in \mathcal{S}'(\mathbb{R}^n)$ is of the form

$$u = \sum_{\text{finite}} x^\alpha D^\beta_x u_{\alpha\beta},\ u_{\alpha\beta} \text{ bounded and continuous.} \tag{2.167}$$

Clearly $\mathcal{C}^0_\infty(\mathbb{R}^n) \subset \langle x\rangle^{1+n}L^2(\mathbb{R}^n)$. Thus as a special case of Theorem 2.4,

$$D^\beta_x : \langle x\rangle^{1+n}L^2(\mathbb{R}^n) \longrightarrow \langle x\rangle^{1+n}H^{-|\beta|}(\mathbb{R}^n)$$

so

LEMMA 2.10.

$$\mathcal{S}'(\mathbb{R}^n) = \bigcup_M \langle x\rangle^M H^{-M}(\mathbb{R}^n). \tag{2.168}$$

The elliptic regularity result we found before can now be refined:

PROPOSITION 2.8. *If $A \in \Psi^m_\infty(\mathbb{R}^n)$ is elliptic then*

$$\begin{gathered} Au \in \langle x\rangle^p H^q(\mathbb{R}^n),\ u \in \langle x\rangle^{p'} H^{q'}(\mathbb{R}^n) \\ \Longrightarrow u \in \langle x\rangle^{p''} H^{q''}(\mathbb{R}^n),\ p'' = \max(p,p'),\ q'' = \max(q+m,q'). \end{gathered} \tag{2.169}$$

PROOF. The existence of a left parametrix for A, $B \in \Psi^{-m}_\infty(\mathbb{R}^n)$,

$$B \cdot A = \operatorname{Id} + G,\ G \in \Psi^{-\infty}_\infty(\mathbb{R}^n)$$

means that

$$u = B(Au) + Gu \in \langle x\rangle^p H^{q+m}(\mathbb{R}^n) + \langle x\rangle^{p'} H^\infty(\mathbb{R}^n) \subset \langle x\rangle^{p''} H^{q+m}(\mathbb{R}^n). \tag{2.170}$$

□

2.15. Polyhomogeneity

So far we have been considering operators $A \in \Psi^m_\infty(\mathbb{R}^n)$ which correspond, via (2.2), to amplitudes satisfying the symbol estimates (2.6), i.e. in $S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$. As already remarked, there are many variants of these estimates and corresponding spaces of pseudodifferential operators. Some *weakening* of the estimates is discussed in the problems below, starting with Problem 2.16. Here we consider a restriction of the spaces, in that we define

$$S^m_{\mathrm{ph}}(\mathbb{R}^p;\mathbb{R}) \subset S^m_\infty(\mathbb{R}^p;\mathbb{R}^n)\,. \tag{2.171}$$

The definition of the subspace (2.171) is straightforward. First we note that if $a \in \mathcal{C}^\infty(\mathbb{R}^p;\mathbb{R}^n)$ is homogeneous of degree $m \in \mathbb{C}$ in $|\xi| \ge 1$, then

$$a(z,t\xi) = t^m a(z,\xi),\ |t|,|\xi| \ge 1 \tag{2.172}$$

where for complex m we always mean the principal branch of t^m for $t > 0$. If it also satisfies the uniform regularity estimates

$$\sup_{z\in\mathbb{R}^n,\ |\xi|\le 2} |D^\alpha_z D^\beta_\xi a(z,\xi)| < \infty\ \forall\ \alpha,\beta\,, \tag{2.173}$$

then in fact

$$a \in S^{\Re m}_\infty(\mathbb{R}^p;\mathbb{R}^n)\,. \tag{2.174}$$

Indeed, (2.173) is exactly the restriction of the symbol estimates to $z \in \mathbb{R}^p$, $|\xi| \le 2$. On the other hand, in $|\xi| \ge 1$, $a(z,\xi)$ is homogeneous so

$$|D^\alpha_z D^\beta_\xi a(z,\xi)| = |\xi|^{m-|\beta|} |D^\alpha_z D^\beta_\xi a(z,\hat\xi)|\,, \quad \hat\xi = \frac{\xi}{|\xi|}$$

from which the symbol estimates follow.

DEFINITION 2.2. *For any $m \in \mathbb{C}$, the subspace of (one-step)*[7] *polyhomogeneous symbols is defined as a subset* (2.171) *by the requirement that $a \in S^m_{\mathrm{ph}}(\mathbb{R}^p;\mathbb{R}^n)$ if*

[7] For a somewhat more general class of polyhomogeneous symbols, see problem 2.8.

and only if there exist elements $a_{m-j}(z,\xi) \in S^{\Re m}_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$ which are homogeneous of degree $m-j$ in $|\xi|\geq 1$, for $j\in\mathbb{N}_0$, such that

$$a \sim \sum_j a_{m-j}\,. \tag{2.175}$$

Clearly

$$S^m_{\mathrm{ph}}(\mathbb{R}^p;\mathbb{R}^n)\cdot S^{m'}_{\mathrm{ph}}(\mathbb{R}^p;\mathbb{R}^n) \subset S^{m+m'}_{\mathrm{ph}}(\mathbb{R}^p;\mathbb{R}^n), \tag{2.176}$$

since the asymptotic expansion of the product is given by the formal product of the asymtotic expansion. In fact there is equality here, because

$$(1+|\xi|^2)^{m/2} \in S^m_{\mathrm{ph}}(\mathbb{R}^p;\mathbb{R}^n) \tag{2.177}$$

and multiplication by $(1+|\xi|^2)^{m/2}$ is an isomorphism of the space $S^0_{\mathrm{ph}}(\mathbb{R}^p;\mathbb{R}^n)$ onto $S^m_{\mathrm{ph}}(\mathbb{R}^p;\mathbb{R}^n)$. Furthermore differentiation with respect to z_j or ξ_l preserves asymptotic homogeneity so

$$\begin{matrix} D_{x_j}: S^m_{\mathrm{ph}}(\mathbb{R}^p;\mathbb{R}^n) \longrightarrow S^m_{\mathrm{ph}}(\mathbb{R}^p;\mathbb{R}^n) \\ D_{\xi_l}: S^m_{\mathrm{ph}}(\mathbb{R}^p;\mathbb{R}^n) \longrightarrow S^{m-1}_{\mathrm{ph}}(\mathbb{R}^p;\mathbb{R}^n) \end{matrix} \quad \forall j=1,\dots,n.$$

It is therefore no great surprise that the polyhomogeneous operators form a subalgebra.

PROPOSITION 2.9. *The spaces $\Psi^m_{\mathrm{ph}}(\mathbb{R}^n)\subset\Psi^m_{\infty}(\mathbb{R}^n)$ defined by the condition that the kernel of $A\in\Psi^m_{\mathrm{ph}}(\mathbb{R}^n)$ should be of the form $I(a)$ for some*

$$a \in (1+|x-y|^2)^{w/2} S^m_{\mathrm{ph}}(\mathbb{R}^{2n};\mathbb{R}^n), \tag{2.178}$$

are such that

$$\Psi^m_{\mathrm{ph}}(\mathbb{R}^n)\circ\Psi^{m'}_{\mathrm{ph}}(\mathbb{R}^n) = \Psi^{m+m'}_{\mathrm{ph}}(\mathbb{R}^n),\ (\Psi^m_{\mathrm{ph}}(\mathbb{R}^n))^* = \Psi^{\bar{m}}_{\mathrm{ph}}(\mathbb{R}^n) \tag{2.179}$$

for all $m,m'\in\mathbb{C}$.

