18.103 TEST 2, 27 APRIL 2004
1:05AM – 2:25PM

This test is closed book. You are not permitted to bring any books, notes or such material with you. You may use theorems, lemmas and propositions from the book or from class.

There are only four questions on the actual Test.

(1) If $f_n \in L^1([0,1])$ is a convergent sequence with respect to the L^1 norm show that there is a subsequence which converges pointwise almost everywhere on $[0,1]$.

(2) (Adams and Guillemin 11, p.129) Let f be a continuously differentiable function on $[0,1]$, show that
$$\sup_{0\le x\le 1} |f(x)-f(y)| \le \|f'\|_{L^2([0,1])}.$$

(3) If $f \in L^1(\mathbb{R})$ show that its Fourier transform
$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-ix\xi}dx$$
is continuous.

(4) Suppose that $f \in L^2([-\pi,\pi])$ has Fourier coefficients satisfying
$$\sum_{k\in\mathbb{Z}} |c_k| < \infty.$$
Show that there is a continuous function on g on $[-\pi,\pi]$ such that $f(x) = g(x)$ for almost all $x \in [-\pi,\pi]$.

(5) Show that if $f : [-\pi,\pi] \longrightarrow \mathbb{C}$ is a bounded measurable function which satisfies $\int_{[-\pi,\pi]} x^k f(x)dx = 0$ for all non-negative integers $k = 0,1,2,\dots$ then $f(x) = 0$ for almost all $x \in [-\pi,\pi]$.

(6) Show that a continuous function f on $[-\pi,\pi]$ which satisfies
$$\int_{-\pi}^{\pi} f(x)e^{ikx}dx = 0 \ \forall\ k \in \mathbb{Z}$$
vanishes identically.