18.100B TEST 2 PRACTIVE, 27 APRIL 2004
11:05AM – 12:25PM

This test is closed book, no books, papers or notes are permitted. You may use theorems, lemmas and propositions from the class and book. Note that where $\mathbb{R}^k$ is mentioned below the standard metric is assumed.

There are 5 questions on the actual test, I think they are mostly easier than these ones.

(1) Consider the function $\alpha : [0, 1] \longrightarrow \mathbb{R}$ defined by

$$\alpha(x) = \begin{cases} \frac{1}{2}x & 0 \le x \le \frac{1}{2} \\ \frac{1}{2}(x+1) & \frac{1}{2} \le x \le 1. \end{cases}$$

Show carefully, using results from class, that any monotonic increasing function $f : [0, 1] \longrightarrow \mathbb{R}$ which is continuous at $x = \frac{1}{2}$ is Riemann-Stieltjes integrable with respect to α.

Solution: Write $\alpha = \alpha_1 + \alpha_2$ where $\alpha_1 = \frac{1}{2}x$ and $\alpha_2 = 0$ in $x \le \frac{1}{2}2$, $\alpha_2 = \frac{1}{2}$ in $\frac{1}{2} < x \le 1$. Then a_1 is continuous and as f is monotonic, $f \in \mathcal{R}(\alpha_1)$ by a result in the book. Since $\alpha_2 = \frac{1}{2}I(x-\frac{1}{2})$ and f is continuous at $\frac{1}{2}$ combinging two results from the books shows that $f \in \mathcal{R}(\alpha_2)$. From this it follows that $f \in \mathcal{R}(\alpha)$.

(2) Let f be a continuous function on $[a, b]$. Explain whether each of the following statements is always true, with brief but precise reasoning.

(a) The function $g(x) = \int_x^b f(y)dy$ is well defined.
Yes, $f \in \mathcal{R}$ for any subinterval.

(b) The function g is continuous.
Yes, the integral is a continous function of the lower limit.

(c) The function g is decreasing.
No, not unless $f \ge 0$.

(d) The function g is uniformly continuous.
Yes, continous on a compact set implies uniformly continuous.

(e) The function g is differentiable.
Yes, g is differentiable since f is continuous.

(f) The derivative $g' = f$ on $[a, b]$.
No, you fiend, it is $g' = -f$ since it is the lower limit!

(3) If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is differntiable and satisfies $f(-10) = 10$, $f(0) = 0$, $f(10) = 10$ show that there is a point where $f'(x) = 1/2$.

Solution: Applying the mean value theorem twice, there are points $z \in (-10, 0)$ where $f'(z) = -1$ and $y \in (0, 10)$ where $f(y) = 1$. From the intermediate value theorem for derivatives there must exist a point $x \in (z, y)$ at which $f'(x) = \frac{1}{2}$.

(4) If f is a strictly positive continuous function on $[-1, 1]$, meaning $\inf_{[-1,1]} f > 0$, show that $g(x) = \sqrt{f(x)}$ is continuous.

Solution: The function $\sqrt{}:(0,\infty) \longrightarrow (0,\infty)$ is continuous since if $x, y > t > 0$, $t < 1$, then

$$|\sqrt{x} - \sqrt{y}| = \frac{|x-y|}{\sqrt{x}+\sqrt{y}} < \epsilon$$

if $|x - y| \le \epsilon t$ (since $\sqrt{t} > t$). The composite of two continuous functions is continuous so $\sqrt{f} : [-1, 1] \longrightarrow (0, \infty)$ is continuous.

(5) (This is basically Rudin Problem 4.14)

Let $f : [0, 1] \longrightarrow [0, 1]$ be continuous.

(a) State why the the map $g(x) = f(x) - x$, from $[0, 1]$ to $\mathbb{R}$ is continuous.
(b) Using this, or otherwise, show that $L = \{x \in [0, 1]; f(x) \le x\}$ is closed and $\{x \in [0, 1]; f(x) < x\}$ is open.
(c) Show that L is not empty.
(d) Suppose that $f(x) \ne x$ for all $x \in [0, 1]$ and conclude that L is open in $[0, 1]$ and that $L \ne [0, 1]$.
(e) Conclude from this, or otherwise, that there must in fact be a point $x \in [0, 1]$ such that $f(x) = x$.

(6) Consider the function

$$f(x) = \frac{-x(x+1)(x-100)}{x^{44}+x^{34}+1}$$

for $x \in [0, 100]$.

(a) Explain why f is differentiable.
(b) Compute $f'(0)$.
(c) Show that there exists $\epsilon > 0$ such that $f(x) > 0$ for $0 < x < \epsilon$.
(d) Show that there must exist a point x with $f'(x) = 0$ and $0 < x < 100$.