PROOF. Since the definition shows that

$$\Psi^m_{\mathrm{ph}}(\mathbb{R}^n)\subset\Psi^{\Re m}_{\infty}(\mathbb{R}^n)$$

we know already that

$$\Psi^m_{\mathrm{ph}}(\mathbb{R}^n)\cdot\Psi^{m'}_{\mathrm{ph}}(\mathbb{R}^n)\subset\Psi^{\Re(m+m')}_{\infty}(\mathbb{R}^n)\,.$$

To see that products are polyhomogeneous it suffices to use (2.176) and (2.178) which together show that the asymptotic formulæ describing the left symbols of $A\in\Psi^m_{\mathrm{ph}}(\mathbb{R}^n)$ and $B\in\Psi^{m'}_{\mathrm{ph}}(\mathbb{R}^m)$, e.g.

$$\sigma_L(A)\sim\sum_{\alpha}\frac{i^{|\alpha|}}{\alpha!}D^\alpha_\xi D^\alpha_y a(x,y,\xi)|_{y=x}$$

imply that $\sigma_L(A)\in S^m_{\mathrm{ph}}(\mathbb{R}^n;\mathbb{R}^n)$, $\sigma_L(B)\in S^m_{\mathrm{ph}}(\mathbb{R}^n;\mathbb{R}^n)$. Then the asymptotic formula for the product shows that $\sigma_L(A\cdot B)\in S^{m+m'}_{\mathrm{ph}}(\mathbb{R}^n;\mathbb{R}^n)$.

The proof of $*$-covariance is similarly elementary, since if $A=I(a)$ then $A^*=I(b)$ with $b(x,y,z)=\overline{a(y,x,\xi)}\in S^m_{\mathrm{ph}}(\mathbb{R}^{2n};\mathbb{R}^n)$. □

In case m is real this subspace is usually denoted simply $\Psi^m(\mathbb{R}^n)$ and its elements are often said to be 'classical' pseudodifferential operators. As a small exercise in the use of the principal symbol map we shall show that

$$\begin{aligned} &A \in \Psi^m_{\mathrm{ph}}(\mathbb{R}^n),\ A \text{ (uniformly) elliptic } \Longrightarrow \exists \text{ a parametrix} \\ &B \in \Psi^{-m}_{\mathrm{ph}}(\mathbb{R}^n),\ A\cdot B - \mathrm{Id},\ B\cdot A - \mathrm{Id} \in \Psi^{-\infty}_{\infty}(\mathbb{R}^n)\,. \end{aligned} \tag{2.180}$$

In fact we already know that $B \in \Psi^{-m}_{\infty}(\mathbb{R}^n)$ exists with these properties, and even that it is unique modulo $\Psi^{-\infty}_{\infty}(\mathbb{R}^n)$. To show that $B \in \Psi^{-m}_{\mathrm{ph}}(\mathbb{R}^n)$ we can use the principal symbol map.

For elements $A \in \Psi^m_{\mathrm{ph}}(\mathbb{R}^n)$ the principal symbol $\sigma_m(A) \in S^{\Re m-[1]}_{\infty}(\mathbb{R}^n;\mathbb{R}^n)$ has a preferred class of representatives, namely the leading term in the expansion of $\sigma_L(A)$

$$\sigma_m(A) = \sigma(\xi)a_m(x,\xi) \qquad \mathrm{mod}\ S^{m-1}_{\mathrm{ph}}(\mathbb{R}^n;\mathbb{R}^n)$$

where $\sigma|\xi| = 1$ in $|\xi| \geq 1$, $\sigma|\xi| = 0$ in $|\xi| \leq 1/2$. It is even natural to identify the principal symbol with $a_m(x,\xi)$ as a *homogeneous* function. Then we can see that

$$\begin{aligned} &A \in \Psi^{\Re m}_{\infty}(\mathbb{R}^n),\ \sigma_{\Re m}(A) \text{ homogeneous of degree m} \\ &\qquad\qquad \Longleftrightarrow \Psi^m_{\mathrm{ph}}(\mathbb{R}^n) + \Psi^{\Re m-1}_{\infty}(\mathbb{R}^n)\,. \end{aligned} \tag{2.181}$$

Indeed, we just subtract from A an element $A_1 \in \Psi^m_{\mathrm{ph}}(\mathbb{R}^n)$ with $\sigma_{\Re m}(A_1) = \sigma_{\Re m}(A)$, then $\sigma_{\Re m}(A - A_1) = 0$ so $A - A_1 \in \Psi^{m-1}_{\infty}(\mathbb{R}^n)$.

So, returning to the proof of (2.180) note straight away that

$$\sigma_{-\Re m}(B) = \sigma_{\Re m}(A)^{-1}$$

has a homogeneous representative, namely $a_m(x,\xi)^{-1}$. Thus we have shown that for $j = 1$

$$B \in \Psi^{-m}_{\mathrm{ph}}(\mathbb{R}^n) + \Psi^{-m-j}_{\infty}(\mathbb{R}^n)\,. \tag{2.182}$$

We take (2.182) as an inductive hypthesis for general j. Writing this decomposition $B = B' + B_j$ it follows from the identity (2.180) that

$$A\cdot B = A\cdot B' + AB_j = \mathrm{Id} \quad \mathrm{mod}\ \Psi^{-\infty}_{\infty}(\mathbb{R}^n)$$

so

$$A\cdot B_j = \mathrm{Id} - AB' \in \Psi^0_{\mathrm{ph}}(\mathbb{R}^n) \cap \Psi^{-j}_{\infty}(\mathbb{R}^n) = \Psi^{-j}_{\mathrm{ph}}(\mathbb{R}^n).$$

Now applying B on the left, or using the principal symbol map, it follows that $B_j \in \Psi^{-m-j}_{\mathrm{ph}}(\mathbb{R}^n) + \Psi^{-m-j-1}_{\infty}(\mathbb{R}^n)$ which gives the inductive hypothesis (2.182) for $j+1$.

It is usually the case that a construction in $\Psi^*_{\infty}(\mathbb{R}^n)$, applied to an element of $\Psi^*_{\mathrm{ph}}(\mathbb{R}^n)$ will yield an element of $\Psi^*_{\mathrm{ph}}(\mathbb{R}^n)$ and when this is the case it can generally be confirmed by an inductive argument like that used above to check (2.180).

2.16. Topologies and continuity of the product

As a subspace[8]

$$S^m_{\mathrm{ph}}(\mathbb{R}^p;\mathbb{R}^n) \subset S^m_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$$

[8]Polyhomogeneous symbols may seem to be quite sophisticated objects but they are really smooth functions on manifolds with boundary; see Problems 2.8–2.7.

is not closed. Indeed, since it contains $S^{-\infty}_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$, its closure contains all of $S^{m'}_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$ for $m' < m$. In fact it is a dense subspace.[9] To capture its properties we can strengthen the topology $S^m_{\text{ph}}(\mathbb{R}^p;\mathbb{R}^n)$ inherits from $S^m_{\infty}(\mathbb{R}^p;\mathbb{R}^n)$.

As well as the symbol norms $\|\cdot\|_{N,m}$ in (2.7) we can add norms on the terms in the expansions in (2.175)

$$\|D^\alpha_x D^\beta_\xi a_{m-j}(x,\xi)\|_{L^\infty(G)},\ G = \mathbb{R}^p \times \{1 \le |\xi| \le 2\}\,. \tag{2.183}$$

We can further add the symbol norms ensuring (2.175), i.e.,

$$\|a - \sum_{j=0}^{k} a_{m-j}\|_{m-k-1,N} \quad \forall\ k, N\,. \tag{2.184}$$

Together these give a countable number of norms on $S^m_{\text{ph}}(\mathbb{R}^p;\mathbb{R}^n)$. With respect to the metric topology defined as in (2.8) the spaces $S^m_{\text{ph}}(\mathbb{R}^p;\mathbb{R}^n)$ are then *complete*.[10].

Since we have shown that the left symbol map is a linear isomorphism $\Psi^m_\infty(\mathbb{R}^n) \longrightarrow S^m_\infty(\mathbb{R}^n;\mathbb{R}^n)$ we give $\Psi^m_\infty(\mathbb{R}^n)$ a topology by declaring this to be a topological isomorphism. Similarly we declare

$$\sigma_L : \Psi^m_{\text{ph}}(\mathbb{R}^n) \longleftrightarrow S^m_{\text{ph}}(\mathbb{R}^n;\mathbb{R}^n) \tag{2.185}$$

to be a topological isomorphism.

Having given the spaces $\Psi^m_\infty(\mathbb{R}^n)$ and $\Psi^m_\infty(\mathbb{R}^n)$ topologies it is natural to ask about the continuity of the operations on them.

Proposition 2.10. *The adjoint and product maps are continuous*

$$\begin{gathered}\Psi^m_\infty(\mathbb{R}^n) \xrightarrow{\ *\ } \Psi^m_\infty(\mathbb{R}^n),\\ \Psi^m_\infty(\mathbb{R}^n) \times \Psi^{m'}_\infty(\mathbb{R}^n) \longrightarrow \Psi^{m+m'}_\infty(\mathbb{R}^n)\end{gathered} \tag{2.186}$$

and similarly for the polyhomogeneous spaces.

Proof. Note that we have put metric topologies on these spaces so it suffices to check sequential continuity. Now the commutative product is continuous, as follows from direct estimation,

$$S^m_\infty(\mathbb{R}^p;\mathbb{R}^n) \times S^{m'}_\infty(\mathbb{R}^p;\mathbb{R}^n) \longrightarrow S^{m+m'}_\infty(\mathbb{R}^p;\mathbb{R}^n) \tag{2.187}$$

as is the 'commutative adjoint', $a(x,y,\xi) \longmapsto \overline{a(y,x\xi)}$ on S^m. The same is true for the polyhomogeneous spaces. From this it follows that it is only necessary to show the continuity of the reduction map

$$S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n) \ni a \longmapsto \sigma_L(I(a)) \in S^m_\infty(\mathbb{R}^n;\mathbb{R}^n). \tag{2.188}$$

Recall that this map is accomplished in two steps, first taking the Taylor series at $y = x$, integrating by parts and taking an asymptotic sum. This constructs $b \in S^m_\infty(\mathbb{R}^n;\mathbb{R}^n)$ so that $q_L(b) - I(a) \in \Psi^{-\infty}_\infty(\mathbb{R}^n)$. Then the case $m = -\infty$ is done directly by estimation. Given a convergent sequence in $S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$, each of the terms in the Taylor series converges and it follows that the asymptotic sums can be arranged to converge, that is if $a_n \to a$ in $S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ then there exists $b_n \to b \in S^m(\mathbb{R}^n;\mathbb{R}^n)$ such that $q_L(b_n) - I(a_n) \to q_L(b) - I(a) \in \Psi^{-\infty}_\infty(\mathbb{R}^n)$. Combined with the case $m = -\infty$ this shows that reduction to the left symbol is continuous. □

[9] See Problem 2.9.

[10] See Problem 2.10.

A result which will be useful later follows from the same argument.

LEMMA 2.11. *Suppose* $\phi_i \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, $i = 1, 2$, *and* $\phi_1 = 1$ *on* $\operatorname{supp}(\phi_2)$ *then*

$$S^m_\infty(\mathbb{R}^n;\mathbb{R}^n) \ni a \longmapsto \sigma_L(\phi_1 q_L(a)(1-\phi_2)) \in S^{-\infty}_\infty(\mathbb{R}^n;\mathbb{R}^n) \tag{2.189}$$

is continuous.

Since we have given topologies to the spaces of pseudodifferential operators the notion of continuous dependence on parameters is well defined. Indeed the same is true of smooth dependence on parameters, since a map $a : [0,1] \longrightarrow \Psi^m_\infty(\mathbb{R}^n)$ is $\mathcal{C}^1$ if it is continuous, the difference quotients $(a(t+s)-a(t))/s$ are continuous down to $s = 0$, and the resulting derivative is smooth. Then smoothness is just iterative regularity in this sense. Essentially by definition this means that $A \in \mathcal{C}^\infty([0,1]_\epsilon;\Psi^m_\infty(\mathbb{R}^n))$ is the left-reduced symbol $a = \sigma_L(A(\epsilon)) \in \mathcal{C}^\infty([0,1];S^m_\infty(\mathbb{R}^n;\mathbb{R}^n))$.

2.17. Linear invariance

It is rather straightforward to see that the algebra $\Psi^\infty_\infty(\mathbb{R}^n)$ is invariant under affine transformations of $\mathbb{R}^n$. In particular if $T_a x = x + a$, for $a \in \mathbb{R}^n$, is translation by a and

$$T_a^* f(x) = f(x+a),\ T_a^* : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$$

is the isomorphism on functions then a new operator is defined by

$$T_a^* A_a f = A T_a^* f \text{ and } A \in \Psi^m_\infty(\mathbb{R}^n) \Longrightarrow A_a \in \Psi^m_\infty(\mathbb{R}^n).$$

In fact the left-reduced symbols satisfy

$$\sigma_L(A_a)(x,\xi) = \sigma_L(A)(x+a,\xi),\ A_a = T^*_{-a} A T^*_a.$$

Similarly if $T \in \mathrm{GL}(n)$ is an invertible linear transformation of $\mathbb{R}^n$ and $A_T f = T^* A (T^*)^{-1} f$ then

$$\begin{aligned} A_T f(x) &= (2\pi)^{-n} \int e^{i(Tx-y)\cdot\xi} a(Tx,\xi) f(T^{-1}y)\, d\xi dy \\ &= (2\pi)^{-n} \int e^{i(Tx-Ty)\cdot\xi} a(Tx,\xi) f(y) |\det(T)|\, d\xi dy \end{aligned} \tag{2.190}$$

so changing dual variable to $(T^t)^{-1}\xi$ shows that

$$\begin{aligned} &A \in \Psi^m_\infty(\mathbb{R}^n) \Longrightarrow A_T \in \Psi^m_\infty(\mathbb{R}^n) \\ &\qquad\qquad \text{and } \sigma_L(A_T)(x,\xi) = \sigma_L(A)(Tx,(T^t)^{-1}\xi) \end{aligned} \tag{2.191}$$

where T^t is the transpose of T (so $Tx \cdot \xi = x \cdot T^t \xi$) and the determinant factors cancel. Thus it suffices to check that

$$S^m_\infty(\mathbb{R}^q;\mathbb{R}^n) \ni a \longmapsto a' = a(Tx, A\xi) \in S^m(\mathbb{R}^q;\mathbb{R}^n) \tag{2.192}$$

for any linear tranformation T on $\mathbb{R}^q$ and invertible linear tranformation A on $\mathbb{R}^n$. Clearly the derivatives of a' are linear combinations of derivatives of a at the image point so it the symbol estimates for a' follow from those for a and the invertibility of A which implies that

$$c|\xi| \le |A\xi| \le C|\xi|,\ c, C > 0. \tag{2.193}$$

This invariance means that we can define the spaces $\Psi^m_\infty(V)$ and $\Psi^m_{\mathrm{ph}}(V)$ for any vector space V (or even affine space) as operators on $\mathcal{S}(V)$.

2.18. Local coordinate invariance

To transfer the definition of pseudodifferential operators to manifolds we need to show not only invariance under linear transformations but also under a diffemorphism $F : \Omega \longrightarrow \Omega'$ between open subsets of $\mathbb{R}^n$. For this to make sense we need to consider an operator on $\mathbb{R}^n$ which acts on functions defined in Ω'. Thus, consider

$$\Psi^m_{\mathrm{c}}(\Omega') = \{A \in \Psi^m_\infty(\mathbb{R}^n) \text{ has kernel satisfying } \operatorname{supp}(A) \Subset \Omega' \times \Omega'\}\,. \tag{2.194}$$

There are plenty of such operators if $\Omega' \neq \emptyset$ since if ϕ, $\psi \in \mathcal{C}^\infty_c(\Omega')$ and $B \in \Psi^m_\infty(\mathbb{R}^n)$ then $A = \phi B \psi \in \Psi^m_{\mathrm{c}}(\Omega')$ since it satisfies (2.194). It follows that if $a \in S^m(\mathbb{R}^n;\mathbb{R}^n)$ has support in $K \times \mathbb{R}^n$ for some $K \Subset \Omega'$ then there exists $A \in \Psi^m_{\mathrm{c}}(\Omega')$ such that $\sigma_L(A) \equiv a$ modulo $S^{-\infty}(\mathbb{R}^n;\mathbb{R}^n)$ – simply take some B with this symbol and then set $A = \phi B \phi$ where $\phi \in \mathcal{C}^\infty_c(\Omega')$ but $\phi = 1$ in a neighbourhood of K.

Proposition 2.11. *If $F : \Omega \longrightarrow \Omega'$ is a diffeomorphism then for $A \in \Psi^m_c(\Omega')$,*

$$A_F u = F^* A (F^{-1})^* (u\big|_\Omega) \text{ defines an isomorphism } \Psi^m_c(\Omega') \longrightarrow \Psi^m_c(\Omega). \tag{2.195}$$

Proof. Since $A \in \Psi^m_\infty(\mathbb{R}^n)$,

$$K_A(x,y) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a(x,\xi) d\xi \tag{2.196}$$

for some $a \in S^m_\infty(\mathbb{R}^n;\mathbb{R}^n)$. Now choose $\psi \in \mathcal{C}^\infty_c(\Omega)$ such that $\psi(x)\psi(y) = 1$ on $\operatorname{supp}(K_A)$, which is possible by (2.194). Then

$$K_A = I\left(\psi(x)\psi(y)a(x,\xi)\right). \tag{2.197}$$

In fact suppose $\mu_\epsilon(x,y) \in \mathcal{C}^\infty(\mathbb{R}^{2n})$ and $\mu \equiv 1$ in $|x-y| < \epsilon$ for $\epsilon > 0$, $\mu(x,y) = 0$ in $|x-y| > 2\epsilon$. Then if

$$K_{A_\epsilon} = I\left(\mu_\epsilon(x,y)\psi(x)\psi(y)a(x,\xi)\right) \tag{2.198}$$

we know that if

$$A'_\epsilon = A - A_\epsilon \text{ then } K_{A'_\epsilon} = \left(1 - \mu_\epsilon(x,y)\right) K_A \in \Psi^{-\infty}_\infty(\mathbb{R}^n). \tag{2.199}$$

Then $A'_\epsilon \in \Psi^{-\infty}_{\mathrm{c}}(\Omega')$ and

$$(A'_\epsilon)_F \in \Psi^{-\infty}_\infty(\mathbb{R}^n). \tag{2.200}$$

So we only need to consider A_ϵ defined by (2.198). Now

$$K_{(A_\epsilon)_F}(x,y) = (2\pi)^{-n} \int e^{i(G(x)-G(y))\cdot\xi} b(G(x),G(y),\xi) \Big|\frac{\partial G}{\partial y}\Big| d\xi \tag{2.201}$$

where $b(x,y,\xi) = \mu_\epsilon(x-y)\psi(x)\psi(y)a(x,\xi)$. Applying Taylor's formula,

$$G(x) - G(y) = (x-y)\cdot T(x,y) \tag{2.202}$$

where $T(x,y)$ is an invertible $\mathcal{C}^\infty$ matrix on $K \times K \cap \{|x-y| < \epsilon\}$ for $\epsilon < \epsilon(K)$, where $\epsilon(K) > 0$ depends on the compact set $K \Subset \Omega'$. Thus we can set

$$\eta = T^t(x,y)\cdot\xi \tag{2.203}$$

and rewrite (2.201) as

$$\begin{gathered} K_{(A_\epsilon)_F}(x,y) = (2\pi)^{-n} \int e^{i(x-y)\cdot\eta} c(x,y,\eta) d\eta \\ c(x,y,\eta) = b\left(G(x),G(y),(T^t)^{-1}(x,y)\eta\right) \Big|\frac{\partial G}{\partial y}\Big| \cdot \big|\det T(x,y)\big|^{-1}. \end{gathered} \tag{2.204}$$

So it only remains to show that $c \in S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ and the proof is complete. We can drop all the $\mathcal{C}^\infty$ factors, given by $\big|\partial G/\partial y\big|$ etc. and proceed to show that

$$\big|D^\alpha_x D^\beta_y D^\gamma_\xi a\,(G(x),G(y),S(x,y)\xi)\,\big| \ \le C(1+|\xi|)^{m-|\gamma|} \quad \text{on } K\times K\times\mathbb{R}^n \tag{2.205}$$

where $K \subset\subset \Omega'$ and S is $\mathcal{C}^\infty$ with $|\det S| \ge \epsilon$. The estimates with $\alpha=\beta=0$ follow easily and the general case by induction:

$$\begin{aligned} &D^\alpha_x D^\beta_y D^\gamma_\xi a\,(G(x),G(y),S(x,y)\xi) \\ &\qquad = \sum_{\substack{|\mu|\le|\alpha|+|\beta|+|\gamma| \\ |\alpha'|\le|\alpha|,|\beta|\le|\beta| \\ |\nu|+|\gamma|\le|\mu|}} M^{\alpha',\rho',\mu'}_{\alpha,\beta,\gamma,\nu}(x,y)\xi^\nu\left(D^{\alpha'}D^{\beta'}D^\mu a\right)(G(x),G(y),S\xi) \end{aligned}$$

where the coefficients are $\mathcal{C}^\infty$ and the main point is that $|\nu|\le|\mu|$. □

2.19. Semiclassical limit

Let us at least pretend to go back to the beginning once more in order to understand the following 'problem'. From the origins of quantum mechanics the relationship between the quantum and related classical system has always been a primary interest. In classical Hamiltonian mechanics the 'energy' (I will keep to one dimension for the moment in the interest only of simplicity) is the sum of the kinetic and potential energies,

$$E(x,\xi) = \frac{1}{2}\hbar\xi^2 + V(x) \tag{2.206}$$

Here $\hbar$ is a 'small parameter' which represents either a coupling constant (the fine structure constant relating the energy change in an atom to the frequence of the light emitted) or else a small 'mass'. The 'corresponding' (one has to be careful about this, the process of quantization does not really work this way) quantum system is

$$q_L(E) = -\frac{1}{2}\hbar\frac{d^2}{dx^2} + V(x). \tag{2.207}$$

For $\hbar > 0$ – which is really the case – this is a perfectly good elliptic (at least locally) differential operator. However something singular clearly happens as $\hbar \downarrow 0$ (although you might ask how a constant is supposed to go to zero; fortunately we have other less frivolous reasons for looking at this).

If we simply set $\hbar = \epsilon^2$ then we can rewrite (2.207) in the form

$$-\frac{1}{2}(\epsilon\frac{d}{dx})^2 + V(x). \tag{2.208}$$

This suggests that to generalize the structure in (2.208) to 'arbitrary symbols' in place of (2.206) we should simply consider operators of the form

$$\begin{aligned} A_\epsilon u(x) &= (2\pi)^{-n}\int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\xi}a(\epsilon,x,y,\epsilon\xi)u(y)dyd\xi \\ &= (2\pi\epsilon)^{-n}\int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot\eta/\epsilon}a(\epsilon,x,y,\eta)u(y)dyd\eta \end{aligned} \tag{2.209}$$

where the second version follows from the first by changing variable to $\eta=\epsilon\xi$ and $a \in \mathcal{C}^\infty([0,1]_\epsilon; S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ is a symbol in the usual sense which may also depend smoothly on ϵ.

DEFINITION 2.3. *Let* $\Psi^m_{\text{sl-}\infty}(\mathbb{R}^n) \subset \mathcal{C}^\infty((0,1]; \Psi^m_\infty(\mathbb{R}^n)$ *(resp.* $\Psi^m_{\text{sl}}(\mathbb{R}^n)) \subset \mathcal{C}^\infty((0,1]; \Psi^m_{\text{ph}}(\mathbb{R}^n))$ *be the subspace consisting of those 1-parameter families which are of the form* (2.209) *for some* $a \in \mathcal{C}^\infty([0,1]; S^m_\infty(\mathbb{R}^{2n}; \mathbb{R}^n))$ *(resp.* $a \in \mathcal{C}^\infty([0,1]; S^m_{\text{ph}}(\mathbb{R}^{2n}; \mathbb{R}^n))$*).*

There is no question about the form of the kernels of these operators. Namely, directly from the second form of the definition

$$A_\epsilon \text{ has kernel of the form } \epsilon^{-n} K_\epsilon(x, \frac{x-y}{\epsilon}) \tag{2.210}$$

where $K_\epsilon(x, x-y)$ is the kernel of a smooth family of pseudodifferential operators in the usual sense, namely

$$K_\epsilon(x, x-y) \text{ is the kernel of } I(a_\epsilon). \tag{2.211}$$

So, as $\epsilon \downarrow 0$ the kernel very much 'bunches up' around the diagonal. This rather explicit description does not tell us directly about the composition properties of these 1-parameter families of operators. However we can work this out fairly easily. First check what happens for the operators of order $-\infty$.

PROPOSITION 2.12. *The space* $\Psi^{-\infty}_{\text{sl}}(\mathbb{R}^n) = \Psi^{-\infty}_{\text{sl-}\infty}(\mathbb{R}^n)$ *is closed under composition and adjoints and there is a short exact multiplicative sequence*

$$\epsilon\Psi^{-\infty}_{\text{sl}}(\mathbb{R}^n) \longrightarrow \Psi^{-\infty}_{\text{sl}}(\mathbb{R}^n) \xrightarrow{\sigma_{\text{sl}}} S^{-\infty}_\infty(\mathbb{R}^n; \mathbb{R}^n). \tag{2.212}$$

PROOF. Already from (2.209) it follows directly that the residual algebra is given by symbols of order $-\infty$, that is

$$\begin{gathered} A_\epsilon \in \bigcap_m \Psi^m_{\text{sl-}\infty}(\mathbb{R}^n) \Longleftrightarrow \\ A_\epsilon \text{ is of the form (2.209) with } a \in \mathcal{C}^\infty([0,1]; S^{-\infty}_\infty(\mathbb{R}^{2n}; \mathbb{R}^n)) \end{gathered} \tag{2.213}$$

since the kernel $K_\epsilon(x, x-y)$ is uniquely determined by A_ϵ. This also shows that the 'residual space' is the same for the classical and non-classical cases.

Thus if $A_\epsilon \in \Psi^{-\infty}_{\text{sl}}(\mathbb{R}^n)$ then there exists $K_\epsilon \in \mathcal{C}^\infty_\infty([0,1] \times \mathbb{R}^n; \mathcal{S}(\mathbb{R}^n))$ such that

$$A_\epsilon \text{ has kernel } \epsilon^{-n} K_\epsilon(\epsilon, x, \frac{x-y}{\epsilon}). \tag{2.214}$$

So the composite – really only for $\epsilon > 0$ – of two such (families of) operators A_ϵ and B_ϵ, where the kernel of B_ϵ is given by (2.214) for a different function L_ϵ, has kernel

$$\begin{aligned} \epsilon^{-n} J_\epsilon(x, \frac{x-y}{\epsilon}) &= \epsilon^{-2n} \int_{\mathbb{R}^n} K(x, \frac{x-z}{\epsilon}) L_\epsilon(z, \frac{z-y}{\epsilon}) dz \\ &= \epsilon^{-n} \int_{\mathbb{R}^n} K(x,t) L_\epsilon(x - \epsilon t, \frac{x-y}{\epsilon} + t) dt \end{aligned} \tag{2.215}$$

where $t = (x-z)/\epsilon$. Thus changing independent variable to $Z = (x-y)/\epsilon$ the kernel of the product (for $\epsilon > 0$) becomes

$$J_\epsilon(x, Z) = \int_{\mathbb{R}^n} K_\epsilon(x,t) L_\epsilon(x - \epsilon t, Z + t) dt. \tag{2.216}$$

Now, it is easy to see that $J_\epsilon(x, Z) \in \mathcal{C}^\infty_\infty([0,1]_\epsilon \times \mathbb{R}^n; \mathcal{S}(\mathbb{R}^n))$. The rapid decay in t in the first factor in the integrand gives rapid convergence of the integral and overall boundness of J_ϵ. Rapid decay in Z follows from the estimate

$$|Z| \le |t| + |Z + t| \tag{2.217}$$

and differentiating with respect to any of the independent variables gives a similar integral with similar bounds.

This shows that the composite is also in $\Psi^{-\infty}_{\mathrm{sl}}(\mathbb{R}^n)$. Notice that at $\epsilon = 0$,

$$J_0(x,Z) = \int_{\mathbb{R}^n} K_0(x,t)L_0(x,Z+t)dt \Longrightarrow c(0,x,\xi) = a(0,x,\xi)b(0,x,\xi). \tag{2.218}$$

by taking the Fourier transform in Z. Thus (2.212) is satisfied by the map

$$\sigma_{\mathrm{sl}}(A_\epsilon) = a(0,x,\xi) \in S^{-\infty}_{\infty}(\mathbb{R}^n;\mathbb{R}^n) = \mathcal{C}^\infty_\infty(\mathbb{R}^n;\mathcal{S}(\mathbb{R}^n)). \tag{2.219}$$

□

It is important to contrast the behaviour of this 'semiclassical symbol' with the usual symbol – with which it is closely related of course. Namely the semiclassical symbol describes in rather complete detail the leading behaviour of the operator at $\epsilon = 0$ and is multiplicative. What this really shows is the basic property of the semiclassical limit, namely that these operators 'become commutative' at $\epsilon = 0$ (where they also fail to exist in the usual sense).[11] As with the principal symbol rather fine results can be proved by iteration. Thus

$$A_\epsilon \in \Psi^{-\infty}_{\mathrm{sl}}(\mathbb{R}^n) \text{ and } \sigma_{\mathrm{sl}}(A_\epsilon) = 0 \Longrightarrow A_\epsilon = \epsilon A^{(1)}_\epsilon,\ A^{(1)}_\epsilon \in \Psi^{-\infty}_{\mathrm{sl}}(\mathbb{R}^n). \tag{2.220}$$

Then if one can arrange repeatedly that $\sigma_{\mathrm{sl}}(A^{(1)}_\epsilon) = 0$ and so on, one may finally conclude that[12]

$$A_\epsilon \in \bigcap_N \epsilon^N \Psi^{-\infty}_{\mathrm{sl}}(\mathbb{R}^n) \Longleftrightarrow A_\epsilon \in \mathcal{C}^\infty([0,1];\Psi^{-\infty}_{\mathrm{sl}}(\mathbb{R}^n)) \text{ and } \frac{d^k}{d\epsilon^k}A_\epsilon\big|_{\epsilon=0} = 0. \tag{2.221}$$

Now we proceed to show that this result extends directly to the operators of finite order.

Theorem 2.5. *The semiclassical families in $\Psi^m_{\mathrm{sl}\text{-}\infty}(\mathbb{R}^n)$ (or $\Psi^m_{\mathrm{sl}}(\mathbb{R}^n)$) form an order-filtered $*$-algebra with two multiplicative symbol maps, one a uniform (perhaps better to say 'rescaled') version of the usual symbol and the second a finite order version of the semiclassical symbol in* (2.219)*:*

$$\begin{gathered}\tilde\sigma_m : \Psi^m_{\mathrm{sl}}(\mathbb{R}^n) \longrightarrow \mathcal{C}^\infty([0,1]\times\times\mathbb{R}^n\times(\mathbb{R}^n\setminus 0)),\ \tilde\sigma_m(A_\epsilon)(x,\eta) = \sigma_m(A_\epsilon)(x,\eta/\epsilon),\\ \sigma_{\mathrm{sl}} : \Psi^m_{\mathrm{sl}}(\mathbb{R}^n) \longrightarrow S^m_{\mathrm{ph}}(\mathbb{R}^n\times\mathbb{R}^n);\end{gathered} \tag{2.222}$$

they are separately surjective and are jointly subject only to the compatibility condition

$$\sigma_{\mathrm{sl}}(A_\epsilon) = \tilde\sigma_m(A_\epsilon)\big|_{\epsilon=0} \text{ in } S^{m-[1]}_{\infty}(\mathbb{R}^n;\mathbb{R}^n). \tag{2.223}$$

Proof. By definition $A_\epsilon \in \Psi^m_{\mathrm{sl}\text{-}\infty}(\mathbb{R}^n)$ means precisely that there is a smooth family $a_\epsilon \in \mathcal{C}^\infty([0,1];S^m_\infty(\mathbb{R}^n;\mathbb{R}^n))$ such that if $K_\epsilon(x,x-y)$ is the family of kernels of $q_L(a_\epsilon)$ then (2.210) holds. Thus the two maps in the statement of the theorem, with

$$\begin{gathered}\tilde\sigma_m(A_\epsilon) = [a_\epsilon] \in \mathcal{C}^\infty([0,1];S^{m-[1]}_\infty(\mathbb{R}^n;\mathbb{R}^n)) \text{ and}\\ \sigma_{\mathrm{sl}}(A_\epsilon) = a_0 \in S^m_\infty(\mathbb{R}^n;\mathbb{R}^n)\end{gathered} \tag{2.224}$$

are certainly well-defined and subject only to the stated compatibility condition.

[11]See Problem 2.22.

[12]See Problem 2.24 for an outline of the proof.

Thus the main issue is multiplicativity. Since a_ϵ can be smoothly approximated by symbols of order $-\infty$ we can use continuity in the symbol topology and start from (2.216). For $\epsilon = 1$

$$
\begin{aligned}
J(x,Z) &= \int_{\mathbb{R}^n} K(x,t)L(x-t,Z+t)dt,\\
K(x,t) &= (2\pi)^{-n}\int e^{it\cdot\xi}b(x,\xi)d\xi,\\
L(x,t) &= (2\pi)^{-n}\int e^{it\cdot\xi}a(x,\xi)d\xi,\\
c(x,\xi) &= \int e^{-iZ\cdot\xi}J(x,Z)dZ
\end{aligned}
\tag{2.225}
$$

reproduces the usual composition formula. Thus we know that this formula extends by continuity to define the jointly continuous product map

$$
S^m_\infty(\mathbb{R}^n;\mathbb{R}^n)\times S^{m'}_\infty(\mathbb{R}^n;\mathbb{R}^n)\longrightarrow S^{m+m'}_\infty(\mathbb{R}^n;\mathbb{R}^n). \tag{2.226}
$$

Now, we can simplify this by assuming that a is constant-coefficient, i.e. is independent of the base variable. The to evaluate $c(0,\xi)$ we only need to know $J(0,Z)$ which is given by the (extension by continuity of) the simplified formula, which therefore, by restriction, defines a continuous map

$$
J(0,Z)=\int_{\mathbb{R}^n}K(t)L(-t,Z+t)dt,\ S^m_\infty(\mathbb{R}^n)\times S^{m'}_\infty(\mathbb{R}^n;\mathbb{R}^n)\longrightarrow S^{m+m'}_\infty(\mathbb{R}^n). \tag{2.227}
$$

Now, from (2.225)

$$
L(-t,Z)=(2\pi)^{-n}\int e^{iZ\cdot\xi}a(-t,\xi)d\xi \tag{2.228}
$$

so in the corresponding formula with ϵ varying

$$
J(0,Z)=\int_{\mathbb{R}^n}K(t)L(-\epsilon t,Z+t)dt \tag{2.229}
$$

$L(-\epsilon t, Z+t)$ corresponds to the symbol $a(-\epsilon t,\xi)\in\mathcal{C}^\infty([0,1];S^{m'}_\infty(\mathbb{R}^n;\mathbb{R}^n))$ as follows easily by direct differentiation. Thus if we fix x in (2.216) at any point in $\mathbb{R}^n$ this shows that the product extends by continuity to the finite order symbol spaces. Then, using the bilinearity, the smooth dependence on x as a parameter can be restored. Thus in fact the same results on composition follow as in the smoothing case, that

$$
\begin{aligned}
\tilde\sigma_{m+m'}(A_\epsilon B_\epsilon) &= \tilde\sigma_m(A_\epsilon)\tilde\sigma_{m'}(B_\epsilon) \text{ and}\\
\sigma_{\text{sl-}\infty}(A_\epsilon B_\epsilon) &= \sigma_{\text{sl-}\infty}(A_\epsilon)\sigma_{\text{sl-}\infty}(B_\epsilon).
\end{aligned}
\tag{2.230}
$$

□

Of course the uniform symbol $\tilde\sigma_m(A)$ is not quite the usual symbol precisely because of rescaling but is equivalent to it for $\epsilon > 0$. Namely

$$
\sigma_m(A_\epsilon)(x,\xi)=\tilde\sigma_m(A_\epsilon)(\epsilon,x,\epsilon\xi). \tag{2.231}
$$

Maybe you like to have things written out explicitly as short exact sequences. There are in fact three such (or more if you allow polyhomogeneous/∞ variants), all of which are also multiplicative. Thus

(2.232)
$$
\begin{gathered}
\Psi_{\mathrm{sl}}^{m-1}(\mathbb{R}^n) \longrightarrow \Psi_{\mathrm{sl}}^{m}(\mathbb{R}^n) \xrightarrow{\tilde\sigma_m} \mathcal{C}^\infty([0,1]; S_{\mathrm{ph}}^{m-[1]}(\mathbb{R}^n;\mathbb{R}^n)),\\
\epsilon\Psi_{\mathrm{sl}}^{m}(\mathbb{R}^n) \longrightarrow \Psi_{\mathrm{sl}}^{m}(\mathbb{R}^n) \xrightarrow{\sigma_{\mathrm{sl}}} S_{\mathrm{ph}}^{m}(\mathbb{R}^n;\mathbb{R}^n),\\
\epsilon\Psi_{\mathrm{sl}}^{m-1}(\mathbb{R}^n) \longrightarrow \Psi_{\mathrm{sl}}^{m}(\mathbb{R}^n) \xrightarrow{(\tilde\sigma_m,\sigma_{\mathrm{sl}})}\\
\left\{(\tilde a, a) \in S_{\mathrm{ph}}^{m}(\mathbb{R}^n;\mathbb{R}^n) \oplus \mathcal{C}^\infty([0,1]; S_{\mathrm{ph}}^{m-[1]}(\mathbb{R}^n;\mathbb{R}^n)); \tilde a = a\big|_{\epsilon=0} \text{ in } S_{\mathrm{ph}}^{m-[1]}(\mathbb{R}^n;\mathbb{R}^n)\right\}.
\end{gathered}
$$

We also want to check coordinate invariance. Note that the semiclassical algebras are mapped into themselves by multiplication of the kernel by an element of $\mathcal{C}^\infty_\infty(\mathbb{R}^{2n}_{x,y})$. In particular we may freely localize on the left or the right by a smooth function of compact support and stay in the algebra. The coordinate invariance of the semiclassical algebra then follows from that of the usual algebra using the same sort of reduction as above.

PROPOSITION 2.13. *If $A_\epsilon \in \Psi^m_{\text{sl-}\infty}(\mathbb{R}^n)$ has kernel with compact support in $\Omega \times \Omega$ for some open $\Omega \subset \mathbb{R}^n$ and $F : \Omega \longrightarrow \Omega'$ is a diffeomorphism then $A_{F,\epsilon} = (F^{-1})^* A_\epsilon F^* \in \Psi^m_{\text{sl-}\infty}(\mathbb{R}^n)$ and*

$$
\begin{aligned}
\tilde\sigma_m(A_{F,\epsilon}) &= (F^*)^*\tilde\sigma_m(A_\epsilon)\\
\sigma_{\mathrm{sl}}(A_{F,\epsilon}) &= (F^*)^*\sigma_{\mathrm{sl}}(A_\epsilon).
\end{aligned}
\tag{2.233}
$$

We will also need some boundedness properties of semiclassical families. The following will suffice for our purposes.

PROPOSITION 2.14. *For $A_\epsilon \in \Psi^0_{\text{sl-}\infty}(\mathbb{R}^n)$,*

$$
\sup_{0<\epsilon\le 1} \|A_\epsilon\|_{L^2(\mathbb{R}^n)} < \infty. \tag{2.234}
$$

PROOF. It is only the uniformity in (6.130) that is at issue, since we know the boundedness for $1 \ge \epsilon \ge \delta$ for any $\delta > 0$. The argument we give is essentially the same as for boundedness. Namely for $C > 0$ large enough we can extract an approximate square-root

$$
C - A_\epsilon^* A_\epsilon = B_\epsilon^2 + E_\epsilon,\ B \in \Psi^0_{\mathrm{sl}}(\mathbb{R}^n),\ E \in \epsilon^\infty \mathcal{C}^\infty([0,1;\Psi^{-\infty}(\mathbb{R}^n)). \tag{2.235}
$$

This can be seen using essentially the same symbolic computation as before but now for both symbols. Thus if $C > \sigma_0(A)^*\sigma_0(A)$ and $C > \sigma_{\mathrm{sl}}(A)^*\sigma_{\mathrm{sl}}(A)$ (and note that the second can well be larger than the first) then be can choose $B \in \Psi^0_{\mathrm{sl}}(\mathbb{R}^n)$ with $B^* = B$, $\sigma_0(B)^2 = C - \sigma_0(A)^*\sigma_0(A)$, $\sigma_{\mathrm{sl}}(B)^2 = C\sigma_{\mathrm{sl}}(A)^*\sigma_{\mathrm{sl}}(A)$ (because the consistency condition is satisfied) and hence

$$
C - A^*A = B^2 + E_1,\ E_1 \in \epsilon\Psi^{-1}_{\mathrm{sl}}(\mathbb{R}^n). \tag{2.236}
$$

Then the construction can be iterated as before to construct a solution to (2.235). The uniform boundedness of E_ϵ is clear – in fact its norm vanishes rapidly as $\epsilon \downarrow 0$ so the uniform boundedness follows. □

2.20. Smooth and holomorphic families

I have gone through the description of 'classical' pseudodifferential operators of complex order here, even though it might seem rather strange – I want to emphasize that these really do arise in practice. In particular we will want to consider the notion of a holomorphic family of complex order $f(z)$ where f is holomorphic.

First consider the issue of continuous or smooth dependence on parameters. Since we have at least implicitly given $\Psi^m_\infty(\mathbb{R}^n)$ and $\Psi^m_{\text{ph}}(\mathbb{R}^n)$ topologies, this is already defined. In fact of course it is just the continuous or smooth dependence of the left-reduced symbol on the parameters, say in some open or smoothly-bounded subset of $\mathbb{R}^p$. Tracking back through the arguments above, it can be seen that the product theorem actually gives continuous dependence of the symbol of a product on the symbols of the factors, although a little thought is needed here because of the asymptotic summation involved see Problem 2.25 for a little more on this point. It is important that the product is unique. For homolormophy say of an element of $\Psi^m_\infty(\mathbb{R}^n)$ in terms of a complex variable $s \in U \subset \mathbb{C}$ open the discussion is essentially the same. Namely a (strongly) holomorphic function into a fixed topological vector space is just a continuous function which satisfies Cauchy criterion, that it integrates to zero around any closed contour. This is actually equivalent to smoothness in s and

$$\overline{\partial} A(s) = 0. \tag{2.237}$$

So, there is nothing very interesting going on here. For polyhomogeneous operators of a fixed order the story is the same, with the spaces of operators and symbols altered appropriately. However if the order itself is allowed to vary then a different notion of 'holomorphy' arises. Namely if $F : U \longrightarrow \mathbb{C}$ is itself a holomorphic function, we may consider polyhomogeneous symbols which are of order $f(s)$. As noted above this can be simplified by writing the (left-reduced) symbol in the form

$$a(s, x, \xi) =< \xi >^{f(s)} b(s, x, \xi) \tag{2.238}$$

where $b \in S^0_{\text{ph}}(\mathbb{R}^n; \mathbb{R}^b)$. Then by holomorphy in this new sense we mean holomorphy of b in the usual sense, as a polyhomogeneous symbol of order 0. We can write $\Psi^f_{\text{hol}}(\mathbb{R}^n)$ for this linear space of operators. Note that we drop the 'ph' since this does not make much sense without it!

Proposition 2.15. *If $A(s) \in \Psi^f_{\text{hol}}(\mathbb{R}^n)$ and $B \in \Psi^g_{\text{hol}}(\mathbb{R}^n)$ for two holomorphic functions f, $g : U \longrightarrow \mathbb{C}$,*

$$A \circ B \in \Psi^{f+g}_{\text{hol}}(\mathbb{R}^n). \tag{2.239}$$

Proof. I suppose I should write one! □

Why bother with such operators? Globally in this sense on $\mathbb{R}^n$ it is difficult to come up with sensible examples but on a compact manifold or for the better 'global' calculi on $\mathbb{R}^n$ discussed below there are natural examples. For instance, getting very much ahead of myself here, if $A \in \Psi^1_{\text{ph}}(M)$ is self-adjoint and elliptic on a compact manifold M then the complex powers A^z for an entire family, so complex in the sense above for $z \in \mathbb{C}$. This was first proved by Seeley and is the starting point for many interesting developments, see Chapters 4, 6 and 7 below.

2.21. Problems

Problem 2.1. Show, in detail, that for each $m \in \mathbb{R}$

$$(1 + |\xi|^2)^{\frac{1}{2}m} \in S^m_\infty(\mathbb{R}^p; \mathbb{R}^n) \tag{2.240}$$

for any p. Use this to show that

$$S^m_\infty(\mathbb{R}^p; \mathbb{R}^n) \cdot S^{m'}_\infty(\mathbb{R}^p; \mathbb{R}^n) = S^{m+m'}_\infty(\mathbb{R}^p; \mathbb{R}^n).$$

PROBLEM 2.2. Consider $w = 0$ and $n = 2$ in the definition of symbols and show that if $a \in S^1_\infty(\mathbb{R}^2)$ is elliptic then for $r > 0$ sufficiently large the integral

$$\int_0^{2\pi} \frac{1}{2\pi} \frac{1}{a(re^{i\theta})} \frac{d}{d\theta} a(re^{i\theta}) d\theta = \frac{1}{2\pi} \int_0^{2\pi} \frac{d}{d\theta} \log a(re^{i\theta}) d\theta,$$

exists and is an integer independent of r, where $z = \xi_1 + i\xi_2$ is the complex variable in $\mathbb{R}^2 = \mathbb{C}$. Conclude that there is an elliptic symbol, a on $\mathbb{R}^2$, such that there does not exist b, a symbol with

$$b \neq 0 \text{ on } \mathbb{R}^2 \text{ and } a(\xi) = b(\xi) \text{ for } |\xi| > r \tag{2.241}$$

for any r.

PROBLEM 2.3. Show that a symbol $a \in S^m_\infty(\mathbb{R}^p_z; \mathbb{R}^n_\xi)$ which satisfies an estimate

$$|a(z,\xi)| \le C(1 + |\xi|)^{m'}, \ m' < m \tag{2.242}$$

is necessarily in the space $S^{m'+\epsilon}_\infty(\mathbb{R}^p_z; \mathbb{R}^n_\xi)$ for all $\epsilon > 0$.

PROBLEM 2.4. Show that if $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^p_z \times \mathbb{R}^n)$ and $\psi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ with $\psi(\xi) = 1$ in $|\xi| < 1$ then

$$c_\phi(z,\xi) = \phi(z, \frac{\xi}{|\xi|})(1 - \psi)(\xi) \in S^0(\mathbb{R}^p_z; \mathbb{R}^n_\xi). \tag{2.243}$$

If $a \in S^m_\infty(\mathbb{R}^p_z; \mathbb{R}^n_\xi)$ define the cone support of a in terms of its complement

$$\begin{aligned} \operatorname{cone\,supp}(a)^\complement = \{(\bar{z}, \bar{\xi}) \in \mathbb{R}^p_z \times (\mathbb{R}^n_\xi \setminus \{0\}); \exists \\ \phi \in \mathcal{C}^\infty_c(\mathbb{R}^p_z; \mathbb{R}^n), \ \phi(\bar{z}, \bar{\xi}) \neq 0, \text{ such that } c_\phi a \in S^{-\infty}_\infty(\mathbb{R}^p; \mathbb{R}^n)\}. \end{aligned} \tag{2.244}$$

Show that if $a \in S^m_\infty(\mathbb{R}^p_z; \mathbb{R}^n_\xi)$ and $b \in S^{m'}_\infty(\mathbb{R}^p_z; \mathbb{R}^n_\xi)$ then

$$\operatorname{cone\,supp}(ab) \subset \operatorname{cone\,supp}(a) \cap \operatorname{cone\,supp}(b). \tag{2.245}$$

If $a \in S^m_\infty(\mathbb{R}^p_z; \mathbb{R}^n_\xi)$ and $\operatorname{cone\,supp}(a)\emptyset$ does it follow that $a \in S^{-\infty}_\infty(\mathbb{R}^p_z; \mathbb{R}^n_\xi)$?

PROBLEM 2.5. Prove that (2.30) is a characterization of functions $a \in (1 + |x - y|^2)^{w/2} S^m(\mathbb{R}^{2n}; \mathbb{R}^n)$. [Hint: Use Liebniz' formula to show instead that the equivalent estimates

$$|D^\alpha_x D^\beta_y D^\gamma_\xi a(x,y,\xi)| \le C_{\alpha,\beta,\gamma}(1 + |x-y|^2)^{w/2}(1 + |\xi|)^{m-|\gamma|} \ \forall \ \alpha,\beta,\gamma \in \mathbb{N}^n_0$$

characterize this space.]

PROBLEM 2.6. Show that $A \in \Psi^{-\infty}_\infty(\mathbb{R}^n)$ if and only if its Schwartz kernel is $\mathcal{C}^\infty$ and satisfies all the estimates

$$|D^\alpha_x D^\beta_y a(x,y)| < C_{\alpha,\beta,N}(1 + |x-y|)^{-N} \tag{2.246}$$

for multiindices $\alpha, \beta \in \mathbb{N}^n_0$ and $N \in \mathbb{N}_0$.

PROBLEM 2.7. Polyhomogeneous symbols as smooth functions.

PROBLEM 2.8. General polyhomogeneous symbols and operators.

PROBLEM 2.9. Density of polyhomogeneous symbols in L^∞ symbols of the same order.

PROBLEM 2.10. Completeness of the spaces of polyhomogeneous symbols.

PROBLEM 2.11. Fourier transform??

PROBLEM 2.12. Show that the kernel of any element of $\Psi^{\infty}_{\infty}(\mathbb{R}^n)$ is $\mathcal{C}^\infty$ away from the diagonal. Hint: Prove that $(x-y)^\alpha K(x,y)$ becomes increasingly smooth as $|\alpha|$ increases.

PROBLEM 2.13. Show that for any $m \geq 0$ the unit ball in $H^m(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$ is *not* precompact, i.e. there is a sequence $f_j \in H^m(\mathbb{R}^n)$ which has $\|f_j\|_m \leq 1$ and has no subsequence convergent in $L^2(\mathbb{R}^n)$.

PROBLEM 2.14. Show that for any $R > 0$ there exists $N > 0$ such that the Hilbert subspace of $H^N(\mathbb{R}^n)$

$$\{u \in H^N(\mathbb{R}^n); u(x) = 0 \text{ in } |x| > R\} \tag{2.247}$$

is compactly included in $L^2(\mathbb{R}^n)$, i.e. the intersection of the unit ball in $H^N(\mathbb{R}^n)$ with the subspace (2.247) is precompact in $L^2(\mathbb{R}^n)$. Hint: This is true for *any* $N > 0$, taking $N >> 0$ will allow you to use the Sobolev embedding theorem and Arzela-Ascoli.

PROBLEM 2.15. Using Problem 2.14 (or otherwise) show that for any $\epsilon > 0$

$$(1+|x|)^\epsilon H^\epsilon(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n)$$

is a compact inclusion, i.e. any infinite sequence f_n such that $(1+|x|^2)^{-\epsilon}$ is bounded in $H^\epsilon(\mathbb{R}^n)$ has a subsequence convergent in $L^2(\mathbb{R}^n)$. Hint: Choose $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ with $\phi(x) = 1$ in $|x| < 1$ and, for each k, consider the sequence $\phi(x/k) f_j$. Show that the Fourier transform converts this into a sequence which is bounded in $(1 + |\xi|^2)^{-\frac{1}{2}\epsilon} H^N(\mathbb{R}^n_\xi)$ for any N. Deduce that it has a convergent subsequence in $L^2(\mathbb{R}^n)$. By diagonalization (and using the rest of the assumption) show that f_j itself has a convergent subsequence.

PROBLEM 2.16. About ρ and δ.

PROBLEM 2.17. Prove the formula (2.191) for the left-reduced symbol of the operator A_T obtained from the pseudodifferential operator A by linear change of variables. How does the right-reduced symbol transform?

PROBLEM 2.18. Density of $\mathcal{S}(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$.

PROBLEM 2.19. Square-root of a positive elliptic symbol is a symbol.

PROBLEM 2.20. Write out a proof to Proposition 4.2. Hint (just to do it elegantly, it is straightforward enough): Write A in right-reduced form as in (2.74) and apply it to $\hat{u}$; this gives a formula for $\hat{Au}$.

PROBLEM 2.21. Show that any continuous linear operator

$$\mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$$

has Schwartz kernel in $\mathcal{S}(\mathbb{R}^{2n})$.

PROBLEM 2.22. Show that if A_ϵ and B_ϵ are as in Proposition 2.12 then they have unique representations as in (2.209) with left-reduced symbols, respectively a, b and for the composite c all in $\mathcal{C}^\infty_\infty([0,1] \times \mathbb{R}^n; \mathcal{S}(\mathbb{R}^n))$ and where in the sense of Taylor series at $\epsilon = 0$,

$$c(\epsilon, x, \eta) \simeq \sum_\alpha \frac{\epsilon^{|\alpha|}}{\alpha!} \partial^\alpha_\eta a(\epsilon, x, \xi) \cdot \partial^\alpha_x b(\epsilon, x, \eta). \tag{2.248}$$

PROBLEM 2.23. Give the details of the reduction argument in the semiclassical setting. Here are some suggestions. First use integration by parts based on the identity

$$\epsilon^2 \Delta_\eta e^{i(x-y)\cdot\eta/\epsilon} = |x-y|^2 e^{i(x-y)\cdot\eta/\epsilon} \tag{2.249}$$

to show that the kernel of a semiclassical family A_ϵ is smooth in $|x-y| > \delta > 0$ in all variables, including ϵ, as a funtion of x and $x-y$, with all x derivatives bounded and rapidly decaying in $x-y$ – that is smoothly cut off in $|x-y > \delta > 0$ it is in $\mathcal{C}^\infty([0,1]_\epsilon; \Psi_\infty^{-\infty}(\mathbb{R}^n)$ and vanishes with all its derivatives at $\epsilon = 0$. Next use the left reduction argument and asymptotic summation to treat the part of the kernel supported in $|x-y| < \delta$.

PROBLEM 2.24. Proof of (2.221).

PROBLEM 2.25. Asymptotic summation of holomorphic families of symbols.